KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

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ABSTRACT. We prove an equivalent relation between Ky Fan-typed inequalities and certain bounds for the differences of means. We also generalize a result of H. Alzer, S. Ruscheweyh and L. Salinas.

1. INTRODUCTION

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^n \omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^n \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \rightarrow 0^+$. We shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$ when there is no risk of confusion.

In this paper, we always assume $0 < x_1 \leq x_2 \leq \cdots \leq x_n$. To any given $\mathbf{x}$ we associate $\mathbf{x}' = (1-x_1, 1-x_2, \cdots, 1-x_n)$ and write $A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1}$. When $1 - x_i \geq 0$ for all i, we define $A_n' = P_{n,1}(\mathbf{x}')$ and similarly for G_n', H_n'. We also let $\sigma_n = \sum_{i=1}^n \omega_i [x_i - A_n]^2$.

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published in the monograph *Inequalities* by Beckenbach and Bellman [7]:

Theorem 1.1. *For* $x_i \in (0, 1/2]$,

$$\frac{A_n'}{G_n'} \leq \frac{A_n}{G_n} \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

In this paper we consider the validity of the following additive Ky Fan-typed inequalities(with $x_1 < x_n < 1$):

$$\frac{x_1}{1-x_1} < \frac{P_{n,r}' - P_{n,s}'}{P_{n,r} - P_{n,s}} < \frac{x_n}{1-x_n} \tag{1.2}$$

Note by a change of variables, $x_i \rightarrow 1 - x_i$, the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). One can deduce(see[9]) theorem 1.1 from the case $r = 1, s = 0, x_n \leq 1/2$ in (1.2) , which is a result of H. Alzer[5]. P.Gao[9] later proved the validity of (1.2) for $r = 1, -1 \leq s < 1, x_n \leq 1/2$.

What's worth mentioning is a nice result of P. Mercer[12], who showed the validity of $r = 1, s = 0$ in (1.2) is a consequence of a result of D.I. Cartwright and M.J. Field[8], who established the validity of $r = 1, s = 0$ for the following bounds for the differences between power means($r > s$):

$$\frac{r-s}{2x_1}\sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2x_n}\sigma_n \tag{1.3}$$

where the constant $(r - s)/2$ is best possible(see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all $r > s$. We refer the reader to the survey article[2] and the references therein for an account of Ky Fan's inequality and to the articles [4],[5],[10],[11] for other interesting refinements and extensions of (1.3).

Mercer's result reveals a close relation between (1.3) and (1.2) and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed r and s. As a consequence of

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this result, we will give a characterization of the validity of (1.3) for $r = 1$ or $s = 1$. A solution of an open problem from [11] is also given.

Among numerous sharpenings of Ky Fan's inequality in the literature, we have the following inequalities connecting the three classical means(with $\omega_i = 1/n$ here):

$$(\frac{H_n}{H'_n})^{n-1}\frac{A_n}{A'_n} \leq (\frac{G_n}{G'_n})^n \leq (\frac{A_n}{A'_n})^{n-1}\frac{H_n}{H'_n} \tag{1.4}$$

The right-hand side inequality of (1.4) is due to P.F.Wang and W.L.Wang[14] and the left-hand side inequality was proved recently by H. Alzer, S. Ruscheweyh and L. Salinas[6].

It is natural to ask whether one can extend the above inequality to the weighted case and by using the same idea as in [6], we will show this indeed is true in section 5.

2. The Main Theorem

Theorem 2.1. *For fixed $r > s$, the following inequalities are equivalent:*
(i). inequality (1.2) for $x_n \leq 1/2$; (ii). inequality (1.2); (iii). inequality (1.3).

Proof. (iii) ⇒ (ii) follows from a similar argument as given in [12], (ii) ⇒ (i) is trivial, so it suffices to show (i)⇒ (iii):

Fix $r > s$, assuming (1.2) holds for $x_n \leq 1/2$. Without loss of generality, we can assume $x_1 < x_n$. For a given $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, let $\mathbf{y} = (\epsilon x_1, \epsilon x_2, \cdots, \epsilon x_n)$. We can choose ϵ small so that $\epsilon x_n \leq 1/2$ and now apply the right-hand side inequality (1.2) for $\mathbf{y}$, we get

$$x_n(P_{n,r}(\mathbf{x}) - P_{n,s}(\mathbf{x})) > \frac{1-\epsilon x_n}{\epsilon^2}(P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}')) \tag{2.1}$$

