ON AN INEQUALITY OF DIANANDA

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Abstract. We consider certain refinements of the arithmetic and geometric means, the results generalize an inequality of P. Diananda.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} q_i x_i^r)^{\frac{1}{r}}$, where $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \rightarrow 0^+$, with $q_i > 0, 1 \leq i \leq n$ are positive real numbers with $\sum_{i=1}^{n} q_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. In this paper, we let $q = \min q_i$ and always assume $n \geq 2, 0 \leq x_1 < x_2 < \cdots < x_n$.

We let $A_n(\mathbf{x}) = P_{n,1}(\mathbf{x}), G_n(\mathbf{x}) = P_{n,0}(\mathbf{x}), H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$ and we shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$ and similarly for other means when there is no risk of confusion.

For mutually distinct numbers r, s, t and any real number α, β, we define

$$\Delta_{r,s,t,\alpha,\beta} = |\frac{P_{n,r}^{\alpha} - P_{n,t}^{\alpha}}{P_{n,r}^{\beta} - P_{n,s}^{\beta}}|$$

where we interpreter $P_{n,r}^0 - P_{n,s}^0$ as $\ln P_{n,r} - \ln P_{n,s}$. When $\alpha = \beta$, we define $\Delta_{r,s,t,\alpha}$ to be $\Delta_{r,s,t,\alpha,\alpha}$. For example $\Delta_{r,s,t,0} = |(\ln \frac{P_{n,r}}{P_{n,t}})/(\ln \frac{P_{n,r}}{P_{n,s}})|$.

Bounds for $\Delta_{r,s,t,\alpha,\beta}$ have been studied by many mathematicians. For the case $\alpha \neq \beta$, we refer the reader to the articles [2, 5, 7] for the detailed discussions. When $\alpha = \beta$, we can bound $\Delta_{r,s,t,\alpha}$ in terms of r, s, t only, due to the following result of H.Hsu[6](see also [1]):

Theorem 1.1. *For $r > s > t > 0$*

$$1 < \Delta_{r,s,t,1} < \frac{s(r-t)}{t(r-s)} \tag{1.1}$$

It is also interesting to consider the following bounds:

$$f_{r,s,t,\alpha}(q) \geq \Delta_{r,s,t,\alpha} \geq g_{r,s,t,\alpha}(q) \tag{1.2}$$

where $f_{r,s,t,\alpha}(q)$ is a decreasing function of q and $q_{r,s,t,\alpha}(q)$ is an increasing function of q.

The case $r = 1, s = 0, t = -1, \alpha = 0$ in (1.2) with $f_{1,0,-1,0}(q) = 1/q, g_{1,0,-1,0}(q) = 1/(1-q)$ is the famous Sierpiński's inequality[9].

Another case, $r = 1, s = \frac{1}{2}, t = 0, \alpha = 1$ with $f_{1,1/2,0,1}(q) = 1/q, g_{1,1/2,0,1}(q) = 1/(1-q)$ was proved by P. Diananda([3], [4])(see also [1],[8]) , originally stated as:

$$\frac{1}{q}\Sigma_n \geq A_n - G_n \geq \frac{1}{1-q}\Sigma_n$$

where $\Sigma_n = \sum_{1 \leq i < j \leq n} q_i q_j (x_i^{\frac{1}{2}} - x_j^{\frac{1}{2}})^2$.

The main purpose of this paper is to generalize Diananda's result, which is given by theorem 3.1 in section 3.

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2. Lemmas

Lemma 2.1. *For $0 \le q \le 1/2$*

$$\frac{r-1}{r} - (1-q^{r-1}) \le 0 \qquad (r \ge 2) \tag{2.1}$$

$$|\frac{r-1}{r}| \ge |1-(1-q)^{r-1}| \qquad (0 < r \le 2) \tag{2.2}$$

with equality holding if and only if $r = 2, q = 1/2$.

Proof. We will prove (2.1) here and the proof for (2.2) is similar. It suffices to prove (2.1) for $q = 1/2$, which is equivalent to $2^r \ge 2r$. Notice the two curves $y = 2^r, y = 2r$ only intersect at $r = 1, r = 2$ in which cases they are equal and the conclusion then follows. □

Lemma 2.2. *For $0 < q \le 1$, the function*

$$f(q) = |\frac{q}{1-(1-q)^{r-1}}| \tag{2.3}$$

is decreasing for $0 < r \neq 1 < 2$ and increasing for $r > 2$.

