

AN INTEGRAL APPROXIMATION IN THREE VARIABLES

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ABSTRACT. In this paper we will investigate a method of approximating an integral in three independent variables. The Ostrowski type inequality is established by the use of Peano kernels and improves a result given by Pachpatte.

1. INTRODUCTION

The numerical estimation of the integral, or multiple integral of a function over some specified interval is important in many scientific applications. Generally speaking, the error bound for the midpoint rule is about one half of the trapezoidal rule and Stewart [14] has a nice geometrical explanation of this generality. The speed of convergence of an integral is also important and Weideman [15] has some pertinent examples illustrating perfect, algebraic, geometric, super-geometric and sub-geometric convergence for periodic functions.

In particular, we shall establish an Ostrowski type inequality for a triple integral which improves a result given by Pachpatte [10].

In 1938 Ostrowski [7] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The following definitions will be used in the following exposition

$$(1.1) \quad \mathcal{M}(f) := \frac{1}{b-a} \int_a^b f(t) dt,$$

$$(1.2) \quad I_T(f) := \frac{f(b) + f(a)}{2}$$

and

$$(1.3) \quad I_M(f) := f\left(\frac{a+b}{2}\right).$$

The Ostrowski result is given by:

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , that is,*

$$\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty.$$

Then we have the inequality

$$(1.4) \quad |f(x) - \mathcal{M}(f)| \leq \left(\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty$$

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for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

Improvements of the result (1.4) has also been obtained by Dedić, Matić and Pearce [2], Pearce, Pečarić, Ujević and Varošaneć [11], Dragomir [3] and Sofo [12]. For a symmetrical point $x \in [a, \frac{a+b}{2}]$, very recently Guessab and Schmeisser [4] studied the more general quadrature formula

$$\mathcal{M}(f) - \left[\frac{f(x) + f(a+b-x)}{2} \right] = E(f; x)$$

where $E(f; x)$ is the remainder.

For $x = \frac{a+b}{2}$ and f defined on $[a, b]$ with Lipschitz constant M , then

$$|\mathcal{M}(f) - I_M(f)| \leq \frac{M(b-a)}{4}.$$

For $x = a$, then

$$|\mathcal{M}(f) - I_T(f)| \leq \frac{M(b-a)}{4}.$$

The following result, which is a generalisation of Theorem 1, was given by Milovanović [6, p. 468] in 1975 concerning a function, f , of several variables.

Theorem 2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function defined on $D = \{(x_1, \dots, x_m) \mid a_i \leq x_i \leq b_i, (i = 1, \dots, m)\}$ and let $\left| \frac{\partial f}{\partial x_i} \right| \leq M_i$ ($M_i > 0, i = 1, \dots, m$) in D . Furthermore, let $x \mapsto p(x)$ be integrable and $p(x) > 0$ for every $x \in D$. Then for every $x \in D$, we have the inequality:*

$$(1.5) \quad \left| f(x) - \frac{\int_D p(y) f(y) dy}{\int_D p(y) dy} \right| \leq \frac{\sum_{i=1}^m M_i \int_D p(y) |x_i - y_i| dy}{\int_D p(y) dy}.$$

In 2001, Barnett and Dragomir [1] obtained the following Ostrowski type inequality for double integrals.

Theorem 3. *Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be continuous on $[a, b] \times [c, d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exist on $(a, b) \times (c, d)$ and is bounded, that is,*

$$\|f''_{s,t}\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty,$$

then we have the inequality:

$$(1.6) \quad \left| \int_a^b \int_c^d f(s,t) ds dt - (b-a) \int_c^d f(x,t) dt - (d-c) \int_a^b f(s,y) ds + (d-c)(b-a)f(x,y) \right| \leq \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \times \left[\frac{(d-c)^2}{4} + \left(y - \frac{c+d}{2} \right)^2 \right] \|f''_{s,t}\|_\infty$$

for all $(x, y) \in [a, b] \times [c, d]$.

Pachpatte [8], obtained an inequality in the vein of (1.6) but used elementary analysis in his proof.

Pachpatte [9] also obtains a discrete version of an inequality with two independent variables. Hanna, Dragomir and Cerone [5] obtained a further complementary result to (1.6) and Sofo [13] further improved the result (1.6).

2. TRIPLE INTEGRALS

In three independent variables Pachpatte obtains several results. For discrete variables he obtains a result in [9] and in [10] for continuous variables he obtained the following.

