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Estimates of the remainder in Taylor's theorem using the Henstock/Kurzweil integral

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Abstract. When a real-valued function of one variable is approximated by its n^{th} degree Taylor polynomial, the remainder is estimated using the Alexiewicz and p-norms in cases where $f^{(n)}$ or $f^{(n+1)}$ are Henstock/Kurzweil integrable. When the only assumption is that $f^{(n)}$ is Henstock/Kurzweil integrable then a modified form of the n^{th} degree Taylor polynomial is used.

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1 Introduction

In this paper the Henstock/Kurzweil integral is used to give various estimates of the remainder in Taylor's theorem in terms of Alexiewicz and p-norms. Let $[a, b]$ be a compact interval in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$. When f is approximated by its n^{th} degree Taylor polynomial, the remainder is estimated using the Alexiewicz and p-norms in the case when $f^{(n+1)}$ is Henstock/Kurzweil integrable. When the only assumption is that $f^{(n)}$ is Henstock/Kurzweil integrable then $f^{(n)}$ need not exist at a. In this case we use a modified form of the Taylor polynomial where $f^{(n)}$ is evaluated at a point $x_0 \in [a, b]$. The resulting modified remainder is then estimated in the Alexiewicz and p-norms. The results extend those in [1].

For the Henstock/Kurzweil integral we have the following version of Taylor's theorem.

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Theorem 1 *Let* $f:[a,b]\to\mathbb{R}$ *and let* n *be a positive integer. If* $f^{(n)}\in ACG_*$ *then for all* $x\in[a,b]$ *we have* $f(x)=P_n(x)+R_n(x)$ *where*

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)(x-a)^k}{k!} \tag{1}$$

and

$$R_n(x)=\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt. \tag{2}$$

A proof under the assumption that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}$ exists nearly everywhere on (a,b) is given in [4]. This is easily modified for the general case of Theorem 1.

The function space ACG_* is defined in [2]. Note that all ACG_* functions are continuous (on $[a,b]$). Amongst continuous functions, ACG_* properly contains the functions differentiable nearly everywhere and is a proper subset of the functions differentiable almost everywhere. And, $AC\subsetneq ACG_*$. Denote the Henstock/Kurzweil integrable functions on $[a,b]$ by $\mathcal{HK}$. If $F:[a,b]\to\mathbb{R}$ and $F\in ACG_*$ then $F'\in\mathcal{HK}$ and $\int_a^x F'=F(x)-F(a)$ for all $x\in[a,b]$ (Theorem 1 with $n=0$). As well, if $f:[a,b]\to\mathbb{R}$ and $f\in\mathcal{HK}$ then the function F defined by $F(x)=\int_a^x f$ is in ACG_*. For the wide Denjoy integral the corresponding function space is $ACG\supsetneq ACG_*$. If $F\in ACG$ then $\int_a^x F'_{\mathrm{ap}}=F(x)-F(a)$ for all $x\in[a,b]$. Here the integral is the wide Denjoy integral and the approximate derivative is used [2]. All the results below have a suitable extension to the wide Denjoy integral.

The Alexiewicz norm of $f\in\mathcal{HK}$ is defined

$$\|f\|=\sup_{a\le x\le b}\left|\int_a^x f\right|. \tag{3}$$

See [3] for a discussion of the Alexiewicz norm and the Henstock/Kurzweil integral.

2 Estimates when $f^{(n-1)}\in ACG_*$

If $f^{(n-1)}\in ACG_*$ then $f^{(n)}$ need only exist almost everywhere on (a,b). If $f^{(n)}$ exists at $x_0\in[a,b]$ then we can modify the Taylor polynomial (1) so that the n^{th} derivative is evaluated at x_0 and then obtain estimates on the resulting remainder term. The following lemma is used.

Lemma 2 *If $f:[a,b]\to\mathbb{R}$ and $f^{(n-1)}\in ACG_*$ then let $x_0\in[a,b]$ such that $f^{(n)}$ exists at x_0. Define the modified Taylor polynomial by*

$$P_{n,x_0}(x)=P_{n-1}(x)+\frac{f^{(n)}(x_0)(x-a)^n}{n!} \tag{4}$$

and define the modified remainder by $R_{n,x_0}(x)=f(x)-P_{n,x_0}(x)$. Then for all $x\in[a,b]$

$$R_{n,x_0}(x)=\frac{1}{(n-1)!}\int_a^x\left[f^{(n)}(t)-f^{(n)}(x_0)\right](x-t)^{n-1}\,dt. \tag{5}$$

The proof is the same as for Lemma 1 in [1] (which is false without the proviso that $f^{(n)}$ exists at x_0). Of course we can take x_0 as close to a as we like.