Proof. We prove the case $1 < r \neq 2$ here and the case $0 < r < 1$ is similar. We have

$$f'(q) = \frac{1-(1-q)^{r-1} - q(r-1)(1-q)^{r-2}}{(1-(1-q)^{r-1})^2}$$

and by the mean value theorem $1-(1-q)^{r-1} = q(r-1)\eta^{r-2}$, where $1-q < \eta < 1$, which implies $f'(q) \le 0$ for $1 < r < 2$ and $f'(q) \ge 0$ for $r > 2$. □

Lemma 2.3. *For $0 < r \neq 1 < 2, 0 < q \le 1/2$,*

$$|\frac{1/2}{1-1/r}| < |\frac{q}{1-(1-q)^{r-1}}| \tag{2.4}$$

If $r > 2$, (2.4) is valid with '$>$' instead of '$<$'.

Proof. We prove the case $1 < r < 2$ here and the other cases are similar. By lemma 2.1 it suffices to show (2.4) for $q = 1/2$. In this case, (2.4) is equivalent to (2.2). □

3. The Main Theorems

Theorem 3.1. *For any $t \neq 0$,*

$$\Delta_{t,\frac{t}{r},0,t} \ge \frac{1}{1-q^{r-1}} \qquad (r \ge 2) \tag{3.1}$$

$$\Delta_{t,\frac{t}{r},0,t} \le |\frac{1}{1-(1-q)^{r-1}}| \qquad (0 < r \neq 1 \le 2) \tag{3.2}$$

with equality holding if and only if $n = 2, x_1 = 0, q_2 = q$ for (3.1), $n = 2, x_1 = 0, q_1 = q$ for (3.2), except in the trivial case $r = n = 2, q_1 = q_2 = 1/2$.

Proof. Since the proofs of (3.1)-(3.2) are very similar, we only prove (3.1) here and we just point out (2.2) is needed for the proof of (3.2). The case $r = 2$ was treated in [3] so we will assume $r > 2$ from now on. Consider the case $t = 1$ first and we define

$$D_n(\mathbf{x}) = (1-q^{r-1})(A_n - G_n) - (A_n - P_{n,1/r})$$

and we then have

$$\frac{1}{q_n}\frac{\partial D_n}{\partial x_n} = (1-q^{r-1})(1-\frac{G_n}{x_n}) - (1-(\frac{P_{n,1/r}}{x_n})^{1-1/r}) \tag{3.3}$$

By a change of variables: $\frac{x_i}{x_n} \to x_i, 1 \le i \le n$, we may assume $0 < x_1 < x_2 < \cdots < x_n = 1$ in (3.3) and rewrite it as

$$g_n(x_1, \cdots, x_{n-1}) := (1 - q^{r-1})(1 - G_n) - (1 - (P_{n,1/r})^{1-1/r}) \tag{3.4}$$

We want to show $g_n \ge 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0, 1]^{n-1}$ be the absolute minimum of g_n. If $\mathbf{a}$ is a boundary point of $[0, 1]^{n-1}$, then $a_1 = 0$, (3.4) is reduced to

$$g_n = 1 - q^{r-1} - (1 - (P_{n,1/r})^{1-1/r})$$

It follows that $g_n \ge 0$ is equivalent to $P_{n,1/r} \ge q^r$ while the last inequality is easily verified with equality holding if and only if $n = 2, a_1 = 0, q_2 = q$. Thus (3.1) holds for this case.

Now we may assume $a_1 > 0$ and $\mathbf{a}$ is an interior point of $[0, 1]^{n-1}$, then we obtain

$$\nabla g_n(a_1, \cdots, a_{n-1}) = 0$$

such that $a_1, \cdots, a_{n-1}$ solve the equation

$$-(1 - q^{r-1})\frac{G_n}{x} + (1 - 1/r)(P_{n,1/r})^{-1/r}(\frac{P_{n,1/r}}{x})^{1-1/r} = 0$$

The above equation has at most one root, so we only need to show $g_n \ge 0$ for the case $n = 2$. Now by letting $0 < x_1 = x < x_2 = 1$ in (3.4), we get

$$\frac{1}{q_1} g_2'(x) = h(x)x^{1/r-1}$$

where

$$h(x) = \frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1 - 1/r}$$

If $1/r \ge q_1$, then

$$h'(x) = \frac{(r-1)(r-2)}{r^2} q_1 x^{1/r-1}(q_1 x^{1/r} + q_2)^{r-3} - (1 - q^{r-1})(q_1 - \frac{1}{r})x^{q_1-1/r-1} \ge 0$$

which implies

$$h(x) \le h(1) = \frac{r-1}{r} - (1 - q^{r-1}) < 0$$

for $r > 2, q \le 1/2$ by lemmas 2.1 and thus $g(x) \ge g(1) = 0$.