Theorem 4. *Let $\Delta := [a, k] \times [b, m] \times [c, n]$ for $a, b, c, k, m, n \in \mathbb{R}^+$ and $f(r, s, t)$ be differentiable on Δ . Denote the partial derivatives by $D_1 f(r, s, t) = \frac{\partial}{\partial r} f(r, s, t)$; $D_2 f(r, s, t) = \frac{\partial}{\partial s} f(r, s, t)$, $D_3 f(r, s, t) = \frac{\partial}{\partial t} f(r, s, t)$ and $D_3 D_2 D_1 f = \frac{\partial^3 f}{\partial t \partial s \partial r}$. Let $F(\Delta)$ be the class of continuous functions $f : \Delta \rightarrow \mathbb{R}$ for which $D_1 f, D_2 f, D_3 f, D_3 D_2 D_1 f$ exist and are continuous on Δ . For $f \in F(\Delta)$ we have*

$$\begin{aligned}
(2.1) \quad & \left| \int_a^k \int_b^m \int_c^n f(r, s, t) dt ds dr \right. \\
& - \frac{1}{8} (k-a)(m-b)(n-c) [f(a, b, c) + f(k, m, n)] \\
& + \frac{1}{4} (m-b)(n-c) \int_a^k [f(r, b, c) + f(r, m, n) + f(r, m, c) + f(r, b, n)] dr \\
& + \frac{1}{4} (k-a)(n-c) \int_b^m [f(a, s, c) + f(k, s, n) + f(a, s, n) + f(k, s, c)] ds \\
& + \frac{1}{4} (k-a)(m-b) \int_c^n [f(a, b, t) + f(k, m, t) + f(k, b, t) + f(a, m, t)] dt \\
& - \frac{1}{2} (k-a) \int_b^m \int_c^n [f(a, s, t) + f(k, s, t)] dt ds \\
& - \frac{1}{2} (m-b) \int_a^k \int_c^n [f(r, b, t) + f(r, m, t)] dt dr \\
& \left. - \frac{1}{2} (n-c) \int_a^k \int_b^m [f(r, s, c) + f(r, s, n)] ds dr \right| \\
& \leq \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(r, s, t)| dt ds dr.
\end{aligned}$$

The following theorem establishes an Ostrowski type identity for an integral in three independent variables.

Theorem 5. *Let $f : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \rightarrow \mathbb{R}$ be a continuous mapping such that the following partial derivatives $\frac{\partial^{i+j+k} f(\cdot, \cdot, \cdot)}{\partial x^i \partial y^j \partial z^k}$; $i = 0, \dots, n-1$, $j = 0, \dots, m-1$; $k = 0, \dots, p-1$ exist and are continuous on $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$. Also, let*

$$(2.2) \quad P_n(x, r) := \begin{cases} \frac{(r-a_1)^n}{n!}; & r \in [a_1, x], \\ \frac{(r-b_1)^n}{n!}; & r \in [x, b_1], \end{cases}$$

$$(2.3) \quad Q_m(y, s) := \begin{cases} \frac{(s - a_2)^m}{m!}; & s \in [a_2, y), \\ \frac{(s - b_2)^m}{m!}; & s \in [y, b_2], \end{cases}$$

and

$$(2.4) \quad S_p(z, t) := \begin{cases} \frac{(t - a_3)^p}{p!}; & t \in [a_3, z), \\ \frac{(t - b_3)^p}{p!}; & t \in [z, b_3], \end{cases}$$

then for all $(x, y, z) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ we have the identity

