ON CERTAIN ENTROPY INEQUALITIES

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ABSTRACT. By using some connections of the entropy function with certain special means of two arguments, the author refines earlier entropy inequalities, or obtains new relations (identities and inequalities) for $H(p,q)$.

1. INTRODUCTION

1. Let p,q be positive real numbers such that $p+q=1$. The entropy of the probability vector (p,q) introduced in [7] is

$$H(p,q) = -p\ln p - q\ln q.$$

In the note [1] a new proof of the following double inequality (see [7]) has been provided:

Theorem 1. *One has*

$$\ln p \cdot \ln q \le H(p,q) \le \frac{(\ln p \cdot \ln q)}{\ln 2}. \tag{1.1}$$

Our aim in what follows is two fold. First, by remaking a connection with the logarithmic mean, we will obtain improvements of (1.1), in fact, a new proof. Secondly, the connection of H with a mean J introduced and studied, for example in [2, 4, 6], will give new relations for the entropy $H(p,q)$.

2. MAIN RESULTS

2. Let p,q be as above. First, we will prove the following relation:

Theorem 2.

$$\begin{aligned} \ln p \cdot \ln q &\le (\sqrt{p}+\sqrt{q}) \ln p \cdot \ln q \\ &\le H(p,q) \\ &\le A(p,q) \ln p \cdot \ln q \\ &\le \frac{(\ln p \cdot \ln q)}{\ln 2}, \end{aligned} \tag{2.1}$$

where $A(p,q) = \frac{2}{q-p}\int_p^q \frac{s-1}{\ln s} ds$.

Proof. We note that, since $q-1=-p$ and $p-1=-q$, one can write

$$H(p,q) = (q-1)\ln p + (p-1)\ln q = (\ln p)(\ln q)\left[\frac{q-1}{\ln q} + \frac{p-1}{\ln p}\right].$$

Now, $\frac{q-1}{\ln q}$ is equal to $L(q,1)$, where $L(x,y)$ is the logarithmic mean of x and y $(x \ne y)$ defined by $L(x,y) = \frac{x-y}{\ln x - \ln y}$. For the mean L there exists an extensive amount of literature. We shall use only the followng relations

$$G < L < A, \tag{2.2}$$

where $G = G(x,y) = \sqrt{xy}$ and $A = A(x,y) = \frac{x+y}{2}$ respectively denote the geometric and arithmetic means of x, y (see e.g. [2, 5]). By the left hand side of (2.2) one has $L(p,1) + L(q,1) > \sqrt{p} + \sqrt{q}$. Here $\sqrt{p} + \sqrt{q} > 1$ since this is equivalent to $\left(\sqrt{p} + \sqrt{q}\right)^2 > 1$, i.e. $p + q + 2\sqrt{pq} = 1 + 2\sqrt{pq} > 1$, which is trivial.

Therefore, the left sides of (2.1) are proved. For the right side, let us introduce the following function: $f(p) = \frac{p-1}{\ln p}$ $(p \in (0,1))$. An easy computation shows that

$$f'(p) = \frac{p \ln p - p + 1}{p \ln^2 p}, \qquad f''(p) = \frac{2(p-1) - (p+1)\ln p}{p^2 \ln^3 p}.$$

By the right side of (2.2), one has $L(p,1) < \frac{p+1}{2}$, i.e. $2(p-1) > (p+1)\ln p$. Since $\ln^3 p < 0$, we get that $f''(p) < 0$. Therefore, f is a *concave* function on $(0,1)$. Now, by the Jensen-Hadamard inequality, one has

$$\frac{f(p) + f(q)}{2} \le \frac{1}{q-p} \int_p^q f(s)\, ds \le f\left(\frac{p+q}{2}\right),$$

so

$$L(p,1) + L(q,1) \le A(p,q) \le 2L\left(\frac{1}{2}, 1\right) = \frac{1}{\ln 2},$$

completing the proof of the right side inequalities in (2.1). ∎

3. We now obtain an interesting connection of the entropy $H(p,q)$ with a mean J, defined by (see [2, 6])

$$J = J(a,b) = \left(a^a \cdot b^b\right)^{\frac{1}{a+b}} \qquad (a, b > 0). \tag{2.3}$$

