NEW INEQUALITIES OF GRÜSS TYPE FOR THE STIELTJES INTEGRAL AND APPLICATIONS

S.S. DRAGOMIR

Abstract. Sharp bounds of two Čebyšev functionals for the Stieltjes integrals and applications for quadrature rules are given.

1. Introduction

Consider the *weighted Čebyšev functional*

$$T_w(f,g) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \cdot \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) g(t)\,dt \tag{1.1}$$

where $f,g,w:[a,b]\to\mathbb{R}$ and $w(t)\geq 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt>0$.

In [2], the authors obtained, among others, the following inequalities:

$$\begin{aligned} &|T_w(f,g)| \\ &\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right| dt \\ &\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) \times \left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \quad (p>1) \\ &\leq \frac{1}{2}(M-m)\, ess \sup_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right| \end{aligned} \tag{1.2}$$

provided

$$-\infty < m \leq f(t) \leq M < \infty \quad \text{for a.e. } t\in[a,b] \tag{1.3}$$

and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (1.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if

$$-\infty < n \leq g(t) \leq N < \infty \quad \text{for a.e. } t\in[a,b], \tag{1.4}$$

Date: June 18, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.

Key words and phrases. Čebyšev functional, Grüss type inequality, Stieltjes integral.

then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(1.5)\quad & |T_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\right. \\
&\qquad\left.\times\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

In this paper, we extend the above results for Riemann-Stieltjes integrals. A quadrature formula is also considered.

For this purpose, we introduce the following Čebyšev functional for the Stieltjes integral

$$
\begin{aligned}
(1.6)\quad T(f,g;u) &:= \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t) \\
&\quad-\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),
\end{aligned}
$$

where $f,g\in C[a,b]$ (are continuous on $[a,b]$) and $u\in BV[a,b]$ (is of bounded variation on $[a,b]$) with $u(b)\neq u(a)$.

For some recent inequalities for Stieltjes integral see [3]-[6].

2. Some Inequalities by Generalised Čebyšev Functional

The following result holds [9].

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ with $u(a)\neq u(b)$. Assume also that there exists the real constants m,M such that*

$$
(2.1)\qquad m\leq f(t)\leq M \quad \text{for each } t\in[a,b].
$$

If u is of bounded variation on $[a,b]$, then we have the inequality

$$
\begin{aligned}
(2.2)\quad |T(f,g;u)| &\leq \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|} \\
&\quad\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u),
\end{aligned}
$$

where $\bigvee_a^b(u)$ denotes the total variation of u in $[a,b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller constant.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$T(f,g;u)=\frac{1}{u(b)-u(a)}\int_a^b\left[f(t)-\frac{m+M}{2}\right]\times\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \tag{2.3}$$

holds.

Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{2.4}$$

provided $p\in C[a,b]$ and $v\in BV[a,b]$, we have, by (2.3), that

$$\begin{aligned}|T(f,g;u)|&\leq\sup_{t\in[a,b]}\left|\left[f(t)-\frac{m+M}{2}\right]\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]\right|\\&\quad\cdot\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u)\\&\quad\left(\text{since }\left|f(t)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}\text{ for any }t\in[a,b]\right)\\&\leq\frac{M-m}{2}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\cdot\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u)\end{aligned}$$

and the inequality (2.2) is proved.

To prove the sharpness of the constant $\frac{1}{2}$ in the inequality (2.2), we assume that it holds with a constant $C>0$, i.e.,

$$|T(f,g;u)|\leq C(M-m)\frac{1}{|u(b)-u(a)|}\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\bigvee_a^b(u). \tag{2.5}$$

Let us consider the functions $f=g$, $f:[a,b]\to\mathbb{R}$, $f(t)=t$, $t\in[a,b]$ and $u:[a,b]\to\mathbb{R}$ given by

$$u(t)=\begin{cases}-1 & \text{if } t=a,\\ 0 & \text{if } t\in(a,b),\\ 1 & \text{if } t=b.\end{cases} \tag{2.6}$$