$$(2.5) \quad \begin{aligned} V &:= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(r, s, t) dt ds dr \\ &- \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} X_i(x) Y_j(y) Z_k(z) \frac{\partial^{i+j+k} f(x, y, z)}{\partial x^i \partial y^j \partial z^k} \\ &+ (-1)^p \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i(x) Y_j(y) \int_{a_3}^{b_3} S_p(z, t) \frac{\partial^{i+j+p} f(x, y, t)}{\partial x^i \partial y^j \partial t^p} dt \\ &+ (-1)^m \sum_{i=0}^{n-1} \sum_{k=0}^{p-1} X_i(x) Z_k(z) \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^{i+m+k} f(x, s, z)}{\partial x^i \partial s^m \partial z^k} ds \\ &+ (-1)^n \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} Y_j(y) Z_k(z) \int_{a_1}^{b_1} P_n(x, r) \frac{\partial^{n+j+k} f(r, y, z)}{\partial r^n \partial y^j \partial z^k} dr \\ &- (-1)^{m+p} \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} \int_{a_3}^{b_3} Q_m(y, s) S_p(z, t) \frac{\partial^{i+m+p} f(x, s, t)}{\partial x^i \partial s^m \partial t^p} dt ds \\ &- (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(y) \int_{a_1}^{b_1} \int_{a_3}^{b_3} P_n(x, r) S_p(z, t) \frac{\partial^{n+j+p} f(r, y, t)}{\partial r^n \partial y^j \partial t^p} dt dr \\ &- (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(z) \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(x, r) Q_m(y, s) \frac{\partial^{n+m+k} f(r, s, z)}{\partial r^n \partial s^m \partial z^k} ds dr \\ &= -(-1)^{n+m+p} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(x, r) Q_m(y, s) S_p(z, t) \\ &\quad \times \frac{\partial^{n+m+p} f(r, s, t)}{\partial r^n \partial s^m \partial t^p} dt ds dr, \end{aligned}$$

where

$$(2.6) \quad X_i(x) := \frac{(b_1 - x)^{i+1} + (-1)^i (x - a_1)^{i+1}}{(i+1)!},$$

$$(2.7) \quad Y_j(y) := \frac{(b_2 - y)^{j+1} + (-1)^j (y - a_2)^{j+1}}{(j+1)!},$$

and

$$(2.8) \quad Z_k(z) := \frac{(b_3 - z)^{k+1} + (-1)^k (z - a_3)^{k+1}}{(k+1)!}.$$

Proof. We have an identity, see [5]

$$(2.9) \quad \int_{a_1}^{b_1} g(r) dr = \sum_{i=0}^{n-1} X_i(x) g^{(i)}(x) + (-1)^n \int_{a_1}^{b_1} P_n(x, r) g^{(n)}(r) dr.$$

Now for the partial mapping $f(\cdot, s, t)$, $s \in [a_2, b_2]$, we have

$$(2.10) \quad \int_{a_1}^{b_1} f(r, s, t) dr = \sum_{i=0}^{n-1} X_i(x) \frac{\partial^i f}{\partial x^i} + (-1)^n \int_{a_1}^{b_1} P_n(x, r) \frac{\partial^n f}{\partial r^n} dr$$

for every $r \in [a_1, b_1]$, $s \in [a_2, b_2]$ and $t \in [a_3, b_3]$.

Now integrate over $s \in [a_2, b_2]$

$$(2.11) \quad \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(r, s, t) ds dr \\ = \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} \frac{\partial^i f}{\partial x^i} ds + (-1)^n \int_{a_1}^{b_1} P_n(x, r) \left(\int_{a_2}^{b_2} \frac{\partial^n f}{\partial r^n} ds \right) dt$$

for all $x \in [a_1, b_1]$.

From (2.9) for the partial mapping $\frac{\partial^i f}{\partial x^i}$ on $[a_2, b_2]$ we have,

$$(2.12) \quad \int_{a_2}^{b_2} \frac{\partial^i f}{\partial x^i}(x, s, t) ds \\ = \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^j}{\partial y^j} \left(\frac{\partial^i f}{\partial x^i} \right) + (-1)^m \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^i f}{\partial x^i} \right) ds \\ = \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds.$$

Also, from (2.8)

$$(2.13) \quad \int_{a_2}^{b_2} \frac{\partial^n f}{\partial r^n} ds = \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{j+n} f}{\partial y^j \partial r^n} + (-1)^m \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^n f}{\partial r^n} \right) ds.$$

From (2.11) substitute (2.12) and (2.13), so that

$$(2.14) \quad \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(r, s, t) ds dr \\ = \sum_{i=0}^{n-1} X_i(x) \left[\sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds \right]$$

$$\begin{aligned}
& + (-1)^n \int_{a_1}^{b_1} P_n(x, r) \left[\sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{j+n} f}{\partial y^j \partial r^n} + (-1)^m \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^n f}{\partial r^n} \right) ds \right] dt \\
& = \sum_{i=0}^{n-1} X_i(x) \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} Q_m(y, s) \frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds \\
& + (-1)^n \sum_{j=0}^{m-1} Y_j(y) \int_{a_1}^{b_1} P_n(x, r) \frac{\partial^{j+n} f}{\partial y^j \partial r^n} \\
& + (-1)^{n+m} \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(x, r) Q_m(y, s) \frac{\partial^{n+m} f}{\partial s^m \partial r^n} ds dr
\end{aligned}$$