Let p, q be as in the introduction. Then (2.3) implies the interesting relation

$$H(p,q) = \ln \frac{1}{J(a,b)}. \tag{2.4}$$

Since there exist many known results for the mean J, by equality (2.4), these give some information on the entropy H. For example, in [6], the following are proved:

$$\sqrt{\frac{a^2+b^2}{2}} \le J(a,b) \le \frac{A^2}{G}; \tag{2.5}$$

$$\frac{A^2 - G^2}{A^2} \le \ln \frac{J}{G} \le \frac{A^2 - G^2}{GA}; \tag{2.6}$$

$$J \le \frac{A\sqrt{2} - G}{\sqrt{2} - 1}, \tag{2.7}$$

where $J = J(a,b)$, etc. By (2.4) – (2.7) the following results are immediate:

Theorem 3.

$$2\ln 2 + \frac{1}{2}\ln(pq) \le H(p,q) \le \frac{1}{2}\ln 2 - \frac{1}{2}\ln(1 - 2pq), \tag{2.8}$$

$$\frac{4pq - 1}{2\sqrt{pq}} - \frac{1}{2}\ln(pq) \le H(p,q) \le 4pq - 1 - \frac{1}{2}\ln(pq), \tag{2.9}$$

$$H(p,q) \ge \ln\left(\frac{2 - \sqrt{2}}{1 - \sqrt{2pq}}\right). \tag{2.10}$$

Proof. Apply (2.5) – (2.7) and remark that $A = \frac{p+q}{2} = \frac{1}{2}$, $G = \sqrt{pq}$. ∎

The following result shows a connection with the so-called identric mean I, defined by $I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ $(a \neq b)$. In the paper [3], the following identity appears:

$$J(a,b) = \frac{I(a^2,b^2)}{I(a,b)}. \tag{2.11}$$

By (2.3), the entropy H is connected to the mean I by

$$H(p,q) = \ln I(p,q) - \ln I(p^2,q^2). \tag{2.12}$$

Since $\frac{4A^2-G^2}{3I} \leq J \leq \frac{A^4}{I^3}$ (see [6]), the following holds.

Theorem 4.

$$4\ln 2 + 3\ln I(p,q) \leq H(p,q) \leq \ln 3 + \ln I(p,q) - \ln(1-pq). \tag{2.13}$$

4. Finally, we shall deduce two series representations for H. In [6], the following representations are proved

$$\ln\frac{J}{A} = \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}\cdot z^{2k}; \tag{2.14}$$

$$\ln\frac{J}{G} = \sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot z^{2k}; \tag{2.15}$$

where $z = \frac{b-a}{b+a}$. Now, let $a = p$, $b = q$ with $p + q = 1$. Then, by (2.3), one can deduce:

Theorem 5.

$$H(p,q) = \ln 2 - \sum_{k=1}^{\infty}\frac{(p-q)^{2k}}{2k(2k-1)} \tag{2.16}$$

and

$$H(p,q) = \frac{1}{2}\ln(pq) - \sum_{k=1}^{\infty}\frac{(p-q)^{2k}}{2k-1}. \tag{2.17}$$

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References

[1] M. Bahramgiri and O.N. Arjomand, A simple proof of the entropy inequalities, *RGMIA Research Report Collection,* **3**(4) (2000), 659-670.
[2] J. Sándor, On the identric and logarithmic means, *Aequationes Math.,* **40** (1990), 261-270.
[3] J. Sándor, On certain identities for means, *Studia Univ. Babeş-Bolyai, Math.,* **38** (1993), No. 4, 7-14.
[4] J. Sándor, On certain inequalities for means, II, *J. Math. Anal. Appl.,* **199** (1996), 629-635.
[5] J. Sándor, On refinements of certain inequalities for means, *Arch. Math.* (Brno), **31** (1995), 279-282.
[6] J. Sándor and I. Raşa, Inequalities for certain means in two arguments, *Nieuw Arch. Wiseunde,* **15** (1997), 51-55.
[7] F. Topsoe, Bounds for the entropy and divergence for distributions over a two-element set, *J. Ineq. Pure and Appl. Math.,* **2**(2) (2001), Article 25.

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