

ON A NEW GENERALIZATION OF MARTINS' INEQUALITY

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ABSTRACT. Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m-1}$ is increasing. Then the following inequality between ratios of the power means and of the geometric means holds:

$$\left(\frac{1}{n} \sum_{i=1}^n a_i^r / \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{n^{+m} \sqrt[n+m]{a_{n+m}!}},$$

where r is a positive number, $a_i!$ denotes the sequence factorial defined by $\prod_{i=1}^n a_i$. The upper bound is the best possible.

1. INTRODUCTION

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^n i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{n^{+1} \sqrt[n+1]{(n+1)!}} \quad (1)$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [1], and the right-hand side Martins's inequality [7].

Alzer's inequality has invoked the interest of several mathematicians, we refer the reader to [5, 9, 16, 18] and the references therein.

Recently, F. Qi and L. Debnath in [15] proved that: Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r} \geq \left(\frac{a_{k+2}}{a_{k+1}}\right)^r \quad (2)$$

for a given positive real number r and $k \in \mathbb{N}$. Then

$$\frac{a_n}{a_{n+m}} \leq \left(\frac{(1/n) \sum_{i=1}^n a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \quad (3)$$

The lower bound of (3) is the best possible.

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In [12, 13, 14, 19, 20, 21], F. Qi and others proved the following inequalities and other more general results:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} / \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}, \quad (4)$$

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k} (ai+b) \right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k} (ai+b) \right]^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \quad (5)$$

where $n, m \in \mathbb{N}$, k is a nonnegative integer, a a positive constant, and b a nonnegative constant. The equalities in (4) and (5) is valid for $n = 1$ and $m = 1$.

In [17], the following monotonicity results for the gamma function were obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \quad (6)$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y , we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \quad (7)$$

In [11, 15], F. Qi proved that: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r / \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r}, \quad (8)$$

where r is a given positive real number. The lower bound is the best possible.

In [4, 18], some more general results for the lower bound of ratio of power means $\left(\frac{1}{n} \sum_{i=1}^n a_i^r / \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r}$ for positive sequence $\{a_i\}_{i \in \mathbb{N}}$ were obtained.

An open problem in [10, 11] asked for the validity of the following inequality:

$$\left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r / \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{n+m \sqrt[n+m]{(n+m+k)!/k!}}, \quad (9)$$

where $r > 0$, $n, m \in \mathbb{N}$, $k \in \mathbb{Z}^+$.

Let $\{a_i\}_{i \in \mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically concave sequence. See [8, p. 284].

In [3], the open problem mentioned above was solved and generalized affirmatively: Let $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and non-constant sequence satisfying $(a_{\ell+1}/a_\ell)^\ell \geq (a_\ell/a_{\ell-1})^{\ell-1}$ for any positive integer $\ell > 1$, then $\left(\frac{1}{n} \sum_{i=1}^n a_i^r / \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r} < \frac{\sqrt[n]{a_n!}}{n+m \sqrt[n+m]{a_{n+m}!}}$, where r is a positive number, $n, m \in \mathbb{N}$, and $a_i!$ denotes the sequence factorial $\prod_{i=1}^n a_i$. The upper bound is best possible.

The purpose of this paper is to give a new generalization of inequality (9) as follows.

Theorem 1. *Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{ i \left[\frac{a_{i+1}}{a_i} - 1 \right] \right\}_{i=1}^{n+m-1}$ is*

increasing. Then the following inequality between ratios of the power means and of the geometric means holds:

$$\left(\frac{\frac{1}{n} \sum_{i=1}^n a_i^r}{\frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r} \right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \quad (10)$$

where r is a positive number and $a_i!$ denotes the sequence factorial $\prod_{i=1}^n a_i$. The upper bound is the best possible.

