A NOTE ON A PAPER BY G.BENNETT AND G. JAMESON

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ABSTRACT. We note that some recent results of G.Bennett and G. Jameson are consequences of the majorization principle. We also generalize a result of H.Alzer.

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be real finite sequences. Then $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ if for all convex functions f, we have

$$\sum_{j=1}^{n} f(x_j) \leq \sum_{j=1}^{n} f(y_j)$$

We write $\mathbf{x} \leq_{maj} \mathbf{y}$ if this occurs. From now on we denote $\mathbf{x}^*$ to be the decreasing rearrangement of $\mathbf{x}$ and the majorization principle states that if (x_j) and (y_j) are decreasing, then $\mathbf{x} \leq_{maj} \mathbf{y}$ is equivalent to

$$\begin{aligned} x_1 + x_2 + \cdots + x_j &\leq y_1 + y_2 + \cdots + y_j \ (1 \leq j \leq n-1) \\ x_1 + x_2 + \cdots + x_n &= y_1 + y_2 + \cdots + y_n \ (n \geq 0) \end{aligned} \tag{1}$$

We refer the reader to [2, Sect. 1.30]for a simple proof of this.

In a recent paper [3], G.Bennett and G. Jameson considered the average of the values of a function at a sequence of n points equally spaced through an interval and by defining:

$$A_n(f) = \frac{1}{n-1}\sum_{r=1}^{n-1} f(\frac{r}{n}) \ (n \geq 2), \ B_n(f) = \frac{1}{n+1}\sum_{r=0}^{n} f(\frac{r}{n}) \ (n \geq 0)$$

they proved the following

Theorem 1. a. *If f is a convex function on the open interval $(0,1)$, then $A_n(f)$ increases with n. If f is concave, $A_n(f)$ decreases with n.*

b. *If f is convex on $[0,1]$, then $B_n(f)$ decreases with n. If f is concave, $B_n(f)$ increases with n.*

We note first that part **a** of theorem 1 was proved by V.I.Levin and S.B.Stečkin in [6] and as they pointed out there, one can also deduce the same result by applying theorem 130 in the famous book *Inequalities* by G.H.Hardy, J.E. Littlewood and G. Pólya[5].

We point out here theorem 1 is a consequence of the majorization principle, by choosing $\mathbf{x}$ to be an $n(n-1)$-tuple, formed by repeating n times each term of the $(n-1)$-tuple: $(\frac{1}{n}, \cdots, \frac{n-1}{n})$ and $\mathbf{y}$ an $n(n-1)$-tuple, formed by repeating $n-1$ times each term of the n-tuple: $(\frac{1}{n+1}, \cdots, \frac{n}{n+1})$. One checks easily $\mathbf{x}^*, \mathbf{y}^*$ satisfy condition (1) and part **a** of theorem 1 follows if we apply the majorization principle to $\mathbf{x}^*, \mathbf{y}^*$ while noticing $-f$ is convex when f is concave.

Similarly, part **b** of theorem 1 follows if we choose $\mathbf{x}$ to be an $(n+1)(n+2)$-tuple, formed by repeating $n+1$ times each term of the $(n+2)$-tuple: $(\frac{0}{n+1}, \cdots, \frac{n+1}{n+1})$ and $\mathbf{y}$ an $(n+1)(n+2)$-tuple, formed by repeating $n+2$ times each term of the $(n+1)$-tuple: $(\frac{0}{n}, \cdots, \frac{n}{n})$ and apply the majorization principle to $\mathbf{x}^*, \mathbf{y}^*$.

We can also obtain some variants of theorem 1 by applying the majorization principle. For example, by choosing $\mathbf{x}$ to be an $n(n+1)$-tuple, formed by repeating $n+1$ times each term of the

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n-tuple: $(\ln \frac{1}{\sqrt[n]{n!}}, \cdots, \ln \frac{n}{\sqrt[n]{n!}})$ and $\mathbf{y}$ an $n(n+1)$-tuple, formed by repeating n times each term of the $(n+1)$-tuple: $(\ln \frac{1}{\sqrt[n+1]{(n+1)!}}, \cdots, \ln \frac{n+1}{\sqrt[n+1]{(n+1)!}})$. By a result of J. Martins[7], the condition (1) is satisfied by $\mathbf{x}^*, \mathbf{y}^*$. Thus we have

Theorem 2. *If f is a convex function on the real line, then $L_n(f)$ increases with n. If f is concave, $L_n(f)$ decreases with n, where*

$$L_n(f) = \frac{1}{n}\sum_{r=1}^{n} f(\ln \frac{r}{\sqrt[n]{n!}}) \ (n \geq 1)$$

By taking $f(x) = e^{rx}$ in the above theorem we immediately get the following result of H.Alzer[1]:

Corollary 1. *Let $n > 0$ be an integer. Then we have for all real numbers $r > 0$:*

$$[(n+1)\sum_{i=1}^{n} i^r/n\sum_{i=1}^{n+1} i^r]^{1/r} \leq \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \leq [n\sum_{i=1}^{n+1} i^{-r}/(n+1)\sum_{i=1}^{n} i^{-r}]^{1/r}$$

We note here one can also generalize theorem 2 hence corollary 1 by using a recent result of T.H.Chan, P.Gao and F.Qi[4] and we will leave this to the reader.

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