INTEGRAL INEQUALITIES OF GRÜSS TYPE VIA PÓLYA-SZEGÖ AND SHISHA-MOND RESULTS

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Abstract. Integral inequalities of Grüss type obtained via Pólya-Szegö and Shisha-Mond results are given. Some applications for Taylor's generalised expansion are also provided.

1. Introduction

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \frac{1}{b-a}\int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2}\int_a^b f(x) dx \cdot \int_a^b g(x) dx, \tag{1.1}$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [11]

$$|T(f, g; a, b)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a, b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [16, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty[a, b]$ and $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş (see for example [16, p. 210]), which states that:

$$|T(f, g; a, b)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{1.4}$$

provided f, g are absolutely continuous and $f', g' \in L_2[a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Grüss type inequalities, see the books [16] and [13], and the papers [2]-[10], where further references are given.

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2. Integral Inequalities of Grüss Type

The following Grüss type inequality holds.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}_+$ be two integrable functions so that*

$$0 < m \le f(x) \le M < \infty \quad \text{and} \quad 0 < n \le g(x) \le N < \infty \tag{2.1}$$

for a.e. $x \in [a, b]$.

Then one has the inequality

$$|T(f, g; a, b)| \le \frac{1}{4} \cdot \frac{(M-m)(N-n)}{\sqrt{mnMN}} \cdot \frac{1}{b-a} \int_a^b f(x)\,dx \cdot \frac{1}{b-a} \int_a^b g(x)\,dx. \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in (2.2) in the sense that it cannot be replaced by a smaller constant.

Proof. We have, by the Cauchy-Buniakowski-Schwartz inequality for double integrals, that

$$\begin{aligned} |T(f, g; a, b)| &= \left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))\,dxdy \right| \\ &\le \frac{1}{2(b-a)^2} \left[\int_a^b \int_a^b (f(x) - f(y))^2\,dxdy \cdot \int_a^b \int_a^b (g(x) - g(y))^2\,dxdy \right]^{\frac{1}{2}} \\ &= \frac{1}{2(b-a)^2} \left[4 \left[(b-a) \int_a^b f^2(x)\,dx - \left(\int_a^b f(x)\,dx \right)^2 \right] \right. \\ &\quad \times \left. \left[(b-a) \int_a^b g^2(x)\,dx - \left(\int_a^b g(x)\,dx \right)^2 \right] \right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a} \int_a^b f^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right)^2 \right]^{\frac{1}{2}} \\ &\quad \times \left[\frac{1}{b-a} \int_a^b g^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Utilising the Pólya-Szegö inequality for integrals [15], i.e.,

$$1 \le \frac{\int_a^b h^2(x)\,dx \int_a^b l^2(x)\,dx}{\left(\int_a^b h(x)\,l(x)\,dx \right)^2} \le \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2, \tag{2.4}$$

provided $0 < m_1 \le h(x) \le M_1 < \infty$, $0 < m_2 \le l(x) \le M_2 < \infty$ for a.e. $x \in [a, b]$, we may state that

$$\frac{(b-a) \int_a^b f^2(x)\,dx}{\left(\int_a^b f(x)\,dx \right)^2} \le \frac{1}{4} \left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}} \right)^2 = \frac{1}{4} \cdot \frac{(M+m)^2}{mM},$$

giving

$$\frac{(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2}{\left(\int_a^b f(x)\,dx\right)^2}\le\frac{1}{4}\cdot\frac{(M+m)^2}{mM}-1=\frac{(M-m)^2}{4mM},$$

that is,

$$(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2\le\frac{(M-m)^2}{4mM}\left(\int_a^b f(x)\,dx\right)^2. \tag{2.5}$$

In a similar fashion, we obtain

$$(b-a)\int_a^b g^2(x)\,dx-\left(\int_a^b g(x)\,dx\right)^2\le\frac{(N-n)^2}{4nN}\left(\int_a^b g(x)\,dx\right)^2. \tag{2.6}$$

Using (2.3), (2.5) and (2.6), we deduce the desired inequality (2.2).

