ON THE CONVERGENCE A SEQUENCE

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Abstract. In this article we consider the problem of the convergence of the sequence $x_0 = 0;\ x_n = (a + bx_{n-1}^l)^{\frac{1}{t}}$ $(n = 1, 2, \cdots)$ defined for some positive a, b, l and real t.

1. Introduction

Let $a > 0$, $s_0(a) = 0$ and

$$s_n(a) = \underbrace{\sqrt{a + \sqrt{a + \cdots \sqrt{a}}}}_{n} \quad (n = 1, 2, \dots).$$

It is clear, that the sequence $\{s_n\}_{n=0}^{\infty}$ satisfies the conditions:

$$s_0(a) = 0,\ s_n(a) = (a + s_{n-1}(a))^{\frac{1}{2}} \quad (n = 1, 2, \dots).$$

Given a,$b > 0$ and real l, t denote

$$x_0 = 0; x_n = (a + bx_{n-1}^l)^{\frac{1}{t}} \quad (n = 1, 2, \dots). \tag{1.1}$$

It is known [1], that there exists $\lim_{n\to\infty} s_n(a)$ and as it was shown in [2], when $b = 1$, $l = 1$ and $t = 3$, there exists $\lim_{n\to\infty} x_n$. The authors of [2] have posed the question whether the previous conclusion remains valid for any real t (under the condition $b = 1$, $l = 1$). In this article we investigate the problem for the sequence (1.1) with some $a, b, l > 0$ and real t.

2. The Convergence For $t > 0$.

In this section we prove the convergence of the sequence (1.1) under the condition $t > l > 0$ and then discuss several corollaries.

Theorem 1. *Let $a, b > 0$ and $t > l > 0$. Then the sequence (1.1) is strictly increasing and convergent.*

Proof. By induction it is easy to verify that the sequence $\{x_n\}_{n=0}^{\infty}$ is strictly increasing. Then

$$x_n = (a + bx_{n-1}^l)^{\frac{1}{t}} < (a + bx_n^l)^{\frac{1}{t}} \quad (n = 1, 2, \dots)$$

and so $\{x_n\}_{n=0}^{\infty}$ satisfies the inequality

$$x_n^t - bx_n^l - a < 0 \quad (n = 0, 1, 2, \dots). \tag{2.1}$$

However, denoting

$$G(x) = x^t - bx^l - a, \quad (x \geq 0) \tag{2.2}$$

one can see that under the conditions of the theorem

$$G(0) = -a < 0, \quad G(+\infty) = +\infty$$

and the derivative

$$G'(x) = tx^{t-1} - blx^{l-1} = x^{l-1}(tx^{t-l} - bl)$$

has the unique positive root $\widetilde{x} = (\frac{bl}{t})^{\frac{1}{t-l}}$, and being strictly negative when $x\epsilon(0;\widetilde{x})$ and strictly positive when $x\epsilon(\widetilde{x};+\infty)$. This yields that the function $G(x)$ $(x \geq 0)$ has the unique root $\widetilde{X}$ $(> \widetilde{x})$, where $G(x) \leq 0$ for $x\epsilon[0,\widetilde{x}]$, and $G(x) > 0$ for $x\epsilon(\widetilde{x},\infty)$.Therefore, $\{x_n\}_{n=0}^{\infty} \subset [0,\widetilde{x}]$ in accordance with (2.1). As the sequence $\{x_n\}_{n=0}^{\infty}$ is monotone and bounded it is convergence. The proof is complete. ∎

Remark 1. *Denoting* $x_\infty = \lim_{n\to\infty} x_n$ *one can find*

$$x_\infty^t - bx_\infty^l - a = 0$$

in view of $x_n^t = a + bx_{n-1}^l$ $(n = 1,2,\dots)$. *This means that* $x_\infty = \widetilde{X}$ *- the unique positive root of* $G(x)$ $(x \geq 0)$. *Thus, for arbitrary* $a, b > 0$ *the sequence* $\{x_n\}_{n=0}^{\infty}$ *approximates the positive root of the function (2.2) (which is not necessarily a polynomial). Analogously, given natural number* k, *any positive numbers* $a, a_1, a_2, \dots, a_k$ *and* $t > l_1 > l_2 > \dots > l_k > 0$ *denote* $x_0 = x_1 = \dots = x_{k-1} = 0$ *and* $x_{n+1} = (a + a_1x_n^{l_1} + a_2x_{n-1}^{l_2} + \dots + a_kx_{n-k+1}^{l_k})^{\frac{1}{t}}$ $(n = k-1, k, k+1, \dots)$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *approximates the positive root of the function* $G(x) = x^t - (a + a_1x^{l_1} + a_2x^{l_2} + \dots + a_kx^{l_k})$ $(x \geq 0)$.

Corollary 1. *.Let the conditions of the Theorem 1 be fulfilled and* $x_\infty = \lim_{n\to\infty} x_n$.

(1) $\max(a^{\frac{1}{t}}, b^{\frac{1}{t-l}}) < x_\infty$;
(2) *if* t, l *are natural numbers, then* $x_\infty \leq 1 + \max(a, b)$;
(3) *if* $l \geq 1$ *and* $a \geq (b+1)^{\frac{1}{t-l}}$ *then* $x_\infty \leq a$;
(5) *if* $a > (b+1)^{\frac{l}{t-l}}$ *then* $x_\infty \leq a^{\frac{1}{t}}$.