Then f,g are continuous on $[a,b]$, u is of bounded variation on $[a,b]$ and

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t)=\frac{b^2+a^2}{2},$$

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)=\frac{b+a}{2},$$

$$\left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right\|_\infty = \sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right| = \frac{b-a}{2}$$

and

$$\bigvee_a^b (u) = 2, \quad M = b, \quad m = a.$$

Inserting these values in (2.5), we get

$$\left| \frac{a^2 + b^2}{2} - \frac{(a+b)^2}{4} \right| \le C(b-a) \cdot \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot 2,$$

giving $C \ge \frac{1}{2}$, and the theorem is thus proved. ∎

The corresponding result for monotonic function u is incorporated in the following theorem [9].

Theorem 2. *Assume that f and g are as in Theorem 1. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then one has the inequality:*

$$\begin{aligned} |T(f,g;u)| \le \frac{1}{2}(M-m) \frac{1}{u(b)-u(a)} \\ \times \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| du(t). \end{aligned} \tag{2.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Using the known inequality

$$\left| \int_a^b p(t)\, dv(t) \right| \le \int_a^b |p(t)|\, dv(t), \tag{2.8}$$

provided $p \in C[a,b]$ and v is a monotonic nondecreasing function on $[a,b]$, we have (by the use of equality (2.3)) that

$$\begin{aligned} |T(f,g;u)| &\le \frac{1}{u(b)-u(a)} \int_a^b \left| f(t) - \frac{m+M}{2} \right| \\ &\quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| du(t) \\ &\le \frac{1}{2}(M-m) \frac{1}{u(b)-u(a)} \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| du(t). \end{aligned}$$

Now, assume that the inequality (2.7) holds with a constant $D > 0$, instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} |T(f,g;u)| \le D(M-m) \frac{1}{u(b)-u(a)} \\ \times \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| du(t). \end{aligned} \tag{2.9}$$

If we choose the same function as in the proof of Theorem 1, we observe that f,g are continuous and u is monotonic nondecreasing on $[a,b]$. Then, for these functions, we have

$$T(f,g;u)=\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}=\frac{(b-a)^2}{4},$$

$$\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t)=\int_a^b\left|t-\frac{a+b}{2}\right|du(t)$$
$$=b-a,$$

and then, by (2.9) we get

$$\frac{(b-a)^2}{4}\leq D(b-a)\frac{1}{2}(b-a)$$

giving $D\geq\frac{1}{2}$, and the theorem is completely proved. ∎

The case when u is a Lipschitzian function is embodied in the following theorem [9].

Theorem 3. *Assume that $f,g:[a,b]\to\mathbb{R}$ are Riemann integrable functions on $[a,b]$ and f satisfies the condition (2.1). If $u:(a,b)\to\mathbb{R}$ $(u(b)\neq u(a))$ is Lipschitzian with the constant L, then we have the inequality*

$$|T(f,g;u)|\leq\frac{1}{2}L(M-m)\frac{1}{|u(b)-u(a)|}\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \tag{2.10}$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. It is well known that if $p:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant L, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b|p(t)|\,dt. \tag{2.11}$$

Using this fact and the identity (2.3), we deduce

$$\begin{aligned}|T(f,g;u)|&\leq\frac{L}{|u(b)-u(a)|}\int_a^b\left|f(t)-\frac{m+M}{2}\right|\\&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt\\&\leq\frac{1}{2}(M-m)\frac{L}{|u(b)-u(a)|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt\end{aligned}$$

and the inequality (2.10) is proved.