By letting ϵ tend to 0, it is easy to verify the limit of the expression on the right-hand side of (2.1) is $(r-s)\sigma_n/2$. We can consider the left-hand side of (2.1) by a similar argument and this completes the proof. □

3. An Application of Theorem 2.1

Lemma 3.1. *If inequality (1.3) holds for $r > s$ then $0 \leq r + s \leq 3$.*

Proof. Let $n = 2$, write $\omega_1 = 1 - q$, $\omega_2 = q$, $x_1 = 1$ and $x_2 = 1 + t$ with $t \geq -1$. Let

$$D(t; r, s, q) = \frac{r-s}{2}\sum_{i=1}^{2} w_i[x_i - A_2]^2 - P_{2,r} + P_{2,s}$$

For $t \geq 0$ then $D(t; r, s, q) \geq 0$ implies the validity of the left-hand side inequality of (1.3) while for $-1 \leq t \leq 0$, $D(t; r, s, q) \leq 0$ implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of $D(t; r, s, q)$ around $t = 0$, it is readily seen that $D(0; r, s, q) = D^{(1)}(0; r, s, q) = D^{(2)}(0; r, s, q) = 0$. Thus by the Lagrangian remainder term of the Taylor expansion:

$$D(t; r, s, q) = \frac{D^{(3)}(\theta t; r, s, q)}{3!}t^3$$

with $0 < \theta < 1$.

Since

$$\lim_{t\to 0^+} D^{(3)}(\theta t; r, s, q) = D^{(3)}(0; r, s, q)$$

a necessary condition for (1.3) to hold is $D^{(3)}(0; r, s, q) > 0$ for $0 \leq q \leq 1$. Calculation yields

$$D^{(3)}(0; r, s, q) = (r-s)q(q-1)((3-2r-2s)q - (3-r-s))$$

It is easy to check that this is equivalent to $0 \leq r + s \leq 3$. □

Theorem 3.1. *If $r = 1$, inequality (1.3) holds if and only if $-1 \leq s < 1$. If $s = 1$, inequality (1.3) holds if and only if $1 < r < 2$.*

Proof. A result of P.Gao[9] shows the validity of (1.2) for $r = 1, -1 \leq s < 1, x_n \leq 1/2$ and a similar result of him[10] shows the validity of (1.2) for $s = 1, 1 < r \leq 2, x_n \leq 1/2$. Thus it follows from theorem 2.1 that (1.3) holds for $r = 1, -1 \leq s < 1$ and $s = 1, 1 < r \leq 2$. This proves the "if" part of the statement and the "only if" part follows from the previous lemma. □

We note here a special case of the above corollary answers an open problem of A. M. Mercer[11], namely, we have shown:

$$\frac{1}{x_1}\sigma_n \geq A_n - H_n \geq \frac{1}{x_n}\sigma_n \tag{3.1}$$

4. Two Lemmas

Lemma 4.1. *Let x, b, u, v be real numbers with $0 < x \leq b, u \geq 1, v \geq 1$, then $f(u, v, x, b) \leq 0$ where*

$$f(u, v, x, b) = \frac{u+v-1}{ux+vb} + \frac{1}{x^2(u/x+v/b)} - \frac{1}{x} - \frac{u+v-2}{b^2(u+v)^2}v(x-b) \tag{4.1}$$

with equality holding if and only if $x = b$ or $u = v = 1$.

Proof. Let $x < b$ and $u > 1, v > 1$. We have

$$\begin{aligned} f(u, v, x, b) &= v(b-x)(-\frac{(u-1)b+(v-1)x}{x(bv+ux)(bu+vx)} + \frac{(u-1)+(v-1)}{b^2(u+v)^2}) \\ &< \frac{v(b-x)}{xb^2(u+v)^2}[((u-1)+(v-1))x - (u-1)b - (v-1)x] \\ &= -\frac{v(u-1)(b-x)^2}{xb^2(u+v)^2} < 0 \end{aligned}$$

Since $b^2(u+v)^2 > (bv+ux)(bu+vx)$. Thus we conclude that $f(u, v, x, b) < 0$ for $0 < x \leq b, u \geq 1, v \geq 1$. □

Lemma 4.2. *Let x, a, b, u, v, s be real numbers with $0 < x \leq a \leq b, u \geq 1, v \geq 1, u+v \geq 2$ and $0 \leq s \leq v$, then*

$$\begin{aligned} &\frac{u+v-1}{ux+sa+(v-s)b} + \frac{1}{x^2(u/x+s/a+(v-s)/b)} - \frac{1}{x} \\ &- \frac{u+v-2}{b^2(u+v)^2}(s(x-a)+(v-s)(x-b)) \leq 0 \end{aligned} \tag{4.2}$$

with equality holding if and only if one of the following cases is true: 1. $x = a = b$; 2. $a = b, u = v = 1$; 3. $s = 0, u = v = 1$; 4. $s = 0, x = b$; 5. $s = 0, x = a$; 6. $s = u = v = 1$.

