ON GENERALIZATIONS OF TWO NEW TYPE HILBERT INTEGRAL INEQUALITIES

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Abstract: In 1998,two new inequalities similar to Hilbert integral inequality were given by B.G.Pachpatte.the main purpose of the present article is to generalize these two new inequalities by Holder integral inequality, Tchebychef integral inequality and Jensen inequality.

Key words: Hilbert integral inequality , Holder integral inequality, Tchebychef integral inequality

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It is well-known that the Hilbert integral inequality is an important inequality in applications. In 1998, B.G.Pachpatte gave two new integral inequalities similar to Hilbert integral inequality(see[2] P.226) in [1].In this paper we will generalize these two new inequalities.

Our main results are given in the following theorems.

THEOREM 1 Let $p \geq 1, q \geq 1$ and $f(\sigma) \geq 0, g(\tau) \geq 0$ for $\sigma \in (0,x), \tau \in (0,y)$, where x,y are positive real numbers and define $F(s) = \int_o^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$ for $s \in (0,x), t \in (0,y)$ and l is the natural number.Then

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)(st)^{2/l}}{(s \cdot t^{1/l})^2 + (t \cdot s^{1/l})^2} dsdt \leq \frac{1}{2} pq(xy)^{(l-1)/l} \Big(\int_0^x \Big(x - s\Big)\Big(F^{p-1}(s)f(s)\Big)^l ds\Big)^{1/l}$$
$$\times \Big(\int_0^y \Big(y - t\Big)\Big(G^{q-1}(t)g(t)\Big)^l dt\Big)^{1/l} \tag{1}$$

Proof From the hypotheses,it is easy to observe that

$$F^p(s) = p \int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma, s \in (0,x)$$

$$G^q(t) = q \int_0^t G^{q-1}(\tau)g(\tau)d\tau, t \in (0,y)$$

Therefore

$$F^p(s)G^q(t) = pq\Big(\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma\Big)\Big(\int_0^t G^{q-1}(\tau)g(\tau)d\tau\Big) \tag{2}$$

On the other hand, by using special case of Tchebychef integral inequality(see[3] P.50)(when all f_i are equal) we have

$$\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma \le s^{(l-1)/l}\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} \tag{3}$$

$$\int_0^t G^{q-1}(\tau)g(\tau)d\tau \le t^{(l-1)/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l} \tag{4}$$

By (2),(3) and (4) yield that

$$\begin{aligned} F^p(s)G^q(t) &\le pq(st)^{(l-1)/l}\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l} \\ &\le \frac{1}{2}pq(s^{2(l-1)/l}+t^{2(l-1)/l})\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} \\ &\quad \times\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l} \end{aligned}$$

Thus

$$\frac{F^p(s)G^q(t)(st)^{1/l}}{(s\cdot t^{1/l})^2+(t\cdot s^{1/l})^2} \le \frac{1}{2}pq\Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l}\Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)^{1/l}$$

Integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using again the special case of Tchebychef integral inequality, we observe that

$$\begin{aligned} \int_0^x\int_0^y \frac{F^p(s)G^q(t)(st)^{2/l}}{(s\cdot t^{1/l})^2+(t\cdot s^{1/l})^2}dsdt &\le \frac{1}{2}pq\Big(\int_0^x \Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)^{1/l} ds\Big) \\ &\quad \times\Big(\int_0^y \Big(\int_0^t \Big(G^{q-1}(\tau)f(\tau)\Big)^l d\tau\Big)^{1/l} dt\Big) \\ &\le \frac{1}{2}pq\cdot x^{(l-1)/l}\Big(\int_0^x \Big(\int_0^s \Big(F^{p-1}(\sigma)f(\sigma)\Big)^l d\sigma\Big)ds\Big)^{1/l} \\ &\quad \times y^{(l-1)/l}\Big(\int_0^y \Big(\int_0^t \Big(G^{q-1}(\tau)g(\tau)\Big)^l d\tau\Big)dt\Big)^{1/l} \\ &= \frac{1}{2}pq(xy)^{(l-1)/l}\Big(\int_0^x \Big(x-s\Big)\Big(F^{p-1}(s)f(s)\Big)^l ds\Big)^{1/l} \\ &\quad \times\Big(\int_0^y \Big(y-t\Big)\Big(G^{q-1}(t)g(t)\Big)^l dt\Big)^{1/l} \end{aligned}$$

The proof is complete.

COROLLARY 1 Under the hypotheses of theorem 1 ,if take $l=2$ in (1),then

$$\int_0^x\int_0^y \frac{F^p(s)G^q(t)}{s+t}dsdt \le \frac{1}{2}pq\sqrt{xy}\left(\int_0^x \Big(x-s\Big)\Big(F^{p-1}(s)f(s)\Big)^2 ds\right)^{1/2}$$
$$\times\left(\int_0^y \Big(y-t\Big)\Big(G^{q-1}(t)g(t)\Big)^2 dt\right)^{1/2}$$

This is just one result which was given by B.G.Pachpatte in[1].