Now, assume that (2.10) holds with a constant $E>0$ instead of $\frac{1}{2}$, i.e.,

$$|T(f,g;u)| \leq EL(M-m)\frac{1}{|u(b)-u(a)|} \times \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt. \tag{2.12}$$

Consider the function $f=g$, $f:[a,b]\to\mathbb{R}$ with

$$f(t) = \begin{cases} -1 & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ 1 & \text{if } t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

and $u:[a,b]\to\mathbb{R}$, $u(t)=t$. Then, obviously, f and g are Riemann integrable on $[a,b]$ and u is Lipschitzian with the constant $L=1$.

Since

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) = \frac{1}{b-a}\int_a^b dt = 1,$$

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t) = \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t) = 0,$$

$$\int_a^b \left| g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt = \int_a^b dt = b-a$$

and

$$M=1, \quad m=1$$

then, by (2.12), we deduce $E\geq\frac{1}{2}$, and the theorem is completely proved. ∎

The following result holds [10].

Theorem 4. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is of $r-H-$Hölder type on $[a,b]$, i.e.,*

$$|f(t)-f(s)| \leq H|t-s|^r \quad \text{for any } t,s\in[a,b], \tag{2.13}$$

and g is continuous on $[a,b]$. If $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ with $u(a)\neq u(b)$, then we have the inequality

$$|T(f,g;u)| \leq \frac{H(b-a)^r}{2^r}\cdot\frac{1}{|u(b)-u(a)|} \times \left\| g - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \tag{2.14}$$

where $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$T(f,g;u) = \frac{1}{u(b)-u(a)}\int_a^b \left[f(t)-f\left(\frac{a+b}{2}\right)\right] \times \left[g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right] du(t) \tag{2.15}$$

holds.

Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v) \tag{2.16}$$

provided $p \in C[a,b]$ and $v \in BV[a,b]$, we have, by (2.15), that

$$\begin{aligned}|T(f,g;u)| &\leq \sup_{t\in[a,b]}\left|\left[f(t)-f\left(\frac{a+b}{2}\right)\right]\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]\right| \\ &\quad\times\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u) \\ &\leq \sup_{t\in[a,b]}\left|f(t)-f\left(\frac{a+b}{2}\right)\right|\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \\ &\quad\times\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u) \\ &\leq L\left(\frac{b-a}{2}\right)^r\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \\ &\quad\times\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u),\end{aligned}$$

and the inequality (2.14) is proved. ∎

The following corollary may be useful in applications [10].

Corollary 1. *Let f be Lipschitzian with the constant $L>0$, i.e.,*

$$|f(t)-f(s)| \leq L|t-s| \quad \text{for any} \quad t,s\in[a,b], \tag{2.17}$$

and u,g are as in Theorem 4. Then we have the inequality

$$\begin{aligned}|T(f,g;u)| &\leq \frac{1}{2}\frac{L(b-a)}{|u(b)-u(a)|} \\ &\quad\times\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\bigvee_a^b(u).\end{aligned} \tag{2.18}$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. The inequality (2.18) follows by (2.14) for $r=1$. It remains to prove only the sharpness of the constant $\frac{1}{2}$.

Consider the functions $f=g$, where $f:[a,b]\to\mathbb{R}$, $f(t)=t$ and $u:[a,b]\to\mathbb{R}$, given by

$$u(t)=\begin{cases}-1 & \text{if } t=a, \\ 0 & \text{if } t\in(a,b), \\ 1 & \text{if } t=b.\end{cases} \tag{2.19}$$

Then, f is Lipschitzian with the constant $L=1$, g is continuous and u is of bounded variation.

If we assume that the inequality (2.18) holds with a constant $C>0$, i.e.,

$$|T(f,g;u)| \leq CL(b-a)\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b (u), \tag{2.20}$$

and since

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t)=\frac{b^2+a^2}{2},$$

$$\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)=\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)=\frac{b+a,}{2}$$

$$\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty=\sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|=\frac{b-a}{2}$$

and $\bigvee_a^b(u)=2$, then, by (2.20), we have

$$\left|\frac{b^2+a^2}{2}-\left(\frac{a+b}{2}\right)^2\right|\leq C\frac{(b-a)}{2}\frac{b-a}{2}\cdot 2,$$

giving $C\geq\frac{1}{2}$. ∎

The following result concerning monotonic function $u:[a,b]\to\mathbb{R}$ also holds [10].