Proof. Let $M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b\}$. Furthermore, we define $H(s, a)$ as the expression on the left-hand side of (4.2), where $(s, a) \in M$. It suffices to show $H(s, a) < 0$. We denote the absolute minimum of H by $m = (s_0, a_0)$. If m is an interior point of M, then we obtain

$$0 = \frac{1}{s}\frac{\partial H}{\partial a} - \frac{1}{a-b}\frac{\partial H}{\partial s}|_{(s,a)=(s_0,a_0)} = \frac{a-b}{x^2a^2b(u/x+s/a+(v-s)/b)^2} < 0$$

Hence, m is a boundary point of M, so that we get

$$m \in \{(s_0, x), (s_0, b), (0, a_0), (v, a_0)\}$$

Using lemma 4.1 we obtain

$$H(s_0, x) = f(u + s_0, v - s_0, x, b) < 0$$

$$H(s_0, b) = H(0, a_0) = f(u, v, x, b) < 0$$

$$H(v, a_0) = f(u, v, x, a_0) < 0$$

Thus, we get: if $(s, a) \in M$, then $H(s, a) \leq 0$. The conditions for equality can be easily checked by using lemma 4.1.

□

5. A sharpening of Ky Fan's inequality

In this section we prove the following theorem:

Theorem 5.1. *For* $0 < x_1 \leq \cdots \leq x_n$, $q = \min\{\omega_i\}$

$$\frac{1-2q}{2x_1^2}\sigma_n \geq (1-q)\ln A_n + q\ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{5.1}$$

$$\frac{1-2q}{2x_1^2}\sigma_n \geq \ln G_n - q\ln A_n - (1-q)\ln H_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{5.2}$$

with equality holding if and only if $q = 1/2$ *or* $x_1 = \cdots = x_n$.

Proof. The proof uses the ideas in [6]. We will prove the right-hand side inequality of (5.1) and the proofs for other inequalities are similar. Fix $0 < x = x_1, x_n = b$ with $x_1 < x_n$, we define

$$f_n(\mathbf{x}_n, q) = (1-q)\ln A_n + q\ln H_n - \ln G_n - \frac{1-2q}{2x_n^2}\sigma_n$$

where we regard A_n, G_n, H_n as functions of $\mathbf{x}_n = (x_1, \cdots, x_n)$.

We then have

$$g_n(x_2, \cdots, x_{n-1}) := \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} = \frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1} - \frac{1-2q}{x_n^2}(x_1 - A_n)$$

We want to show $g_n \leq 0$. Let $D = \{(x_2, \cdots, x_{n-1}) \in R^{n-2} | 0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b\}$. Let $\mathbf{a} = (a_2, \cdots, a_{n-1}) \in D$ be the absolute minimum of g_n. Next, we show that

$$\mathbf{a} = (x, \cdots, x, a, \cdots, a, b, \cdots, b) \text{ with } x < a < b \tag{5.3}$$

where the numbers x, a, and b appear r, s, and t times, respectively, with $r, s, t \geq 0, r+s+t = n-2$.

Suppose not, this implies two components of $\mathbf{a}$ have different values and are interior points of D. We denote these values by a_k and a_l. Partial differentiation leads to

$$\frac{B}{a_i^2} + C = 0 \tag{5.4}$$

for $i = k, l$, where

$$B = q\frac{H_n^2}{x_1^2}, C = -\frac{1-q}{A_n^2} + \frac{1-2q}{x_n^2}$$

Since $z \mapsto B/z^2 + C$ is strictly monotonic for $z > 0$, (5.4) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus (5.3) is valid and it suffices to show $g_n \leq 0$ for the case $n = 2, 3$.

When $n = 2$, by setting $x_1 = x, x_2 = b, \omega_1/q = u, \omega_2/q = v$, we can identify g_2/q as (4.1) and the result follows from lemma 4.1.

When $n = 3$, by setting $x_1 = x, x_2 = a, x_3 = b, \omega_1/q = u, \omega_2/q = s, \omega_3/q = v - s$, we can identify g_3/q as (4.2) and the result follows from lemma 4.2.