Theorem 2 in [1] gives pointwise estimates of R_n in terms of p-norms of $f^{(n)}$ when $f^{(n-1)}\in AC$ and $f^{(n)}\in L^p$ $(1\le p\le\infty)$. We have the following analogue when $f^{(n-1)}\in ACG_*$. Note that if $f^{(n-1)}\in ACG_*\setminus AC$ then for each $1\le p\le\infty$ we have $f^{(n)}\notin L^p$. However, we can use the Alexiewicz norm.

Theorem 3 *With the notation and assumptions of Lemma 2,*

$$\|R_{n,x_0}\|\le\frac{(b-a)^n}{n!}\|f^{(n)}(\cdot)-f^{(n)}(x_0)\|. \tag{6}$$

For all $x\in[a,b]$,

$$|R_{n,x_0}(x)|\le\frac{(x-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot)-f^{(n)}(x_0)\|. \tag{7}$$

And,

$$\|R_{n,x_0}\|_p\le\begin{cases}\frac{(b-a)^{n-1+1/p}}{(n-1)!\,[(n-1)p+1]^{1/p}}\|f^{(n)}(\cdot)-f^{(n)}(x_0)\|, & 1\le p<\infty\\ \frac{(b-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot)-f^{(n)}(x_0)\|, & p=\infty.\end{cases} \tag{8}$$

Proof: Let $a\le c\le b$. Using Lemma 2 and the reversal of integrals criterion in [2, Theorem 57, p. 58],

$$\int_a^c R_{n,x_0} = \frac{1}{(n-1)!}\int_a^c\left[f^{(n)}(t)-f^{(n)}(x_0)\right]\int_t^c(x-t)^{n-1}\,dx\,dt \tag{9}$$

$$= \frac{(c-a)^n}{n!}\int_a^\xi\left[f^{(n)}(t)-f^{(n)}(x_0)\right]dt \quad \text{for some } \xi\in[a,c] \tag{10}$$

Equation (10) comes from the second mean value theorem for integrals [2]. Taking the supremum over $c \in [a, b]$ now gives (6).

Similarly,

$$\begin{aligned} |R_{n,x_0}(x)| &= \frac{(x-a)^{n-1}}{(n-1)!}\left|\int_a^\xi \left[f^{(n)}(t) - f^{(n)}(x_0)\right]\,dt\right| \quad \text{for some } \xi \in [a, x] \\ &\le \frac{(x-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|. \qquad (11) \end{aligned}$$

This gives (7). The other estimates follow from this. ■

Note that if $f^{(n)} \in ACG_*$ then $\|f^{(n+1)}\| = \max_{a\le x\le b}|f^{(n)}(x) - f^{(n)}(a)|$.

An alternative approach in Theorem 3 is to assume $f^{(n-1)}$ is ACG_* on $[a, b]$ and $f^{(n)}$ exists at a. Then, using (1) and (2), equation (6) is replaced by $\|R_n\| \le (b-a)^n\|f^{(n)}(\cdot) - f^{(n)}(a)\|/n!$ with similar changes in (7) and (8).

3 Estimates when $f^{(n)} \in ACG_*$

Corollary 1 in [1] gives a pointwise estimate of R_n in terms of p-norms of $f^{(n+1)}$ when $f^{(n)} \in AC$ and $f^{(n+1)} \in L^p$. When $f^{(n)} \in ACG_* \setminus AC$ then for no $1 \le p \le \infty$ do we have $f^{(n+1)} \in L^p$. But, we can estimate R_n using the Alexiewicz norm of $f^{(n+1)}$ and p-norms of $f^{(n)}$.

Theorem 4 *If $f : [a, b] \to \mathbb{R}$ such that $f^{(n)} \in ACG_*$ then*

$$\|R_n\| \le \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n+1)}\|. \qquad (12)$$

For all $x \in [a, b]$,

$$|R_n(x)| \le \frac{(x-a)^n}{n!}\|f^{(n+1)}\|. \qquad (13)$$

And,

$$\|R_n\|_p \le \begin{cases} \frac{(b-a)^{n+1/p}}{n!\,(np+1)^{1/p}}\|f^{(n+1)}\|, & 1 \le p < \infty \\ \frac{(b-a)^n}{n!}\|f^{(n+1)}\|, & p = \infty. \end{cases} \qquad (14)$$