Now integrate (2.14) for $t \in [a_3, b_3]$

$$\begin{aligned}
(2.15) \quad & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(r, s, t) dt ds dr = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i(x) Y_j(y) \int_{a_3}^{b_3} \frac{\partial^{i+j} f}{\partial x^i \partial y^j} dt \\
& + (-1)^m \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} Q_m(y, s) \left(\int_{a_3}^{b_3} \frac{\partial^{i+m} f}{\partial x^i \partial s^m} dt \right) ds \\
& + (-1)^n \sum_{j=0}^{m-1} Y_j(y) \int_{a_2}^{b_2} P_n(x, r) \left(\int_{a_3}^{b_3} \frac{\partial^{j+n} f}{\partial y^j \partial r^n} dt \right) dr \\
& + (-1)^{n+m} \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(x, r) Q_m(y, s) \left(\int_{a_3}^{b_3} \frac{\partial^{n+m} f}{\partial s^m \partial r^n} dt \right) ds dr.
\end{aligned}$$

From (2.9),

$$\begin{aligned}
(2.16) \quad & \int_{a_3}^{b_3} \frac{\partial^{i+j} f}{\partial x^i \partial y^j} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right) \\
& + (-1)^p \int_{a_3}^{b_3} S_p(z, t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right) dt,
\end{aligned}$$

$$\begin{aligned}
(2.17) \quad & \int_{a_3}^{b_3} \frac{\partial^{i+m} f}{\partial x^i \partial s^m} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m} \right) \\
& + (-1)^p \int_{a_3}^{b_3} S_p(z, t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m} \right) dt,
\end{aligned}$$

$$\begin{aligned}
& \int_{a_3}^{b_3} \frac{\partial^{j+n} f}{\partial y^j \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n} \right) \\
& + (-1)^p \int_{a_3}^{b_3} S_p(z, t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n} \right) dt,
\end{aligned}$$

and

$$(2.18) \quad \int_{a_3}^{b_3} \frac{\partial^{n+m} f}{\partial s^m \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m} \right) \\ + (-1)^p \int_{a_3}^{b_3} S_p(z, t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m} \right) dt.$$

Putting (2.16), (2.17) and (2.18) into (2.15) we arrive at the identity (2.5). ■

At the midpoint of the interval

$$\bar{x} = \frac{a_1 + b_1}{2}, \quad \bar{y} = \frac{a_2 + b_2}{2}, \quad \bar{z} = \frac{a_3 + b_3}{2}$$

we have the following corollary.

Corollary 1. *Under the assumptions of Theorem 5, we have the identity*

$$(2.19) \quad \bar{V} := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(r, s, t) dt ds dr \\ - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} X_i(\bar{x}) Y_j(\bar{y}) Z_k(\bar{z}) \frac{\partial^{i+j+k} f(\bar{x}, \bar{y}, \bar{z})}{\partial x^i \partial y^j \partial z^k} \\ + (-1)^p \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i(\bar{x}) Y_j(\bar{y}) \int_{a_3}^{b_3} S_p(\bar{z}, t) \frac{\partial^{i+j+p} f(\bar{x}, \bar{y}, t)}{\partial x^i \partial y^j \partial t^p} dt \\ + (-1)^m \sum_{i=0}^{n-1} \sum_{k=0}^{p-1} X_i(\bar{x}) Z_k(\bar{z}) \int_{a_2}^{b_2} Q_m(\bar{y}, s) \frac{\partial^{i+m+k} f(\bar{x}, s, \bar{z})}{\partial x^i \partial s^m \partial z^k} ds \\ + (-1)^n \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} Y_j(\bar{y}) Z_k(\bar{z}) \int_{a_1}^{b_1} P_n(\bar{x}, r) \frac{\partial^{n+j+k} f(r, \bar{y}, \bar{z})}{\partial r^n \partial y^j \partial z^k} dr \\ - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(\bar{x}) \int_{a_2}^{b_2} \int_{a_3}^{b_3} Q_m(\bar{y}, s) S_p(\bar{z}, t) \frac{\partial^{i+m+p} f(\bar{x}, s, t)}{\partial x^i \partial s^m \partial t^p} dt ds \\ - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(\bar{y}) \int_{a_1}^{b_1} \int_{a_3}^{b_3} P_n(\bar{x}, r) S_p(\bar{z}, t) \frac{\partial^{n+j+p} f(r, \bar{y}, t)}{\partial r^n \partial y^j \partial t^p} dt dr \\ - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(\bar{z}) \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(\bar{x}, r) Q_m(\bar{y}, s) \frac{\partial^{n+m+k} f(r, s, \bar{z})}{\partial r^n \partial s^m \partial z^k} ds dr \\ = - (-1)^{n+m+p} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(\bar{x}, r) Q_m(\bar{y}, s) S_p(\bar{z}, t) \\ \times \frac{\partial^{n+m+p} f(r, s, t)}{\partial r^n \partial s^m \partial t^p} dt ds dr.$$

