ON A FEW NEW INEQUALITIES SIMILAR TO HILBERT'S INEQUALITY

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Abstract. In this paper, we first establish a new inequality similar to Hilbert's inequality. Then the integral inequality similar to the new inequality is also given.

1. Introduction

The well-known inequality due to Hilbert can be stated as follows [1].

Theorem 1. *If $p > 1$, $p' > \frac{p}{p-1}$ and $\sum a_m^p \le A$, $\sum b_n^{p'} \le B$, the summations running from 1 to ∞, then*

$$\sum\sum \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} A^{\frac{1}{p}} B^{\frac{1}{p'}} \tag{1.1}$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

The inequality in Theorem 1 was studied extensively and numerous variants, generalizations and extensions appeared in the literature [1] – [7]. The main purpose of the present article is to first establish a new inequality similar to the Hilbert inequality given in Theorem 1. Then the integral inequality of our main result is also given.

The following two lemmas are easily proved.

Lemma 1. *If $a_k \ge 0$, $(k = 1, 2, \ldots, n)$ and m is a natural number, then*

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^m \le \frac{1}{n}\sum_{k=1}^{n} a_k^m.$$

Lemma 2. *If $f(x) \ge 0$ and $f(x)$ is an integrable function for $a < x < b$, then*

$$\left(\int_a^b f(x)\,dx\right)^n \le (b-a)^{n-1}\int_a^b f^n(x)\,dx.$$

2. A New Inequality Similar to (1.1)

Our main result is given in the following theorems.

Theorem 2. *Let $p \ge 1$, $q \ge 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ where k, r and e are*

Key words and phrases. Hilbert's inequality, Integral inequality, Integrable function.

the natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ *and* $B_n = \sum_{i=1}^{n} b_t$. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q (mn)^{\frac{2}{e}}}{\left(m\cdot n^{\frac{1}{e}}\right)^2+\left(n\cdot m^{\frac{1}{e}}\right)^2} \leq \frac{1}{2}pq\,(kr)^{\frac{e-1}{e}} \times \left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^e\right)^{\frac{1}{e}} \times \left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^e\right)^{\frac{1}{e}} \tag{2.1}$$

Proof. By using the following inequality [8]

$$\left(\sum_{m=1}^{n} Z_m\right)^{\alpha} \leq \alpha \sum_{m=1}^{n} Z_m \left(\sum_{k=1}^{m} Z_k\right)^{\alpha-1},$$

where $\alpha \geq 1$ is a constant and $Z_m \geq 0$ $(m = 1, 2, \dots)$ it is easy to observe that

$$A_m^p = \left(\sum_{s=1}^{m} a_s\right)^p \leq p\sum_{s=1}^{m} a_s A_s^{p-1}, \quad m = 1, 2, \dots, k$$

$$B_n^q = \left(\sum_{t=1}^{n} b_t\right)^q \leq q\sum_{t=1}^{n} b_t B_t^{q-1}, \quad n = 1, 2, \dots, r.$$

Hence

$$A_m^p B_n^q \leq pq\left(\sum_{s=1}^{m} a_s A_s^{p-1}\right)\left(\sum_{t=1}^{n} b_t B_t^{q-1}\right). \tag{2.2}$$

Moreover, by Lemma 1, we have

$$\sum_{s=1}^{m} a_s A_s^{p-1} \leq m^{\frac{e-1}{e}}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^e\right)^{\frac{1}{e}}, \tag{2.3}$$

$$\sum_{t=1}^{n} b_t B_t^{q-1} \leq n^{\frac{e-1}{e}}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^e\right)^{\frac{1}{e}}. \tag{2.4}$$

Therefore, by (2.2), (2.3) and (2.4) we get

$$A_m^p B_n^q \leq pq\cdot m^{\frac{e-1}{e}}\cdot n^{\frac{e-1}{e}}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^e\right)^{\frac{1}{e}}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^e\right)^{\frac{1}{e}} \leq \frac{1}{2}pq\left(m^{\frac{2(e-1)}{e}}+n^{\frac{2(e-1)}{e}}\right)\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^e\right)^{\frac{1}{e}}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^e\right)^{\frac{1}{e}}. \tag{2.5}$$