As an easy consequence of Theorem 1 by taking $\{a_i\}_{i=1}^{n+m} = \{(i+k+b)^\alpha\}_{i=1}^{n+m}$ for a positive constant α , we have

Corollary 1. *Let α be a positive real number, k a nonnegative integer and b a real number such that $k+b > 0$, and $m, n \in \mathbb{N}$. If the sequence*

$$\left\{ i \left[\left(1 + \frac{1}{i+k+b} \right)^\alpha - 1 \right] \right\}_{i=1}^{n+m-1} \quad (11)$$

is increasing, then for any real number $r > 0$, we have

$$\left(\frac{\frac{1}{n} \sum_{i=k+1}^{n+k} [(i+b)^\alpha]^r}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} [(i+b)^\alpha]^r} \right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} (i+b)^\alpha}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} (i+b)^\alpha}}. \quad (12)$$

The upper bound is the best possible.

Remark 1. By letting $\alpha = 1$ and $b = 0$ in (12), we recover inequality (9).

Taking $\alpha = 2$ in Corollary 1 leads to the following

Corollary 2. *Let k be a nonnegative integer, and b a real number such that $k+b \geq \frac{1}{2}$, and $m, n \in \mathbb{N}$. Then, for any real number $r > 0$, we have*

$$\left(\frac{\frac{1}{n} \sum_{i=k+1}^{n+k} [(i+b)^2]^r}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} [(i+b)^2]^r} \right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k} (i+b)^2}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} (i+b)^2}}. \quad (13)$$

The upper bound is the best possible.

Considering $\{a_i\}_{i \in \mathbb{N}} = \{e^{i^\alpha}\}_{i \in \mathbb{N}}$ in Theorem 1 and standard argument gives us the following

Corollary 3. *Let $m, n \in \mathbb{N}$. If the constant $0 < \alpha < 1$ such that inequality*

$$\frac{e^{(1+x)^\alpha} - e^{x^\alpha}}{x^{\alpha-1} - (1+x)^{\alpha-1}} \geq \alpha x e^{(1+x)^\alpha} \quad (14)$$

holds with x on $[1, \infty)$, then, for any real number $r > 0$, we have

$$\left(\frac{\frac{1}{n} \sum_{i=1}^n e^{i^{\alpha r}}}{\frac{1}{n+m} \sum_{i=1}^{n+m} e^{i^{\alpha r}}} \right)^{1/r} < \exp \left[\frac{1}{n} \sum_{i=1}^n i^\alpha - \frac{1}{n+m} \sum_{i=1}^{n+m} i^\alpha \right]. \quad (15)$$

The upper bound is the best possible.

2. LEMMAS

To prove our main results, the following lemmas are necessary.

Lemma 1. *Let $n, m \in \mathbb{N}$, and $\{a_i\}_{i=1}^{n+m+1}$ a nonconstant positive sequence such that the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m}$ is increasing, then the sequence*

$$\left\{ \frac{\sqrt[i]{a_i!}}{a_{i+1}} \right\}_{i=1}^{n+m} \quad (16)$$

is decreasing. As a simple consequence, we have the following

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} > \frac{a_{n+1}}{a_{n+m+1}}, \quad (17)$$

where $a_n!$ denotes the sequence factorial defined by $\prod_{i=1}^n a_i$.

Proof. For $1 \leq i \leq n+m-1$, the monotonicity of the sequence (16) is equivalent to the following

$$\begin{aligned} & \frac{\sqrt[i]{a_i!}}{a_{i+1}} \geq \frac{\sqrt[i+1]{a_{i+1}!}}{a_{i+2}}, \quad (18) \\ \iff & \left(\prod_{k=1}^i \frac{a_k}{a_{i+1}} \right)^{1/i} \geq \left(\prod_{k=1}^{i+1} \frac{a_k}{a_{i+2}} \right)^{1/(i+1)}, \\ \iff & \frac{1}{i} \sum_{k=1}^i \ln \frac{a_k}{a_{i+1}} \geq \frac{1}{i+1} \sum_{k=1}^{i+1} \ln \frac{a_k}{a_{i+2}}, \\ \iff & \frac{i}{i+1} \sum_{k=1}^{i+1} \ln \frac{a_k}{a_{i+2}} \leq \sum_{k=1}^i \ln \frac{a_k}{a_{i+1}}. \quad (19) \end{aligned}$$