The identity (2.5) will now be utilised to establish an inequality for a function of three independent variables which will furnish a refinement for the inequality (2.1) given by Pachpatte.

Theorem 6. *Let $f : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \rightarrow \mathbb{R}$ be continuous on $(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ and the conditions of Theorem 5 apply. Then we have the inequality*

$$|V| \leq \left\{ \begin{array}{l} \left[\frac{(x-a_1)^{n+1} + (b_1-x)^{n+1}}{(n+1)!} \right] \left[\frac{(y-a_2)^{m+1} + (b_2-y)^{m+1}}{(m+1)!} \right] \\ \quad \times \left[\frac{(z-a_3)^{p+1} + (b_3-z)^{p+1}}{(p+1)!} \right] \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty} \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\infty}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]); \\ \\ \frac{1}{n!m!p!} \left[\frac{(x-a_1)^{n\beta+1} + (b_1-x)^{n\beta+1}}{n\beta+1} \right]^{\frac{1}{\beta}} \left[\frac{(y-a_2)^{m\beta+1} + (b_2-y)^{m\beta+1}}{m\beta+1} \right]^{\frac{1}{\beta}} \\ \quad \times \left[\frac{(z-a_3)^{p\beta+1} + (b_3-z)^{p\beta+1}}{p\beta+1} \right]^{\frac{1}{\beta}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha} \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\alpha}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]), \quad \alpha > 1, \quad \alpha^{-1} + \beta^{-1} = 1; \\ \\ \frac{1}{8n!m!p!} [(x-a_1)^n + (b_1-x)^n + |(x-a_1)^n - (b_1-x)^n|] \\ \quad \times [(y-a_2)^m + (b_2-y)^m + |(y-a_2)^m - (b_2-y)^m|] \\ \quad \times [(z-a_3)^p + (b_3-z)^p + |(z-a_3)^p - (b_3-z)^p|] \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1 \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]); \end{array} \right.$$

for all $(x, y, z) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, where

$$\left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty} = \sup_{(r,s,t) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]} \left| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right| < \infty,$$

and

$$(2.20) \quad \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha} = \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right|^{\alpha} dt ds dr \right)^{\frac{1}{\alpha}} < \infty.$$

Proof.

$$\begin{aligned} |V| &= \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(x, r) Q_m(y, s) S_p(z, t) \frac{\partial^{n+m+p} f(r, s, t)}{\partial r^n \partial s^m \partial t^p} dt ds dr \right| \\ &\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)| \left| \frac{\partial^{n+m+p} f(r, s, t)}{\partial r^n \partial s^m \partial t^p} \right| dt ds dr. \end{aligned}$$

Using Hölder's inequality and property of the modulus and integral, then we have that

$$(2.21) \quad \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)| \left| \frac{\partial^{n+m+p} f(r, s, t)}{\partial r^n \partial s^m \partial t^p} \right| dt ds dr$$

$$\leq \begin{cases} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)| dt ds dr, \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)|^{\beta} dt ds dr \right)^{\frac{1}{\beta}}, \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1 \sup_{(r,s,t) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]} |P_n(x, r) Q_m(y, s) S_p(z, t)|. \end{cases}$$