In fact, *(1)* follows from the inequalities $G(a^{\frac{1}{t}}) = -ba^{\frac{l}{t}} < 0$, $G(b^{\frac{1}{t-l}}) = -a < 0$. Note, that $a^{\frac{1}{t}} = x_1 < x_n (< x_\infty)$ $(n = 2, 3, \dots)$, meanwhile $b^{\frac{1}{t-l}} < x_n(< x_\infty)$ holds for $n > n_0$ with some natural n_0.

For any polynomial $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ $(a_0 \neq 0)$ it is known [3] that if $|x| > 1 + \frac{A}{a_0}$, where $A = \max(|a_1|, |a_2|, \dots, |a_n|)$, then $|a_0x^n| > |a_1x^{n-1} + \dots + a_n|$. Therefore, if t, l are natural numbers and $P(x) = G(x)$, then the inequality $x^t > a + bx^l$ ($x > 1 + \max(a, b)$) means $G(x) > 0$ $(x > 1 + \max(a, b))$, and *(2)* is valid.

Under the conditions of *(3)* one can see that $G(a) = a^l(a^{t-l} - b) - a \geq a^l(b + 1 - b) - a = a^l - a \geq 0$ and hence *(3)* is valid.

Analogously, *(4)* follows from the inequality $G(a^{\frac{1}{t}}) > 0$.

3. The Convergence For $t < 0$

Given $a > 0, b = l = 1$ and $t < 0$ we will consider two subsequences $\Lambda_0 = \{x_{2k}\}_{k=0}^{\infty}$ and $\Lambda_1 = \{x_{2k+1}\}_{k=0}^{\infty}$ of the sequence (1.1). Denoting $s = -t$ one can see that for $k = 1, 2, \dots$:

$$x_{2k} = \frac{1}{\sqrt[s]{a + x_{2k-1}}}, \qquad x_{2k+1} = \frac{1}{\sqrt[s]{a + x_{2k}}} \tag{3.1}$$

$$x_{2k} = \frac{1}{\sqrt[s]{a + \frac{1}{\sqrt[s]{a+x_{2k-2}}}}}, \qquad x_{2k+1} = \frac{1}{\sqrt[s]{a + \frac{1}{\sqrt[s]{a+x_{2k-1}}}}}. \tag{3.2}$$

Obviously,

$$0 = x_0 < x_2 < x_4 < \cdots < x_{2k} < x_{2k+1} < x_{2k-1} < \cdots < x_1 = \frac{1}{\sqrt[s]{a}} \tag{3.3}$$

Therefore the sequences Λ_0 and Λ_1 are convergent; let $\underline{x} = \lim\limits_{k\to\infty} x_{2k}$, and $\overline{x} = \lim\limits_{k\to\infty} x_{2k-1}$. Note that, on taking the limit in (3.1) one obtains

$$\overline{x}\sqrt[s]{a+\underline{x}} = 1, \qquad \underline{x}\sqrt[s]{a+\overline{x}} = 1 \tag{3.4}$$

Theorem 2. *If $a > 0, b = l = 1$ and $t \leq -1$ then the sequence (1.1) is convergent.*

Proof. It is sufficient to verify the equality $\underline{x} = \overline{x}$. Let $s = -t$ (≥ 1) and

$$\varphi(\tau) = \frac{a+\tau}{a+\underline{x}} - \frac{\tau^s}{\underline{x}^s} \qquad (\tau \geq 0)$$

In view of (3.4)

$$\varphi(\underline{x}) = \varphi(\overline{x}) = 0.$$

For $s = 1$ the function $\varphi(\tau)$ is strictly decreasing , so $\underline{x} = \overline{x}$ in view of (3.3). If $s > 1$ and $\underline{x} < \overline{x}$ then there exists $\tau_0 \epsilon (\underline{x}; \overline{x})$ such that $\varphi'(\tau_0) = 0$. However, direct computation leads to

$$\tau_0^{s-1} = \frac{\underline{x}^s}{s(a+\underline{x})} < \frac{\underline{x}^{s-1}}{s} < \underline{x}^{s-1},$$

so $\tau_0 < \underline{x}$ in contrast with $\tau_0 \in (\underline{x}; \overline{x})$. The proof is complete. ∎

Remark 2. *It is easy to see that if $t \epsilon (-1; 0)$ and $at \leq -1$ then the sequence (1.1) is convergence. In fact, if $\underline{x} < \overline{x}$ and $\varphi'(\tau_0) = 0$, $\tau_0 \in (\underline{x}; \overline{x})$, then*

$$\underline{x}^{1-s} < \tau_0^{1-s} = \frac{s(a+\underline{x})}{\underline{x}^s} < \overline{x}^{1-s}$$

which implies that

$$\underline{x} < s(a+\underline{x}) < \underline{x}^s \overline{x}^{1-s}$$

According to (3.4), this means that

$$\underline{x}\overline{x}^s < s < \underline{x}^s \overline{x},$$

hence $s < (\frac{1}{\sqrt[s]{a}})^s \frac{1}{\sqrt[s]{a}}$, or $as < \frac{1}{\sqrt[s]{a}}$. However this is not possible as $a > as \geq 1$.

References

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