Dividing both sides of (2.5) by $\left(m^{\frac{2(e-1)}{e}}+n^{\frac{2(e-1)}{e}}\right)$ and then taking the sum over n from 1 to r first and then the sum over m from 1 to k and using again Lemma 1, and then interchanging the order of the summations [8]. We observe that

$$
\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q (mn)^{\frac{2}{e}}}{\left(m\cdot n^{\frac{1}{e}}\right)^2+\left(n\cdot m^{\frac{1}{e}}\right)^2}\\
&\leq \frac{1}{2}pq\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^e\right)^{\frac{1}{e}}\right)\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^e\right)^{\frac{1}{e}}\right)\\
&\leq \frac{1}{2}pq\cdot k^{\frac{e-1}{e}}\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^e\right)\right)^{\frac{1}{e}}\\
&\quad\times r^{\frac{e-1}{e}}\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^e\right)\right)^{\frac{1}{e}}\\
&=\frac{1}{2}pq\,(kr)^{\frac{e-1}{e}}\left(\sum_{s=1}^{k}\left(a_sA_s^{p-1}\right)^e\left(\sum_{m=s}^{k}1\right)\right)^{\frac{1}{e}}\\
&\quad\times\left(\sum_{t=1}^{r}\left(b_tB_t^{q-1}\right)^e\left(\sum_{n=t}^{r}1\right)\right)^{\frac{1}{e}}\\
&=\frac{1}{2}pq\,(kr)^{\frac{e-1}{e}}\times\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^e\right)^{\frac{1}{e}}\\
&\quad\times\left(\sum_{n=1}^{r}(k-n+1)\left(b_nB_n^{q-1}\right)^e\right)^{\frac{1}{e}}.
\end{aligned}
$$

This completes the proof. ∎

According to Theorem 2, we shall get some important results.

Corollary 1. *Under the hypotheses of Theorem 2, if $e=2$, then*

$$
\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{m+n}\leq\frac{1}{2}pq\sqrt{kr}&\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^2\right)^{\frac{1}{2}}
\end{aligned}
\tag{2.6}
$$

Corollary 2. *Under the hypotheses of corollary 1, we have*

$$
\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{m+n}\leq\frac{1}{4}pq\sqrt{kr}&\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^2\right.\\
&\left.+\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^2\right)
\end{aligned}
\tag{2.7}
$$

Corollary 3. *Under the hypotheses of Theorem 2, if $e = 2$ and $p = q = 1$, then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m B_n}{m+n} \le \frac{1}{2}\sqrt{kr}\left(\sum_{m=1}^{k}(k-m+1)\,a_m^2\right)^{\frac{1}{2}} \times\left(\sum_{n=1}^{r}(r-n+1)\,b_n^2\right)^{\frac{1}{2}} \tag{2.8}$$

3. An Integral Inequality Similar to (2.1)

Theorem 3. *Let $p \ge 1$, $q \ge 1$ and $f(x) \ge 0$, $g(y) \ge 0$ for $x \in (0,a)$, $y \in (0,b)$, where a, b are positive real numbers and define $F(s) = \int_0^s f(x)\,dx$ and $G(t) = \int_0^t g(y)\,dy$, for $s \in (0,a)$, $t \in (0,b)$. Then*

$$\int_0^a\int_0^b \frac{F^p(s)\,G^q(t)\,(st)^{\frac{2}{e}}}{\left(s\cdot t^{\frac{1}{e}}\right)^2+\left(t\cdot s^{\frac{1}{e}}\right)^2}dsdt \le \frac{1}{2}pq\,(ab)^{\frac{e-1}{e}}\left(\int_0^a (a-s)\left(F^{p-1}(s)\,f(s)\right)^e ds\right)^{\frac{1}{e}} \times\left(\int_0^b (b-t)\left(G^{q-1}(t)\,g(t)\right)^e dt\right)^{\frac{1}{e}},\quad e\in\mathbb{N}, \tag{3.1}$$

Proof. From the hypotheses, it is easy to observe that

$$F^p(s) = p\int_0^s F^{p-1}(x)\,f(x)\,dx, \qquad s\in(0,a),$$
$$G^q(t) = q\int_0^t G^{q-1}(y)\,g(y)\,dy, \qquad t\in(0,b).$$

Therefore

$$F^p(s)\,G^q(t) = pq\left(\int_0^s F^{p-1}(x)\,f(x)\,dx\right)\left(\int_0^t G^{q-1}(y)\,g(y)\,dy\right). \tag{3.2}$$

On the other hand, according to Lemma 2, we have for $e = 1, 2, \ldots,$

$$\int_0^s F^{p-1}(x)\,f(x)\,dx \le s^{\frac{e-1}{e}}\left(\int_0^s \left(F^{p-1}(x)\,f(x)\right)^e dx\right)^{\frac{1}{e}}, \tag{3.3}$$

$$\int_0^t G^{q-1}(y)\,g(y)\,dy \le t^{\frac{e-1}{e}}\left(\int_0^t \left(G^{q-1}(y)\,g(y)\right)^e dy\right)^{\frac{1}{e}}. \tag{3.4}$$