$\alpha > 1, \alpha^{-1} + \beta^{-1} = 1;$

From (2.21) and using (2.2), (2.3) and (2.4)

$$\begin{aligned} & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)| dt ds dr \\ &= \int_{a_1}^{b_1} |P_n(x, r)| dr \int_{a_2}^{b_2} |Q_m(y, s)| ds \int_{a_3}^{b_3} |S_p(z, t)| dt \\ &= \left[\int_{a_1}^x \frac{(r-a_1)^n}{n!} dr + \int_x^{b_1} \frac{(b_1-r)^n}{n!} dr \right] \left[\int_{a_2}^y \frac{(s-a_2)^m}{m!} ds + \int_y^{b_2} \frac{(b_2-s)^m}{m!} ds \right] \\ &\quad \times \left[\int_{a_3}^z \frac{(t-a_3)^p}{p!} dt + \int_z^{b_3} \frac{(b_3-t)^p}{p!} dt \right] \\ &= \frac{[(x-a_1)^{n+1} + (b_1-x)^{n+1}] [(y-a_2)^{m+1} + (b_2-y)^{m+1}]}{(n+1)!(m+1)!} \\ &\quad \times \frac{[(z-a_3)^{p+1} + (b_3-z)^{p+1}]}{(p+1)!} \end{aligned}$$

giving the first inequality in (2.20).

Now, if we again use (2.21) we have

$$\begin{aligned} & \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x, r) Q_m(y, s) S_p(z, t)|^{\beta} dt ds dr \right)^{\frac{1}{\beta}} \\ &= \left(\int_{a_1}^{b_1} |P_n(x, r)|^{\beta} dr \right)^{\frac{1}{\beta}} \left(\int_{a_2}^{b_2} |Q_m(y, s)|^{\beta} ds \right)^{\frac{1}{\beta}} \left(\int_{a_3}^{b_3} |S_p(z, t)|^{\beta} dt \right)^{\frac{1}{\beta}} \\ &= \frac{1}{n!m!p!} \left[\int_{a_1}^x (r-a_1)^{n\beta} dr + \int_x^{b_1} (b_1-r)^{n\beta} dr \right]^{\frac{1}{\beta}} \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{a_2}^y (s - a_2)^{m\beta} ds + \int_y^{b_2} (b_2 - s)^{m\beta} ds \right]^{\frac{1}{\beta}} \left[\int_{a_3}^z (t - a_3)^{p\beta} dt + \int_z^{b_3} (b_3 - t)^{p\beta} dt \right]^{\frac{1}{\beta}} \\
& = \frac{1}{n!m!p!} \left[\frac{(x - a_1)^{n\beta+1} + (b_1 - x)^{n\beta+1}}{n\beta + 1} \right]^{\frac{1}{\beta}} \left[\frac{(y - a_2)^{m\beta+1} + (b_2 - y)^{m\beta+1}}{m\beta + 1} \right]^{\frac{1}{\beta}} \\
& \times \left[\frac{(z - a_3)^{p\beta+1} + (b_3 - z)^{p\beta+1}}{p\beta + 1} \right]^{\frac{1}{\beta}}
\end{aligned}$$

producing the second inequality in (2.20).

Finally, we have

$$\begin{aligned}
& \sup_{(r,s,t) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]} |P_n(x, r) Q_m(y, s) S_p(z, t)| \\
& = \sup_{r \in [a_1, b_1]} |P_n(x, r)| \sup_{s \in [a_2, b_2]} |Q_m(y, s)| \sup_{t \in [a_3, b_3]} |S_p(z, t)| \\
& = \max \left\{ \frac{(x - a_1)^n}{n!}, \frac{(b_1 - x)^n}{n!} \right\} \max \left\{ \frac{(y - a_2)^m}{m!}, \frac{(b_2 - y)^m}{m!} \right\} \max \left\{ \frac{(z - a_3)^p}{p!}, \frac{(b_3 - z)^p}{p!} \right\} \\
& = \frac{1}{n!m!p!} \left[\frac{(x - a_1)^n + (b_1 - x)^n}{2} + \left| \frac{(x - a_1)^n - (b_1 - x)^n}{2} \right| \right] \\
& \times \left[\frac{(y - a_2)^m + (b_2 - y)^m}{2} + \left| \frac{(y - a_2)^m - (b_2 - y)^m}{2} \right| \right] \\
& \times \left[\frac{(z - a_3)^p + (b_3 - z)^p}{2} + \left| \frac{(z - a_3)^p - (b_3 - z)^p}{2} \right| \right],
\end{aligned}$$

giving us the third inequality in (2.20) and we have used the fact that for $A > 0$, $B > 0$ then

$$\max \{A, B\} = \frac{A + B}{2} + \left| \frac{A - B}{2} \right|.$$

Hence the theorem is completely solved. ■

The following corollary is a consequence of Theorem 6.