Now, assume that the inequality in (2.2) holds with a constant $c>0$, i.e.,

$$\begin{aligned}|T(f,g;a,b)|\\ \le c\cdot\frac{(M-m)(N-n)}{\sqrt{mnMN}}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.\end{aligned} \tag{2.7}$$

We choose the functions $f=g$ with

$$f(x)=\begin{cases} m, & x\in\left[a,\frac{a+b}{2}\right]\\ M, & x\in\left(\frac{a+b}{2},b\right]\end{cases},\ 0<m<M<\infty.$$

Then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2&=\frac{m^2+M^2}{2}-\left(\frac{m+M}{2}\right)^2\\ &=\frac{1}{4}(M-m)^2,\end{aligned}$$

and by (2.7) we deduce

$$\frac{1}{4}(M-m)^2\le c\cdot\frac{(M-m)^2}{mM}\cdot\left(\frac{m+M}{2}\right)^2$$

from where we get

$$mM\le c(M-m)^2 \tag{2.8}$$

for any $0<m<M<\infty$.

If in (2.8) we consider $m=1-\varepsilon$, $M=1+\varepsilon$, $\varepsilon\in(0,1)$, then we get $1-\varepsilon^2\le 4c$ for any $\varepsilon\in(0,1)$, which shows that $c\ge\frac{1}{4}$. ∎

The second result of Grüss type is embodied in the following theorem.

Theorem 2. *Assume that f and g are as in Theorem 1. Then one has the inequality:*

$$|T(f,g;a,b)| \leq \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx}. \tag{2.9}$$

The constant $c=1$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. We shall use the Shisha-Mond inequality [17] (see also [13, p. 121])

$$\frac{\sum_{i=1}^n z_i^2}{\sum_{i=1}^n z_i y_i}-\frac{\sum_{i=1}^n z_i y_i}{\sum_{i=1}^n y_i^2}\leq\left(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}}\right)^2, \tag{2.10}$$

provided $0<m_1\leq z_i\leq M_1<\infty$ and $0<m_2\leq y_i\leq M_2<\infty$ for all $i\in\{1,\dots,n\}$.

Applying a standard procedure for Riemann sums instead of z_i, y_i, i.e.,

$$\frac{\frac{b-a}{n}\sum_{i=0}^n h^2\left(a+\frac{i}{n}(b-a)\right)}{\frac{b-a}{n}\sum_{i=0}^n h\left(a+\frac{i}{n}(b-a)\right)l\left(a+\frac{i}{n}(b-a)\right)} - \frac{\frac{b-a}{n}\sum_{i=0}^n h\left(a+\frac{i}{n}(b-a)\right)l\left(a+\frac{i}{n}(b-a)\right)}{\frac{b-a}{n}\sum_{i=0}^n l^2\left(a+\frac{i}{n}(b-a)\right)}\leq\left(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}}\right)^2,$$

provided h,l are Riemann integrable on $[a,b]$ and $0<m_1\leq h(x)\leq M_1<\infty$, $0<m_2\leq l(x)\leq M_2<\infty$, we may deduce, by letting $n\to\infty$, the integral inequality

$$\frac{\int_a^b h^2(x)\,dx}{\int_a^b h(x)\,l(x)\,dx}-\frac{\int_a^b h(x)\,l(x)\,dx}{\int_a^b l^2(x)\,dx}\leq\left(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}}\right)^2, \tag{2.11}$$

which is the integral version of the Shisha-Mond inequality (2.10).

From (2.11) we may easily deduce

$$0\leq\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \leq\left(\sqrt{M}-\sqrt{m}\right)^2\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.12}$$

and

$$0\leq\frac{1}{b-a}\int_a^b g^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2 \leq\left(\sqrt{N}-\sqrt{n}\right)^2\frac{1}{b-a}\int_a^b g(x)\,dx. \tag{2.13}$$

Finally, by making use of (2.3), (2.12) and (2.13), we obtain the desired inequality (2.9).

To prove the sharpness of the constant, assume that (2.9) holds with a constant $c > 0$, i.e.,

$$|T(f,g;a,b)| \leq c\left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx}. \tag{2.14}$$