By (3.2), (3.3) and (3.4) yield that

$$F^p(s)\,G^q(t) \le pq\,(st)^{\frac{e-1}{e}}\left(\int_0^s \left(F^{p-1}(x)\,f(x)\right)^e dx\right)^{\frac{1}{e}}\left(\int_0^t \left(G^{q-1}(y)\,g(y)\right)^e dy\right)^{\frac{1}{e}}$$
$$\le \frac{1}{2}pq\left(s^{\frac{2(e-1)}{e}}+t^{\frac{2(e-1)}{e}}\right)\left(\int_0^s \left(F^{p-1}(x)\,f(x)\right)^e dx\right)^{\frac{1}{e}} \times\left(\int_0^t \left(G^{q-1}(y)\,g(y)\right)^e dy\right)^{\frac{1}{e}}.$$

Thus

$$\frac{F^{p}(s)\,G^{q}(t)\,(st)^{\frac{2}{e}}}{\left(s\cdot t^{\frac{1}{e}}\right)^{2}+\left(t\cdot s^{\frac{1}{e}}\right)^{2}}$$
$$\leq \frac{1}{2}pq\left(\int_{0}^{s}\left(F^{p-1}(x)\,f(x)\right)^{e}dx\right)^{\frac{1}{e}}\left(\int_{0}^{t}\left(G^{q-1}(y)\,g(y)\right)^{e}dy\right)^{\frac{1}{e}}.$$

Integrating over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and using again the Lemma 2, we observe that

$$\int_{0}^{a}\int_{0}^{b}\frac{F^{p}(s)\,G^{q}(t)\,(st)^{\frac{2}{e}}}{\left(s\cdot t^{\frac{1}{e}}\right)^{2}+\left(t\cdot s^{\frac{1}{e}}\right)^{2}}dsdt$$
$$\leq \frac{1}{2}pq\left(\int_{0}^{a}\left(\int_{0}^{s}\left(F^{p-1}(x)\,f(x)\right)^{e}dx\right)^{\frac{1}{e}}ds\right)$$
$$\times\left(\int_{0}^{b}\left(\int_{0}^{t}\left(G^{q-1}(y)\,g(y)\right)^{e}dy\right)^{\frac{1}{e}}dt\right)$$
$$\leq \frac{1}{2}pq\cdot a^{\frac{e-1}{e}}\left\{\int_{0}^{a}\left(\int_{0}^{s}\left(F^{p-1}(x)\,f(x)\right)^{e}dx\right)ds\right\}^{\frac{1}{e}}$$
$$\times\, b^{\frac{e-1}{e}}\left\{\int_{0}^{b}\left(\int_{0}^{t}\left(G^{q-1}(y)\,g(y)\right)^{e}dy\right)dt\right\}^{\frac{1}{e}}$$
$$=\frac{1}{2}pq\,(ab)^{\frac{e-1}{e}}\left(\int_{0}^{a}(a-s)\left(F^{p-1}(s)\,f(s)\right)^{e}ds\right)^{\frac{1}{e}}$$
$$\times\left(\int_{0}^{b}(b-t)\left(G^{q-1}(t)\,g(t)\right)^{e}dt\right)^{\frac{1}{e}}.$$

The proof is complete. ∎

Similarly, we can also get the following.

Corollary 4. *Under the hypotheses of Theorem 3, if* $e=2$, *then*

$$\int_{0}^{a}\int_{0}^{b}\frac{F^{p}(s)\,G^{q}(t)}{s+t}dsdt\leq\frac{1}{2}pq\sqrt{ab}\left(\int_{0}^{a}(a-s)\left(F^{p-1}(s)\,f(s)\right)^{2}ds\right)^{\frac{1}{2}}$$
$$\times\left(\int_{0}^{b}(b-t)\left(G^{q-1}(t)\,g(t)\right)^{2}dt\right)^{\frac{1}{2}},$$

Corollary 5. *Under the hypotheses of Corollary 4, then*

$$\int_{0}^{a}\int_{0}^{b}\frac{F^{p}(s)\,G^{q}(t)}{s+t}dsdt\leq\frac{1}{4}pq\sqrt{ab}\left(\int_{0}^{a}(a-s)\left(F^{p-1}(s)\,f(s)\right)^{2}ds\right.$$
$$\left.+\int_{0}^{b}(b-t)\left(G^{q-1}(t)\,g(t)\right)^{2}dt\right),$$

Corollary 6. *Under the hypotheses of Theorem 3, if* $e = 2$, *and* $p = q = 1$, *then*

$$\int_0^a \int_0^b \frac{F(s)\,G(t)}{s+t} dsdt \le \frac{1}{2}\sqrt{ab} \left(\int_0^a (a-s)\, f^2(s)\, ds \right)^{\frac{1}{2}} \left(\int_0^b (b-t)\, g^2(t)\, dt \right)^{\frac{1}{2}},$$

Acknowledgement 1. *We wish to express my gratitude to Professor Li Wenrong for his valuable help in writing this paper.*

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