Corollary 2. *Under the assumptions of Corollary 1, we have the inequality*

$$|\bar{V}| \leq \begin{cases} \left[\frac{(b_1 - a_1)^{n+1} (b_2 - a_2)^{m+1} (b_3 - a_3)^{p+1}}{2^{n+m+p} (n+1)! (m+1)! (p+1)!} \right] \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty}, \\ \frac{1}{2^{n+m+p} n! m! p!} \left[\frac{(b_1 - a_1)^{n\beta+1} (b_2 - a_2)^{m\beta+1} (b_3 - a_3)^{p\beta+1}}{(n\beta + 1) (m\beta + 1) (p\beta + 1)} \right]^{\frac{1}{\beta}} \\ \quad \times \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha}, \\ \frac{1}{2^{n+m+p} n! m! p!} (b_1 - a_1)^n (b_2 - a_2)^m (b_3 - a_3)^p \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \end{cases}$$

where $\|\cdot\|_{\alpha}$ ($\alpha \in [1, \infty)$) are the Lebesgue norms on $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

The following two corollaries concern the estimation of V at the end points.

Corollary 3. *Under the assumptions of Theorem 6 we have, for $x = a_1$, $y = a_2$ and $z = a_3$, the inequality*

$$\begin{aligned}
& |V(a_1, a_2, a_3)| : \\
= & \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(r, s, t) dt ds dr - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} X_i(a_1) Y_j(a_2) Z_k(a_3) \frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \right. \\
& + (-1)^p \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i(a_1) Y_j(a_2) \int_{a_3}^{b_3} \bar{S}_p(a_3, t) \frac{\partial^{i+j+p} f}{\partial x^i \partial y^j \partial t^p} dt \\
& + (-1)^m \sum_{i=0}^{n-1} \sum_{k=0}^{p-1} X_i(a_1) Z_k(a_3) \int_{a_2}^{b_2} \bar{Q}_m(a_2, s) \frac{\partial^{i+m+k} f}{\partial x^i \partial s^m \partial z^k} ds \\
& + (-1)^n \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} Y_j(a_2) Z_k(a_3) \int_{a_1}^{b_1} \bar{P}_n(a_1, r) \frac{\partial^{n+j+k} f}{\partial r^n \partial y^j \partial z^k} dr \\
& - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(a_1) \int_{a_2}^{b_2} \int_{a_3}^{b_3} \bar{Q}_m(a_2, s) \bar{S}_p(a_3, t) \frac{\partial^{i+m+p} f}{\partial x^i \partial s^m \partial t^p} dt ds \\
& - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(a_2) \int_{a_1}^{b_1} \int_{a_3}^{b_3} \bar{P}_n(a_1, r) \bar{S}_p(a_3, t) \frac{\partial^{n+j+p} f}{\partial r^n \partial y^j \partial t^p} dt dr \\
& \left. - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(a_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \bar{P}_n(a_1, r) \bar{Q}_m(a_2, s) \frac{\partial^{n+m+k} f}{\partial r^n \partial s^m \partial z^k} ds dr \right| \\
(2.22) & \leq \left\{ \begin{array}{l} \frac{(b_1 - a_1)^{n+1} (b_2 - a_2)^{m+1} (b_3 - a_3)^{p+1}}{(n+1)! (m+1)! (p+1)!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty}, \\ \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\infty}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]); \\ \frac{(b_1 - a_1)^{n+\frac{1}{\beta}}}{n! (n\beta + 1)^{\frac{1}{\beta}}} \cdot \frac{(b_2 - a_2)^{m+\frac{1}{\beta}}}{m! (m\beta + 1)^{\frac{1}{\beta}}} \cdot \frac{(b_3 - a_3)^{p+\frac{1}{\beta}}}{p! (p\beta + 1)^{\frac{1}{\beta}}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha}, \\ \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\alpha}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]), \quad \alpha > 1, \quad \alpha^{-1} + \beta^{-1} = 1; \\ \frac{(b_1 - a_1)^n (b_2 - a_2)^m (b_3 - a_3)^p}{n! m! p!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\ \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]), \end{array} \right.
\end{aligned}$$