Theorem 5. *Assume that f and g are as in Theorem 4. If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ with $u(b)>u(a)$, then we have the inequalities:*

$$\begin{aligned}|T(f,g;u)| &\leq \frac{H}{u(b)-u(a)}\int_a^b\left|t-\frac{a+b}{2}\right|^r \\ &\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t) \\ &\leq \frac{H(b-a)^r}{2^r[u(b)-u(a)]} \\ &\quad\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t).\end{aligned}\tag{2.21}$$

Proof. Using the known inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\int_a^b|p(t)|\,dv(t), \tag{2.22}$$

provided $p \in C[a,b]$ and v is monotonic nondecreasing on $[a,b]$, we have, by (2.15), the following estimate:

$$
\begin{aligned}
|T(f,g;u)| &\leq \frac{1}{u(b)-u(a)}\int_a^b \left|\left(f(t)-f\left(\frac{a+b}{2}\right)\right)\right. \\
&\quad\times\left.\left(g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right)\right| du(t) \\
&\leq \frac{H}{u(b)-u(a)}\int_a^b \left|t-\frac{a+b}{2}\right|^r \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{H}{u(b)-u(a)}\sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|^r \\
&\quad\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t)
\end{aligned}
$$

which simply provides (2.21). ∎

The particular case of Lipschitzian functions that is relevant for applications is embodied in the following corollary [10].

Corollary 2. *Assume that f is $L-$Lipschitzian, g is continuous and u is monotonic nondecreasing on $[a,b]$ with $u(b)>u(a)$. Then we have the inequalities*

$$
\begin{aligned}
|T(f,g;u)| &\leq \frac{L}{u(b)-u(a)}\int_a^b\left|t-\frac{a+b}{2}\right| \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{1}{2}\cdot\frac{L(b-a)}{u(b)-u(a)} \\
&\quad\times\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t).
\end{aligned} \tag{2.23}
$$

The first inequality is sharp. The constant $\frac{1}{2}$ in the second inequality cannot be replaced by a smaller constant.

Proof. The inequality (2.23) follows by (2.21) on choosing $r = 1$. Assume that (2.23) holds with the constants $D, E > 0$, i.e.,

$$\begin{aligned} (2.24) \quad & |T(f,g;u)| \\ & \leq \frac{LD}{u(b)-u(a)} \int_a^b \left| t - \frac{a+b}{2} \right| \\ & \quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| du(t) \\ & \leq \frac{LE(b-a)}{u(b)-u(a)} \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| du(t). \end{aligned}$$

Consider the functions $f = g$, where $f : [a,b] \to \mathbb{R}$, $f(t) = t$ and u is as given by (2.19). Then, obviously, f is Lipschitzian with the constant $L = 1$, g is continuous and u is monotonic nondecreasing on $[a,b]$.

Since, we know, for these functions

$$T(f,g;u) = \frac{(b-a)^2}{4},$$

and

$$\int_a^b \left| t - \frac{a+b}{2} \right| \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| du(t) = \frac{(b-a)^2}{2},$$

$$\int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\,du(s) \right| du(t) = b-a,$$

then by (2.24) we deduce

$$\frac{(b-a)^2}{4} \leq \frac{D}{2} \cdot \frac{(b-a)^2}{2} \leq \frac{E(b-a)^2}{2}$$

giving $D \geq 1$ and $E \geq \frac{1}{2}$. ∎

Another natural possibility to obtain bounds for the functional $T(f,g;u)$, where u is Lipschitzian with the constant $K > 0$, is embodied in the following theorem [10].