If $q_1 > 1/r$, we have:

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} (\frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1-1/r}) > 0 \tag{3.5}$$

and

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} (\frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1-1/r}) = \frac{r-1}{r} - (1 - q^{r-1}) < 0 \tag{3.6}$$

Notice here any positive root of $h(x)$ also satisfies the equation:

$$P(x) = q_1 x^{1/r} + q_2 - (Cx^{q_1-1/r})^{\frac{1}{r-2}} = 0$$

where $C = r(1 - q^{r-1})/(r - 1)$.

It is easy to see that $P'(x)$ can have at most one positive root. Thus by Rolle's theorem, $P(x)$ hence $h(x)$ can have at most two roots in $(0, 1)$. (3.5) and (3.6) further implies $h(x)$ hence $g_2'(x)$ has exactly one root x_0 in $(0, 1)$. Since (3.6) shows $g_2'(1) < 0$, $g_2(x)$ takes its maximum value at x_0. Thus $g_2(x) \ge min\{g_2(0), g_2(1)\} = 0$.

Thus we have shown $g_n \ge 0$, hence $\frac{\partial D_n}{\partial x_n} \ge 0$ with equality holding if and only if $n = 1$ or $n = 2, x_1 = 0, q_2 = q$. By letting x_n tend to x_{n-1}, we have $D_n \ge D_{n-1}$(with weights $q_1, \cdots, q_{n-2}, q_{n-1} + q_n$). Since $1 - q^{r-1}$ is a decreasing function of q, it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0, q_2 = q$ in D_2 or else $D_n > D_{n-1} > \cdots > D_1 = 0$. Since we assume $n > 2$ in this paper, this completes the proof for $t = 1$.

Now for an arbitrary t, a change of variables $x_i \to x_i^t$ in the above cases leads to the desired conclusion. □

We remark here the constants in (3.1)-(3.2) are best possible by considering the case $n = 2, x_1 = 0, q_2 = q$ or $q_1 = q$. Also when $n = 2$, we conclude from the proof of lemma 2.1 and $\lim_{x_1 \to x_2} \Delta_{t,\frac{t}{r},0,t} = r/(r-1)$ that an upper bound in the form of (3.2) does not hold for $\Delta_{1,\frac{1}{r},0,1}$ when $r > 2$. Similarly, a lower bound in the form of (3.1) doesn't hold for $1 < r < 2$.

For $t = 1$, rewrite (3.1) as

$$A_n - G_n \geq \frac{1}{1-q^{r-1}}(A_n - P_{n,1/r}) \tag{3.7}$$

When $n = 2$ we have

$$\lim_{x_1 \to x_2} \frac{(A_2 - P_{2,1/2})/(1-q)}{(A_2 - P_{2,1/r'})/(1-q^{r'-1})} = \frac{1/2/(1-q)}{(1-1/r')/(1-q^{r'-1})}$$

by considering $q = 0, 1/2$, we find that the right hand sides of (3.7) are not comparable for $r = 2$ and any $r' > 2$.

However, for the comparison of the left hand sides of (3.2), we have

Theorem 3.2. *For any $t \neq 0$, $0 < r \neq 1 < 2$, $q > 0$*

$$\left|\frac{q}{1-(1-q)^{r-1}}\right| \geq \Delta_{t,\frac{t}{r},\frac{t}{2},t} \tag{3.8}$$

If $r \geq 2$, (3.8) is valid with '$\leq$' instead '$\geq$' with equality holding in all the cases if and only if $n = 2, x_1 = 0, q_1 = q$.

Proof. Since the proofs are similar, we only prove the case $1 < r < 2$ here. Notice by lemma 2.2, $\frac{q}{1-(1-q)^{r-1}}$ is decreasing with respect to q so we can prove by induction as we did in the proof of theorem 3.1. Consider the case $t = 1$ first and define

$$E_n(\mathbf{x}) = q(A_n - P_{n,1/r}) - (1-(1-q)^{r-1})(A_n - P_{n,1/2})$$

so

$$\frac{1}{q_n}\frac{\partial E_n}{\partial x_n} = q\left(1 - \left(\frac{P_{n,1/r}}{x_n}\right)^{1-1/r}\right) - (1-(1-q)^{r-1})\left(1 - \left(\frac{P_{n,1/2}}{x_n}\right)^{1/2}\right) \tag{3.9}$$

By a change of variables: $\frac{x_i}{x_n} \to x_i, 1 \leq i \leq n$, we may assume $0 < x_1 < x_2 < \cdots < x_n = 1$ in (3.9) and rewrite it as

$$h_n(x_1, \cdots, x_{n-1}) := q(1 - (P_{n,1/r})^{1-1/r}) - (1-(1-q)^{r-1})(1 - P_{n,1/2}^{1/2}) \tag{3.10}$$