where

$$X_i(a_1) := \frac{(b_1 - a_1)^{i+1}}{(i+1)!}, \quad Y_j(a_2) := \frac{(b_2 - a_2)^{j+1}}{(j+1)!}, \quad Z_k(a_3) := \frac{(b_3 - a_3)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n(a_1, r) = \frac{(r - b_1)^n}{n!}, \quad r \in [a_1, b_1]; \quad \bar{Q}_m(a_2, s) = \frac{(s - b_2)^m}{m!}, \quad s \in [a_2, b_2]$$

and

$$\bar{S}_p(a_3, t) = \frac{(t - b_3)^p}{p!}; \quad t \in [a_3, b_3].$$

Corollary 4. *Under the assumptions of Theorem 6 we have, for $x = b_1$, $y = b_2$ and $z = b_3$, the inequality*

$$\begin{aligned} & |V(b_1, b_2, b_3)| : \\ = & \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(r, s, t) dt ds dr - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} X_i(a_1) Y_j(a_2) Z_k(a_3) \frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \right. \\ & + (-1)^p \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i(b_1) Y_j(b_2) \int_{a_3}^{b_3} \bar{S}_p(b_3, t) \frac{\partial^{i+j+p} f}{\partial x^i \partial y^j \partial t^p} dt \\ & + (-1)^m \sum_{i=0}^{n-1} \sum_{k=0}^{p-1} X_i(b_1) Z_k(b_3) \int_{a_2}^{b_2} \bar{Q}_m(b_2, s) \frac{\partial^{i+m+k} f}{\partial x^i \partial s^m \partial z^k} ds \\ & + (-1)^n \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} Y_j(b_2) Z_k(b_3) \int_{a_1}^{b_1} \bar{P}_n(b_1, r) \frac{\partial^{n+j+k} f}{\partial r^n \partial y^j \partial z^k} dr \\ & - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(b_1) \int_{a_2}^{b_2} \int_{a_3}^{b_3} \bar{Q}_m(b_2, s) \bar{S}_p(b_3, t) \frac{\partial^{i+m+p} f}{\partial x^i \partial s^m \partial t^p} dt ds \\ & - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(b_2) \int_{a_1}^{b_1} \int_{a_3}^{b_3} \bar{P}_n(b_1, r) \bar{S}_p(b_3, t) \frac{\partial^{n+j+p} f}{\partial r^n \partial y^j \partial t^p} dt dr \\ & \left. - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(b_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \bar{P}_n(b_1, r) \bar{Q}_m(b_2, s) \frac{\partial^{n+m+k} f}{\partial r^n \partial s^m \partial z^k} ds dr \right| \\ (2.23) & \leq \left\{ \begin{array}{l} \frac{(b_1 - a_1)^{n+1} (b_2 - a_2)^{m+1} (b_3 - a_3)^{p+1}}{(n+1)! (m+1)! (p+1)!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\infty}, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\infty}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]); \\ \frac{(b_1 - a_1)^{n+\frac{1}{\beta}}}{n! (n\beta + 1)^{\frac{1}{\beta}}} \cdot \frac{(b_2 - a_2)^{m+\frac{1}{\beta}}}{m! (m\beta + 1)^{\frac{1}{\beta}}} \cdot \frac{(b_3 - a_3)^{p+\frac{1}{\beta}}}{p! (p\beta + 1)^{\frac{1}{\beta}}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_{\alpha}, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_{\alpha}([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]), \quad \alpha > 1, \quad \alpha^{-1} + \beta^{-1} = 1; \\ \frac{(b_1 - a_1)^n (b_2 - a_2)^m (b_3 - a_3)^p}{n! m! p!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]), \end{array} \right. \end{aligned}$$

where

$$X_i(b_1) := \frac{(-1)^i (b_1 - a_1)^{i+1}}{(i+1)!}, \quad Y_j(b_2) := \frac{(-1)^j (b_2 - a_2)^{j+1}}{(j+1)!},$$

$$Z_k(b_3) := \frac{(-1)^k (b_3 - a_3)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n(b_1, r) = \frac{(r - a_1)^n}{n!}; \quad r \in [a_1, b_1],$$

$$\bar{Q}_m(b_2, s) = \frac{(s - a_2)^m}{m!}; \quad s \in [a_2, b_2]$$

and

$$\bar{S}_p(b_3, t) = \frac{(t - a_3)^p}{p!}; \quad t \in [a_3, b_3].$$

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