Since $\ln x$ is concave on $(0, \infty)$, by definition of concaveness, it follows that, for $1 \leq k \leq i$,

$$\begin{aligned} & \frac{k}{i+1} \ln \frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1} \ln \frac{a_k}{a_{i+2}} \\ & \leq \ln \left(\frac{k}{i+1} \cdot \frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1} \cdot \frac{a_k}{a_{i+2}} \right) \quad (20) \\ & = \ln \left(\frac{ka_{k+1} + (i-k+1)a_k}{(i+1)a_{i+2}} \right). \end{aligned}$$

Since the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m}$ is increasing, we have, for $1 \leq i \leq n+m-1$ and $1 \leq k \leq i$, the following

$$\begin{aligned} & \frac{(i+1)a_{i+2}}{a_{i+1}} - (i+1) \geq \frac{ia_{i+1}}{a_i} - i, \\ \iff & \frac{(i+1)a_{i+2}}{a_{i+1}} - (i+1) \geq \frac{ka_{k+1}}{a_k} - k, \\ \iff & \frac{ka_{k+1} + (i-k+1)a_k}{a_k} \leq \frac{(i+1)a_{i+2}}{a_{i+1}}, \\ \iff & \frac{ka_{k+1} + (i-k+1)a_k}{(i+1)a_{i+2}} \leq \frac{a_k}{a_{i+1}}. \end{aligned}$$

Combining the last line above with (20) yields

$$\frac{k}{i+1} \ln \frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1} \ln \frac{a_k}{a_{i+2}} \leq \ln \frac{a_k}{a_{i+1}}. \quad (21)$$

Summing up on both sides of (21) with k from 1 to i and simplifying reveals inequality (19). The monotonicity follows.

Since $\{a_i\}_{i=1}^{n+m+1}$ is a nonconstant positive sequence, there exists at least one number $1 \leq i_0 \leq n+m-1$ such that $a_{i_0} \neq a_{i_0+1}$. The function $\ln x$ is strictly concave on $(0, \infty)$. Then, for any i such that $i_0 \leq i \leq n+m-1$, we have

$$\begin{aligned} & \frac{i_0}{i+1} \ln \frac{a_{i_0+1}}{a_{i+2}} + \frac{i-i_0+1}{i+1} \ln \frac{a_{i_0}}{a_{i+2}} \\ & < \ln \left(\frac{i_0}{i+1} \cdot \frac{a_{i_0+1}}{a_{i+2}} + \frac{i-i_0+1}{i+1} \cdot \frac{a_{i_0}}{a_{i+2}} \right) \\ & = \ln \left(\frac{i_0 a_{i_0+1} + (i-i_0+1) a_{i_0}}{(i+1) a_{i+2}} \right) \\ & \leq \ln \frac{a_{i_0}}{a_{i+1}}, \end{aligned} \quad (22)$$

notice that the last line follows from the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m}$ being increasing. Therefore, for any i such that $i_0 \leq i \leq n+m-1$, inequality (18) is strict. Inequality (17) is proved.

The proof is complete. \square

Lemma 2. *Let $n > 1$ be a positive integer and $\{a_i\}_{i=1}^n$ an increasing nonconstant positive sequence such that $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n-1}$ is increasing. Then the sequence*

$$\left\{ \frac{a_i}{(a_i!)^{1/i}} \right\}_{i=1}^n \quad (23)$$

is increasing, and, for any positive integer ℓ satisfying $1 \leq \ell < n$,

$$\frac{a_\ell}{a_n} < \frac{(a_\ell!)^{1/\ell}}{(a_n!)^{1/n}}, \quad (24)$$

where $a_n!$ denotes the sequence factorial $\prod_{i=1}^n a_i$.

Proof. For $1 \leq \ell \leq n-1$, the monotonicity of the sequence (23) is equivalent to

$$\begin{aligned} & \frac{a_\ell}{(a_\ell!)^{1/\ell}} \leq \frac{a_{\ell+1}}{(a_{\ell+1}!)^{1/(\ell+1)}}, \\ \Leftrightarrow & \left(\prod_{j=1}^{\ell} \frac{a_j}{a_\ell} \right)^{\frac{1}{\ell}} \geq \left(\prod_{j=1}^{\ell+1} \frac{a_j}{a_{\ell+1}} \right)^{\frac{1}{\ell+1}}, \\ \Leftrightarrow & \frac{1}{\ell} \sum_{j=1}^{\ell-1} \ln \frac{a_j}{a_\ell} \geq \frac{1}{\ell+1} \sum_{j=1}^{\ell} \ln \frac{a_j}{a_{\ell+1}}, \\ \Leftrightarrow & \sum_{j=1}^{\ell-1} \ln \frac{a_j}{a_\ell} \geq \frac{\ell}{\ell+1} \sum_{j=1}^{\ell} \ln \frac{a_j}{a_{\ell+1}}. \end{aligned} \quad (25)$$