THEOREM 2 Let f,g,F,G be as in theorem 1 .Let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0,x), \tau \in (0,y)$ and define $P(s) = \int_0^s p(\sigma)d\sigma$, $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in (0,x), t \in (0,y)$ where x , y are positive real numbers and l is the natural number and p, q are real numbers and $\frac{1}{p}+\frac{1}{q} = 1, p > 1$. Let ϕ and ψ be two real-valued nonnegative, conves, and submultiplicative functions defined on $R_+ = [0,+\infty)$ Then

$$\int_0^x\int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)(st)^{2/l}}{(s\cdot t^{1/l})^2+(t\cdot s^{1/l})^2}dsdt \le L(x,y,p)\left(\int_0^x\left(\left(\int_0^s \Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)})\Big)^l d\sigma\right)^{1/l}\right)^q ds\right)^{1/q}$$
$$\times\left(\int_0^y\left(\left(\int_0^t \Big(q(\tau)\psi(\frac{g(\tau)}{q(\tau)})\Big)^l d\tau\right)^{1/l}\right)^q dt\right)^{1/q} \tag{5}$$

where

$$L(x,y,p) = \frac{1}{2}\left(\int_0^x\Big(\frac{\phi\Big(P(s)\Big)}{P(s)}\Big)^p ds\right)^{1/p}\left(\int_0^y\Big(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\Big)^p dt\right)^{1/p}$$

Proof From the hypotheses and by using Jensen inquality and the special case of tchebychef integral inequality , it is easy to observe that

$$\phi\left(F(s)\right) = \phi\left(\frac{P(s)\int_0^s p(\sigma)\frac{f(\sigma)}{p(\sigma)}d\sigma}{\int_0^s p(\sigma)d\sigma}\right) \le \frac{\phi\left(P(s)\right)}{P(s)}\int_0^s p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma$$
$$\le \left(\frac{\phi\Big(P(s)\Big)}{P(s)}\right)s^{(l-1)/l}\left(\int_0^s\Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)})\Big)^l d\sigma\right)^{1/l} \tag{6}$$

and similarly,

$$\psi\Big(G(t)\Big) \le \left(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\right)t^{(l-1)/l}\left(\int_0^t\Big(q(\tau)\psi(\frac{g(\tau)}{q(\tau)})\Big)^l d\tau\right)^{1/l} \tag{7}$$

By (6) and (7),we get that

$$\frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)(st)^{2/l}}{(s\cdot t^{1/l}+t\cdot s^{1/l})^2} \le \frac{1}{2}\left(\frac{\phi\Big(P(s)\Big)}{P(s)}\right)\left(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\right)\left(\int_0^s\Big(p(\sigma)\phi(\frac{f(\sigma)}{p(\sigma)})\Big)^l d\sigma\right)^{1/l}$$

$$\left(\int_0^t \left(q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right)^l d\tau\right)^{1/l} \tag{8}$$

Integrating two sides of (8) over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Holder integral inequality(see[3] P.69) we observe that

$$\int_0^x \int_0^y \frac{\phi\big(F(s)\big)\psi\big(G(t)\big)(st)^{2/l}}{(s\cdot t^{1/l}+t\cdot s^{1/l})^2}dsdt \le \frac{1}{2}\left(\int_0^x \frac{\phi\big(P(s)\big)}{P(s)}\left(\int_0^s \left(p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^l d\sigma\right)^{1/l} ds\right)$$

$$\times\left(\int_0^y \frac{\psi\big(Q(t)\big)}{Q(t)}\left(\int_0^t \left(q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right)^l d\tau\right)^{1/l} dt\right)$$

$$\le \frac{1}{2}\left(\int_0^x \left(\frac{\phi\big(P(s)\big)}{P(s)}\right)^p ds\right)^{1/p}\left(\int_0^x \left(\left(\int_0^s \left(p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^l d\sigma\right)^{1/l}\right)^q ds\right)^{1/q}$$

$$\times\left(\int_0^y \left(\frac{\psi\big(Q(t)\big)}{Q(t)}\right)^p dt\right)^{1/p}\left(\int_0^y \left(\left(\int_0^t \left(q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right)^l d\tau\right)^{1/l}\right)^q dt\right)^{1/q}$$

$$= L(x,y,p)\left(\int_0^x \left(\left(\int_0^s \left(p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^l d\sigma\right)^{1/l}\right)^q ds\right)^{1/q}$$

$$\times\left(\int_0^y \left(\left(\int_0^t \left(q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right)^l d\tau\right)^{1/l}\right)^q dt\right)^{1/q}$$

Q.E.D.

COROLLARY 2 Under the hypothese of theorem 2 ,if $l = 2$, $p = q$, then the inequality(5) reduces to the following inequality.

$$\int_0^x \int_0^y \frac{\phi\big(F(s)\big))\psi\big(G(t)\big)}{s+t}dsdt \le L(x,y)\left(\int_0^x (x-s)\left(p(s)\phi\left(\frac{f(s)}{p(s)}\right)\right)^2 ds\right)^{1/2}$$

$$\times\left(\int_0^y (y-t)\left(q(t)\psi\left(\frac{g(t)}{q(t)}\right)\right)^2 dt\right)^{1/2} \tag{9}$$

where

$$L(x,y) = \frac{1}{2}\left(\int_0^x \left(\frac{\phi\big(P(s)\big)}{P(s)}\right)^2 ds\right)^{1/2}\left(\int_0^y \left(\frac{\psi\big(Q(t)\big)}{Q(t)}\right)^2 dts\right)^{1/2}$$

This is just another result which was given by B.G.Pachpatte in [1].

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