Now, let us choose $f = g$ and

$$f(x) = \begin{cases} m, & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ M, & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then from (2.14) we deduce (see also Theorem 1) that

$$\frac{1}{4}(M-m)^2 \leq c\left(\sqrt{M}-\sqrt{m}\right)^2 \frac{m+M}{2}, \quad 0 < m < M < \infty$$

that is,

$$\frac{1}{4}\left(\sqrt{M}-\sqrt{m}\right)^2\left(\sqrt{M}+\sqrt{m}\right)^2 \leq c\left(\sqrt{M}-\sqrt{m}\right)^2 \frac{m+M}{2},$$

giving for any $0 < m < M < \infty$ that

$$\left(\sqrt{M}+\sqrt{m}\right)^2 \leq 2c(m+M). \tag{2.15}$$

If in (2.15) we choose $m = 1-\varepsilon$, $M = 1+\varepsilon$, $\varepsilon \in (0,1)$, we get $\left(\sqrt{1-\varepsilon}+\sqrt{1+\varepsilon}\right)^2 \leq 4c$. Letting $\varepsilon \to 0+$, we deduce $c \geq 1$, and the theorem is proved. ∎

By the classical Grüss' inequality, we obviously have

$$|T(f,g;a,b)| \leq \frac{1}{4}(M-m)(N-n). \tag{2.16}$$

It is natural to compare the bounds provided by (2.2), (2.9) and (2.16).

Proposition 1. *The bounds provided by (2.2), (2.9) and (2.16) are not related. This means that one is better than the others depending on the different choices of functions f and g.*

Proof. (1) With the assumptions in Theorem 2, consider, for $f = g$, $n = m$, $N = M$, the quantity

$$U := \frac{\left(\int_a^b f(x)\,dx\right)^2}{(b-a)^2 mM} > 0.$$

We want to compare this quantity with 1.

Choose $a = 0$, $b = 3$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,2], \\ k & \text{if } x \in (2,3], \quad k \geq 1. \end{cases}$$

Then $\int_a^b f(x)\,dx = 1+k$, $m = 1$, $M = k$ and thus

$$U(k) = U = \frac{(k+2)^2}{9k}.$$

We observe that

$$U(k)-1=\frac{(k-1)(k-4)}{9k},$$

showing that if $k\in(0,1]\cup[4,\infty)$, $U(k)\geq 1$ while for $k\in(1,4)$, $U(k)<1$.

In conclusion, for the above choice, if $k\in(1,4)$, the bound provided by (2.2) is better than the bound provided by (2.16), while for $k\in(4,\infty)$ this bound is worse than that provided by the Grüss inequality.

(2) With the assumptions in Theorem 2, consider, for $f=g$, $n=m$, $N=M$, the quantity

$$I_1:=\frac{1}{4}(M-m)^2,\quad I_2:=\left(\sqrt{M}-\sqrt{m}\right)^2\frac{1}{b-a}\int_a^b f(x)\,dx.$$

If we assume that $m=0$, $M=1$, then $I_1=\frac{1}{4}$, $I_2=\frac{1}{b-a}\int_a^b f(x)\,dx$, provided $0\leq f(x)\leq 1$, $x\in[a,b]$.

Now, if we choose f so that $\frac{1}{b-a}\int_a^b f(x)\,dx<\frac{1}{4}$, then the bound provided by (2.9) is better than the one provided by (2.16). If $\frac{1}{b-a}\int_a^b f(x)\,dx>\frac{1}{4}$, then Grüss' inequality provides a better bound.

(3) With the assumptions in Theorem 2, consider, for $f=g$, $n=m$, $N=M$, the quantities

$$\begin{aligned}J_1 &:= \frac{1}{4}\frac{(M-m)^2}{mM}\cdot\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2,\\ J_2 &:= \left(\sqrt{M}-\sqrt{m}\right)^2\cdot\frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

If we choose $m=1$, $M=4$, we get

$$J_1=\frac{9}{16}y^2,\quad J_2=y\text{ where }y:=\frac{1}{b-a}\int_a^b f(x)\,dx\in[1,4].$$

Now, observe that

$$J_1-J_2=\frac{y(9y-16)}{16},$$

showing that for $y\in\left[1,\frac{16}{9}\right]$ the bound provided by (2.2) is better than the bound provided by (2.9) while for $y\in\left(\frac{16}{9},4\right]$, the conclusion is the other way around.

∎

3. Some Pre-Grüss Type Inequalities and Applications

If there is no information available about the upper and lower bounds of the function g, but the integrals

$$\int_a^b g^2(x)\,dx\quad\text{and}\quad\int_a^b g(x)\,dx$$

can be exactly computed, then the following pre-Grüss type result may be stated.