Since $\ln x$ is concave on $(0, \infty)$, by definition of concaveness, it follows that, for $1 \leq j \leq \ell$,

$$\begin{aligned} & \frac{j}{\ell+1} \ln \frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1} \ln \frac{a_j}{a_{\ell+1}} \\ & \leq \ln \left(\frac{j}{\ell+1} \cdot \frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1} \cdot \frac{a_j}{a_{\ell+1}} \right) \\ & = \ln \left(\frac{ja_{j+1} + (\ell-j+1)a_j}{(\ell+1)a_{\ell+1}} \right). \end{aligned} \quad (26)$$

Straightforward computation gives us

$$\begin{aligned} & \sum_{j=1}^{\ell} \left[\frac{j}{\ell+1} \ln \frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1} \ln \frac{a_j}{a_{\ell+1}} \right] \\ & = \frac{\ell}{\ell+1} \sum_{j=1}^{\ell} \ln \frac{a_j}{a_{\ell+1}} + \sum_{j=1}^{\ell} \left[\frac{j}{\ell+1} \ln \frac{a_{j+1}}{a_{\ell+1}} \right] - \sum_{j=1}^{\ell} \frac{j-1}{\ell+1} \ln \frac{a_j}{a_{\ell+1}} \\ & = \frac{\ell}{\ell+1} \sum_{j=1}^{\ell} \ln \frac{a_j}{a_{\ell+1}} + \sum_{j=2}^{\ell+1} \left[\frac{j-1}{\ell+1} \ln \frac{a_j}{a_{\ell+1}} \right] - \sum_{j=1}^{\ell} \frac{j-1}{\ell+1} \ln \frac{a_j}{a_{\ell+1}} \\ & = \frac{\ell}{\ell+1} \sum_{j=1}^{\ell} \ln \frac{a_j}{a_{\ell+1}}. \end{aligned} \quad (27)$$

From combining of (25), (26) and (27), it suffices to prove for $1 \leq j \leq \ell$

$$\begin{aligned} & \frac{ja_{j+1} + (\ell-j+1)a_j}{(\ell+1)a_{\ell+1}} \leq \frac{a_j}{a_{\ell}}, \\ \iff & \frac{ja_{j+1} + (\ell-j+1)a_j}{a_j} \leq \frac{(\ell+1)a_{\ell+1}}{a_{\ell}}, \\ \iff & \frac{ja_{j+1}}{a_j} + \ell - j + 1 \leq \frac{(\ell+1)a_{\ell+1}}{a_{\ell}}, \\ \iff & (\ell+1) \left[\frac{a_{\ell+1}}{a_{\ell}} - 1 \right] \geq j \left[\frac{a_{j+1}}{a_j} - 1 \right]. \end{aligned} \quad (28)$$

Since the sequences $\{a_i\}_{i=1}^n$ and $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n-1}$ are increasing, the inequality (28) holds.

Moreover, the sequence $\{a_i\}_{i=1}^n$ is nonconstant positive, then there exists at least one number $1 \leq i_1 \leq n-1$ such that $a_{i_1} \neq a_{i_1+1}$. The function $\ln x$ is strictly concave on $(0, \infty)$. Then, for any ℓ such that $i_1 < \ell \leq n-1$, we have

$$\begin{aligned} & \frac{i_1}{\ell+1} \ln \frac{a_{i_1+1}}{a_{\ell+1}} + \frac{\ell-i_1+1}{\ell+1} \ln \frac{a_{i_1}}{a_{\ell+1}} \\ & < \ln \left(\frac{i_1}{\ell+1} \cdot \frac{a_{i_1+1}}{a_{\ell+1}} + \frac{\ell-i_1+1}{\ell+1} \cdot \frac{a_{i_1}}{a_{\ell+1}} \right) \\ & = \ln \left(\frac{i_1 a_{i_1+1} + (\ell-i_1+1)a_{i_1}}{(\ell+1)a_{\ell+1}} \right) \\ & \leq \ln \frac{a_{i_1}}{a_{\ell}}. \end{aligned} \quad (29)$$