Thus we have shown that $g_n = \frac{\partial f_n}{\partial x_n} \leq 0$ with equality holding if and only if $n = 1$ or $n = 2, q = 1/2$. By letting x_1 tend to x_2, we have

$$f_n(\mathbf{x}_n, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q')$$

where $\mathbf{x}_{n-1} = (x_2, \cdots, x_n)$ with weights $\omega_1 + \omega_2, \cdots, \omega_{n-1}, \omega_n$ and $q' = min\{\omega_1 + \omega_2, \cdots, \omega_n\}$. Here we have used the following inequality, which is a consequence of (3.1)(see [9]):

$$\ln A_n - \ln H_n \geq \frac{1}{x_n^2}\sigma_n$$

It then follows by induction that $f_n > f_{n-1} > \cdots > f_2 = 0$ when $q = 1/2$ in f_2 or else $f_n > f_{n-1} > \cdots > f_1 = 0$ and this completes the proof. □

We note the above theorem gives a sharpening of Sierpiński's inequality[13], originally states for the unweighted case($\omega_i = 1/n$) as:

$$H_n^{n-1}A_n \leq G_n \leq A_n^{n-1}H_n$$

The following corollary gives refinements of (1.4):

Corollary 5.1. *For* $0 < x_1 \leq \cdots \leq x_n < 1$, $q = \min\{\omega_i\}$

$$(\frac{A_n^{'(1-q)}H_n^{'q}}{G_n'})^{\frac{(1-x_1)^2}{x_1^2}} \geq \frac{A_n^{1-q}H_n^q}{G_n} \geq (\frac{A_n^{'(1-q)}H_n^{'q}}{G_n'})^{\frac{(1-x_n)^2}{x_n^2}} \tag{5.5}$$

$$(\frac{G_n'}{A_n^{'q}H_n^{'(1-q)}})^{\frac{(1-x_1)^2}{x_1^2}} \geq \frac{G_n}{A_n^qH_n^{1-q}} \geq (\frac{G_n'}{A_n^{'q}H_n^{'(1-q)}})^{\frac{(1-x_n)^2}{x_n^2}} \tag{5.6}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ *or* $q = 1/2$.

Proof. This is a direct consequence of theorem 5.1, following from a similar argument as in [12]. □

6. Conclusion Remarks

We note that if for $x_n \leq 1/2$ one has

$$(\frac{x_1}{1-x_1})^\beta < \frac{P_{n,r}' - P_{n,s}'}{P_{n,r} - P_{n,s}} < (\frac{x_n}{1-x_n})^\alpha$$

then $\beta \geq 1, \alpha \leq 1$, otherwise by letting ϵ tend to 0 in (2.1), we will get contradictions.

It was conjectured that an additive companion of (1.4) is true(see [1]):

$$n(G_n - G_n') \leq (n-1)(A_n - A_n') + H_n - H_n' \tag{6.1}$$

In [3], H. Alzer asked if the above conjecture is true, whether there exists a weighted version or not. Based on what we've got in this paper, it is natural to give the following conjecture of the weighed version of (6.1):

Conjecture 6.1. *For* $0 < x_1 \leq \cdots \leq x_n \leq 1/2$, $q = \min\{\omega_i\}$

$$G_n - G_n' \leq (1-q)(A_n - A_n') + q(H_n - H_n') \tag{6.2}$$

Recently, H. Alzer, S. Ruscheweyh and L. Salinas[6] asked the following question: What is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

$$\alpha(A_n - A_n') + (1-\alpha)(H_n - H_n') \leq G_n - G_n' \leq \beta(A_n - A_n') + (1-\beta)(H_n - H_n')$$

for all $x_i \in (0, 1/2](j = 1, \cdots, n)$?

We note here $\alpha \leq 0$, since the left-hand side inequality above can be written as:

$$\alpha A_n + (1-\alpha)H_n - G_n \leq \alpha A_n' + (1-\alpha)H_n' - G_n' \tag{6.3}$$

By a similar argument as in the proof of theorem 2.1, replacing $(x_1, \cdots, x_n)$ by $(\epsilon x_1, \cdots, \epsilon x_n)$ and letting ϵ tend to 0 in (6.3), we find that (6.3) implies:

$$\alpha A_n + (1-\alpha)H_n - G_n \leq 0 \tag{6.4}$$

for any $\mathbf{x}$. If we further let x_1 tend to 0 in (6.4), we get

$$\alpha A_n \leq 0$$

which implies $\alpha \leq 0$.

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