Theorem 3. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions such that there exist $m, M > 0$ with*

$$0 < m \leq f(x) \leq M < \infty \tag{3.1}$$

and $g \in L_2[a,b]$. Then one has the inequality

$$|T(f,g;a,b)| \leq \frac{1}{2} \cdot \frac{(M-m)}{\sqrt{mM}} \cdot \frac{1}{b-a} \int_a^b f(x)\,dx \times \left[\frac{1}{b-a} \int_a^b g^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right)^2 \right]^{\frac{1}{2}}. \tag{3.2}$$

The constant $\frac{1}{2}$ is best possible.

The proof is similar to the one incorporated in Theorem 1 and we omit the details.

Similarly, we may state the corresponding pre-Grüss inequality that may be deduced from Shisha-Mond's result.

Theorem 4. *With the assumption of Theorem 3, we have*

$$|T(f,g;a,b)| \leq \left(\sqrt{M} - \sqrt{m}\right) \sqrt{\frac{1}{b-a} \int_a^b f(x)\,dx} \times \left[\frac{1}{b-a} \int_a^b g^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right)^2 \right]^{\frac{1}{2}}. \tag{3.3}$$

The constant $c = 1$ is best possible in the sense that it cannot be replaced by a smaller constant.

Following Matić et al. [12], we may say that the sequence of polynomials $\{P_n(x)\}_{n \in \mathbb{N}}$ is a *harmonic sequence* if

$$P_n'(x) = P_{n-1}(x) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad P_0(x) = 1.$$

In the above mentioned paper [12], the authors considered the following particular instances of harmonic polynomials:

$$P_n(t) = \frac{(t-x)^n}{n!}, \quad n \geq 0;$$
$$P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n, \quad n \geq 0;$$
$$P_n(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \quad P_0(t) = 1, \ n \geq 2;$$

where $B_n(t)$ are the well known Bernoulli polynomials, and

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right), \quad P_0(t) = 1, \ n \geq 1,$$

where $E_n(t)$ are the Euler polynomials.

The following perturbed version of the generalised Taylor's formula was obtained in [12].

Theorem 5. *Let $\{P_n(x)\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose that $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalised Taylor's formula:*

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x), \tag{3.4}$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right]$$

and

$$\left[f^{(n)};a,x\right] = \frac{f^{(k)}(x) - f^{(k)}(a)}{x-a}.$$

For $x \geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimation

$$\left|\tilde{G}_n(f;a,x)\right| \leq \frac{x-a}{2}(\Gamma(x) - \gamma(x)) [T(P_n,P_n)]^{\frac{1}{2}}, \tag{3.5}$$

where

$$T(P_n,P_n;a,x) := \frac{1}{x-a}\int_a^x P_n^2(t)\,dt - \left(\frac{1}{x-a}\int_a^x P_n(t)\,dt\right)^2$$

and

$$\gamma(x) = \inf_{t\in[a,x]} f^{(n+1)}(t), \quad \Gamma(x) = \sup_{t\in[a,x]} f^{(n+1)}(t).$$

Using Theorems 3 and 4, we may point out the following bounds for the remainder $\tilde{G}(f;a,x)$ as well.

Theorem 6. *Assume that $\{P_n(x)\}_{n\in\mathbb{N}}$ and f are as in Theorem 5. Moreover, if $\gamma(x) > 0$, then we have the representation (3.4) and the remainder $\tilde{G}(f;a,x)$ satisfies the bounds*

$$\left|\tilde{G}_n(f;a,x)\right| \leq \begin{cases} \dfrac{1}{2}\cdot\dfrac{\Gamma(x)-\gamma(x)}{\sqrt{\gamma(x)\Gamma(x)}}\left[f^{(n)};a,x\right][T(P_n,P_n;a,x)]^{\frac{1}{2}}(x-a) \\[2ex] \left(\sqrt{\Gamma(x)} - \sqrt{\gamma(x)}\right)\sqrt{\left[f^{(n)};a,x\right]}\,[T(P_n,P_n;a,x)]^{\frac{1}{2}}(x-a) \end{cases} \tag{3.6}$$

for any $x \geq a$.

The proof is similar to the one in Theorem 3, [12] and we omit the details.

Remark 1. *If we choose the above particular instances of harmonic polynomials, then we may produce a number of particular Taylor-like formulae whose remainder will obey similar bounds to those incorporated in (3.6). We omit the details.*

Remark 2. *As shown by Proposition 1, the bounds provided by (3.5) and (3.6) cannot be compared in general.*

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