Therefore, for any ℓ such that $i_1 + 1 \leq \ell < n$, inequality (25) is strict, and

$$\frac{a_\ell}{a_{\ell+1}} < \frac{(a_\ell!)^{1/\ell}}{(a_{\ell+1}!)^{1/(\ell+1)}}, \quad (30)$$

and then inequality (24) is strict. The proof is complete. \square

Remark 2. Some problems similar to Lemma 1 and Lemma 2 were discussed in [19] by the author and B.-N. Guo.

The methods proving Lemma 1 and Lemma 2 had been used in [18] and others.

Lemma 3 (König's inequality [2, p. 149] and [7, 22]). *Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be decreasing nonnegative n -tuples such that*

$$\prod_{i=1}^k b_i \leq \prod_{i=1}^k a_i, \quad 1 \leq k \leq n, \quad (31)$$

then, for $r > 0$, we have

$$\sum_{i=1}^k b_i^r \leq \sum_{i=1}^k a_i^r, \quad 1 \leq k \leq n. \quad (32)$$

Remark 3. Lemma 3 is a well-known result due to König used to give a proof of Weyl's inequality (cf. Corollary 1.b.8 of [6, p. 24]).

3. PROOFS OF THEOREM 1

Inequality (10) holds for $n = 1$ by the power mean inequality and its case of equality.

For $n \geq 2$, inequality (10) is equivalent to

$$\left(\frac{1}{n} \sum_{i=1}^n a_i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} a_i^r \right)^{1/r} < \frac{\sqrt[n]{a_n!}}{n+1 \sqrt[n+1]{a_{n+1}!}}, \quad (33)$$

which is equivalent to

$$\frac{1}{n} \sum_{i=1}^n \left(\frac{a_i}{\sqrt[n]{a_n!}} \right)^r < \frac{1}{n+1} \sum_{i=1}^{n+1} \left(\frac{a_i}{n+1 \sqrt[n+1]{a_{n+1}!}} \right)^r. \quad (34)$$

Set

$$b_{jn+1} = b_{jn+2} = \cdots = b_{jn+n} = \frac{a_{n+1-j}}{n+1 \sqrt[n+1]{a_{n+1}!}}, \quad 0 \leq j \leq n; \quad (35)$$

$$c_{j(n+1)+1} = c_{j(n+1)+2} = \cdots = c_{j(n+1)+(n+1)} = \frac{a_{n-j}}{\sqrt[n]{a_n!}}, \quad 0 \leq j \leq n-1. \quad (36)$$

Direct calculation yields

$$\begin{aligned} \sum_{i=1}^{n(n+1)} b_i^r &= \sum_{j=0}^n \sum_{k=1}^n b_{jn+k}^r \\ &= n \sum_{j=0}^n \left(\frac{a_{n+1-j}}{n+1 \sqrt[n+1]{a_{n+1}!}} \right)^r \\ &= n \sum_{i=1}^{n+1} \left(\frac{a_i}{n+1 \sqrt[n+1]{a_{n+1}!}} \right)^r \end{aligned} \quad (37)$$

and

$$\sum_{i=1}^{n(n+1)} c_i^r = (n+1) \sum_{i=1}^n \left(\frac{a_i}{\sqrt[n]{a_n!}} \right)^r. \quad (38)$$

Since $\{a_i\}_{i=1}^{n+1}$ is increasing, the sequence $\{b_i\}_{i=1}^{n(n+1)}$ and $\{c_i\}_{i=1}^{n(n+1)}$ are decreasing. Therefore, by Lemma 3, to obtain inequality (34), it is sufficient to prove inequality

$$b_m! \geq c_m! \quad (39)$$

for $1 \leq m \leq n(n+1)$.