We want to show $h_n \geq 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0,1]^{n-1}$ be the absolute minimum of h_n. If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and we can regard h_n as a function of $a_2, \cdots, a_{n-1}$, then we obtain

$$\nabla h_n(a_2, \cdots, a_{n-1}) = 0$$

Otherwise $a_1 > 0$, $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$ and

$$\nabla h_n(a_1, \cdots, a_{n-1}) = 0$$

In either case $a_2, \cdots, a_{n-1}$ solve the equation

$$-q(1-1/r)(P_{n,1/r})^{-1/r}\left(\frac{P_{n,1/r}}{x}\right)^{1-1/r} + \frac{1}{2}(1-(1-q)^{r-1})x^{-1/2} = 0$$

The above equation has at most one root, so we only need to show $h_n \geq 0$ for the case $n = 3$ with $0 = x_1 < x_2 = x < x_3 = 1$ in (3.10). In this case we regard h_3 as a function of x and we get

$$\frac{1}{q_2}h_3'(x) = -q\frac{r-1}{r}(q_2x^{1/r} + q_3)^{r-2}x^{1/r-1} + \frac{1}{2}(1-(1-q)^{r-1})x^{-1/2}$$

Let x be a critical point, then $h_3'(x) = 0$. Similar to the proof of theorem 3.1, there can be at most two roots in $[0, 1]$ for $h_3'(x) = 0$.

Further notice that

$$\lim_{x\to 1^-} \frac{1}{q_2} h_3'(x) = -q\frac{r-1}{r}(1-q_1)^{r-2} + \frac{1-(1-q)^{r-1}}{2} < 0$$

by lemma 2.3 and

$$\lim_{x\to 0^+} \frac{1}{q_2} h_3'(x) = +\infty$$

It then follows that $h_3'(x)$ has exactly one root x_0 in $(0, 1)$ and $h_3'(1) < 0$ implies $h_3(x)$ takes its maximum value at x_0. Thus $h_3(x) \geq min\{h_3(0), h_3(1)\} \geq 0$ where the last inequality follows from lemma 2.2. Thus $D_n \geq 0$ with equality holding if and only if $n = 2, x_1 = 0, q_1 = q$ and a change of variables $x_i \to x_i^t$ completes the proof. □

Notice here for $1 < r < 2$, by setting $t = 1$ and letting $q \to 0$ in (3.8) while noticing $\frac{q}{1-(1-q)^{r-1}}$ is a decreasing function of q , we get

$$\Delta_{1,\frac{1}{r},\frac{1}{2},1} \leq \frac{1}{r-1}$$

a special case of theorem 1.1, which shows in this case theorem 3.2 refines theorem 1.1.

We end the paper by refining a result of the author[5]:

Theorem 3.3. *If $x_1 \neq x_n, n \geq 2$, then for $1 > s \geq 0$*

$$\frac{P_{n,s}^{1-s} - x_1^{1-s}}{2x_1^{1-s}(A_n - x_1)}\sigma_{n,1} - q\frac{(A_n - P_{n,s})^2}{2(A_n - x_1)} > A_n - P_{n,s} > \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}(x_n - A_n)}\sigma_{n,1} + q\frac{(A_n - P_{n,s})^2}{2(x_n - A_n)} \tag{3.11}$$

Proof. We will prove the right-hand inequality and the left-hand side inequality is similar. let

$$F_n(\mathbf{x}) = (x_n - A_n)(A_n - P_{n,s}) - \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}}\sigma_{n,1} - q(A_n - P_{n,s})^2/2$$

We want to show by induction that $F_n \geq 0$. We have

$$\begin{aligned}
\frac{\partial F_n}{\partial x_n} &= (1 - q_n - qq_n(1 - (\frac{P_{n,s}}{x_n})^{1-s}))(A_n - P_{n,s}) - \frac{1-s}{2x_n}(\frac{P_{n,s}}{x_n})^{1-s}(1 - (\frac{x_n}{P_{n,s}})^s q_n)\sigma_{n,1} \\
&\geq (1 - q_n)(\frac{P_{n,s}}{x_n})^{1-s}(A_n - P_{n,s} - \frac{1-s}{2x_n}\sigma_{n,1}) \geq 0
\end{aligned}$$

where the last inequality holds by a theorem of the author[5]. Thus by a similar induction process as the one in the proof of theorem 3.1, we have $F_n \geq 0$. Since not all the x_i's are equal, we get the desired result. □

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