Also,

$$\|R_n\|_p \le \begin{cases} \frac{(b-a)^{n+1/p}}{n!\,(np+1)^{1/p}}|f^{(n)}(a)| + A_i, & 1 \le p < \infty \\ \frac{(b-a)^n}{n!}|f^{(n)}(a)| + \frac{(b-a)^n}{n!}\|f^{(n)}\|_\infty, & p = \infty, \end{cases} \qquad (15)$$

where $1/\alpha + 1/\beta = 1$ *and*

$$\begin{aligned}
A_1 &= \frac{(b-a)^{n-1+1/p}\|f^{(n)}\|}{(n-1)!\,[(n-1)p+1]^{1/p}}, \quad \textit{for } n \geq 1 \\
A_1 &= \|f(\cdot) - f(a)\|_p, \quad \textit{for } n = 1 \\
A_2 &= \frac{(b-a)^{n-1+1/p+1/\beta}\|f^{(n)}\|_\alpha}{(n-1)!\,[(n-1)\beta+1]^{1/\beta}[(n-1+1/\beta)p+1]^{1/p}}, \quad \textit{for } 1 < \alpha \leq \infty \\
A_2 &= \frac{(b-a)^{n-1+1/p}\|f^{(n)}\|_1}{(n-1)!\,[(n-1)p+1]^{1/p}}, \quad \textit{for } \alpha = 1 \\
A_3 &= \frac{(b-a)^{1-1/p}}{(n-1)!\,[(n-1)p+1]^{1/p}} \left(\int_a^b |f^{(n)}(t)|^p (b-t)^{(n-1)p+1}\, dt \right)^{1/p} \\
A_4 &= \frac{(b-a)^{n(1-1/p)}}{(n-1)!\,n} \left(\int_a^b |f^{(n)}(t)|^p (b-t)^n\, dt \right)^{1/p}.
\end{aligned}$$

Proof: The proof of (12) is very similar to the proof of (6), except that we begin with the remainder in the form of (2).

Using the second mean value theorem we have

$$|R_n(x)| = \frac{(x-a)^n}{n!} \left| \int_a^\xi f^{(n+1)}(t)\, dt \right| \quad \text{for some } \xi \in [a,b]. \tag{16}$$

The estimates in (13) and (14) now follow directly.

Integrate (2) by parts to get

$$R_n(x) = -\frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{1}{(n-1)!} \int_a^x f^{(n)}(t)(x-t)^{n-1}\, dt. \tag{17}$$

Then

$$\|R_n\|_p \leq \frac{|f^{(n)}(a)|}{n!} I_1^{1/p} + \frac{1}{(n-1)!} I_2^{1/p} \tag{18}$$

where

$$I_1 = \int_a^b (x-a)^{np}\, dx = \frac{(b-a)^{np+1}}{np+1} \tag{19}$$

and

$$I_2 = \int_a^b \left| \int_a^x f^{(n)}(t)(x-t)^{n-1}\, dt \right|^p dx. \tag{20}$$

A_1 is obtained from I_2 using the second mean value theorem and A_2 using Hölder's inequality.

Writing

$$I_2 \leq \int_a^b \left(\int_a^x |f^{(n)}(t)|(x-t)^{n-1} \frac{dt}{x-a} \right)^p (x-a)^p \, dx, \tag{21}$$

Jensen's inequality and Fubini's theorem give

$$\begin{aligned} I_2 &\leq \int_a^b |f^{(n)}(t)|^p \int_t^b (x-t)^{(n-1)p}(x-a)^{p-1} dx \, dt && (22) \\ &\leq \frac{(b-a)^{p-1}}{(n-1)p+1} \int_a^b |f^{(n)}(t)|^p (b-t)^{(n-1)p+1} dt. && (23) \end{aligned}$$

¿From this we obtain A_3.

Using Jensen's inequality in the form

$$\begin{aligned} I_2 &\leq \int_a^b \left(\int_a^x |f^{(n)}(t)| \frac{(x-t)^{n-1} n \, dt}{(x-a)^n} \right)^p \frac{(x-a)^{np}}{n^p} dx && (24) \\ &\leq \frac{1}{n^{p-1}} \int_a^b \int_a^x |f^{(n)}(t)|^p (x-t)^{n-1}(x-a)^{n(p-1)} dt \, dx, && (25) \end{aligned}$$

we can apply Fubini's theorem to get

$$I_2 \leq \frac{(b-a)^{n(p-1)}}{n^p} \int_a^b |f^{(n)}(t)|^p (b-t)^n \, dt, \tag{26}$$

which gives A_4. The case $p = \infty$ follows directly from (17). ∎

Note that A_2, A_3 and A_4 all lead to estimates of form $A_k \leq C_{n,p}(b-a)^n \|f^{(n)}\|_p$ where $C_{n,p}$ is independent of f and $k = 2, 3, 4$.

The integral over x in (22) can be evaluated using hypergeometric functions. However, this does not markedly improve the estimate for A_3. Similarly with A_4.

In (7) and (13) we can get a slightly more refined estimate by replacing the Alexiewicz norm over $[a, b]$ by the Alexiewicz norm over $[a, x]$. Similarly with the α-norm in A_2.

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