It is easy to see that $b_{n(n+1)!} = c_{n(n+1)!} = 1$. Thus, inequality (39) is equivalent to

$$\prod_{i=m}^{n(n+1)} b_i \leq \prod_{i=m}^{n(n+1)} c_i \quad (40)$$

for $2 \leq m \leq n(n+1)$.

For $0 \leq \ell \leq n$ and $0 \leq j \leq n-2$, we have $2 \leq (n-\ell)n + (n-j) = (n-\ell)(n+1) + (\ell-j) \leq n(n+1)$. Then

$$\prod_{i=(n-\ell)n+(n-j)}^{n(n+1)} b_i = \frac{(a_{\ell+1})^{j+1} (a_\ell!)^n}{(a_{n+1}!)^{\frac{\ell n+j+1}{n+1}}}; \quad (41)$$

$$\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i = \frac{(a_\ell)^{n-\ell+j+2} (a_{\ell-1}!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell > j; \quad (42)$$

$$\begin{aligned} \prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i &= \prod_{i=(n-\ell-1)(n+1)+(n+1+\ell-j)}^{n(n+1)} c_i \\ &= \frac{(a_{\ell+1})^{j-\ell+1} (a_\ell!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell \leq j; \end{aligned} \quad (43)$$

where $a_0 = 1$.

The last term in (43) is bigger than the right term in (42), so, without loss of generality, we can assume $j < \ell$. Therefore, from formulae (41) and (42), inequality (40) is reduced to

$$\frac{(a_{\ell+1})^{j+1} (a_\ell!)^n (a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_{n+1}!)^\ell} \leq \frac{(a_\ell)^{n-\ell+j+2} (a_{\ell-1}!)^{n+1}}{(a_n!)^\ell (a_n!)^{\frac{j+1}{n}}}, \quad (44)$$

that is

$$\frac{(a_{\ell+1})^{j+1} (a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_\ell!) (a_\ell)^{j-\ell+1}} \leq \frac{(a_{n+1})^\ell (a_n!)^{-\frac{\ell}{n}}}{(a_n!)^{\frac{j-\ell+1}{n}}}, \quad (45)$$

this is further equivalent to

$$\frac{(a_{\ell+1})^{j+1} (a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{a_\ell! (a_\ell)^{j-\ell+1} (a_n!)^{\frac{\ell-j-1}{n}}} \leq \frac{(a_{n+1})^\ell}{(a_n!)^{\frac{\ell}{n}}}, \quad (46)$$

which can be rearranged as

$$\left(\frac{a_{\ell+1}}{a_\ell} \cdot \frac{\sqrt[n]{a_n!}}{n+1 \sqrt[n+1]{a_{n+1}!}} \right)^{\frac{j+1}{\ell}} \leq \frac{\sqrt[\ell]{a_\ell!}}{a_\ell} \cdot \frac{a_{n+1}}{n+1 \sqrt[n+1]{a_{n+1}!}}, \quad j+1 \leq \ell \leq n. \quad (47)$$

Utilizing Lemma 2 and the logarithmical concaveness of the sequence $\{a_i\}_{i=1}^{n+1}$ yields

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > \frac{a_n}{a_{n+1}} \geq \frac{a_\ell}{a_{\ell+1}}. \quad (48)$$

Since $\frac{j+1}{\ell} \leq 1$ and $\frac{a_{\ell+1}}{a_\ell} \cdot \frac{\sqrt[n+1]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > 1$ by (48), thus, to obtain (47), it suffices to prove

$$a_{\ell+1} \sqrt[n]{a_n!} < a_{n+1} \sqrt[\ell]{a_\ell!}, \quad (49)$$

this follows from Lemma 1.

Since the sequence $\{a_i\}_{i=1}^{n+m}$ is nonconstant, the inequality (10) is strict.

By the L'Hospital rule, easy calculation produces

$$\lim_{r \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n a_i^r \bigg/ \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r} = \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \quad (50)$$

thus, the upper bound is the best possible. The proof is complete.

Remark 4. Recently, some new inequalities for the ratios of the mean values of functions were established in [23].

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