Theorem 6. *Assume that $f : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ is Lipschitzian with* *the constant* $K>0$ *and* $u(a)\neq u(b)$, *then one has the inequalities:*

$$\begin{aligned}
(2.25)\qquad & |T(f,g;u)| \\
&\leq \frac{HK}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right|^r \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt \\
&\leq \begin{cases}
\frac{HK(b-a)^{r+1}}{2^r(r+1)|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty; \\[2ex]
\frac{HK(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p \\
\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\frac{HK(b-a)^{r}}{2^r|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_1.
\end{cases}
\end{aligned}$$

Proof. Using the identity (2.15), we have successively

$$\begin{aligned}
(2.26)\qquad |T(f,g;u)| &\leq \frac{K}{|u(b)-u(a)|}\int_a^b \left|f(t)-f\left(\frac{a+b}{2}\right)\right| \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt \\
&\leq \frac{KH}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right|^r \\
&\quad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt
\end{aligned}$$

and the first inequality in (2.25) is proved.

Since

$$\begin{aligned}
&\int_a^b \left|t-\frac{a+b}{2}\right|^r\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt \\
&\leq \left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\int_a^b \left|t-\frac{a+b}{2}\right|^r dt \\
&= \frac{(b-a)^{r+1}}{2^r(r+1)}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty,
\end{aligned}$$

then by (2.26) we deduce the first part in the second inequality in (2.25).

By Hölder's integral inequality we have

$$\int_a^b \left|t-\frac{a+b}{2}\right|^r \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt$$
$$\leq \left(\int_a^b \left|t-\frac{a+b}{2}\right|^{qr} dt\right)^{\frac{1}{q}} \left(\int_a^b \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|^p dt\right)^{\frac{1}{p}}$$
$$=\left[\frac{(b-a)^{qr+1}}{2^{qr}(qr+1)}\right]^{\frac{1}{q}} \left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p$$
$$=\frac{(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}} \left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p .$$

Using (2.26), we deduce the second part of the second inequality in (2.25).

Finally, since

$$\left|t-\frac{a+b}{2}\right|^r \leq \left(\frac{b-a}{2}\right)^r, \quad t\in[a,b],$$

we deduce

$$\int_a^b \left|t-\frac{a+b}{2}\right|^r \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \leq \frac{(b-a)^r}{2^r}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_1$$

and the theorem is completely proved. ∎

The following particular case is useful in applications [10].

Corollary 3. *If f is Lipschitzian with the constant L and g and u are as in Theorem 6, then we have the inequalities:*

$$|T(f,g;u)| \leq \frac{LK}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right| \times \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \tag{2.27}$$
$$\leq \begin{cases} \dfrac{LK(b-a)^2}{4|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty ; \\[2ex] \dfrac{LK(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{LK(b-a)}{2|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_1 . \end{cases}$$

The first inequality in (2.27) is sharp.

The constants $\frac{1}{4}$ and $\frac{1}{2}$ in the second branch of the second inequality cannot be replaced by smaller constants, respectively.

Proof. The inequality (2.27) follows obviously from (2.25) on choosing $r = 1$.

Now, assume that the following inequalities hold

$$\begin{aligned} (2.28)\qquad & |T(f,g;u)| \\ & \le \frac{CLK}{|u(b)-u(a)|}\int_a^b \left|t-\frac{a+b}{2}\right| \\ & \quad \times \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt \\ & \le \begin{cases} \dfrac{DLK(b-a)^2}{|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\displaystyle\int_a^b g(s)\,du(s)\right\|_\infty; \\[2ex] \dfrac{ELK(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}|u(b)-u(a)|}\left\|g-\dfrac{1}{u(b)-u(a)}\displaystyle\int_a^b g(s)\,du(s)\right\|_p \\ \qquad\qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \end{aligned}$$

with $C, D, E > 0$.

Consider the functions $f, g, u : [a,b] \to \mathbb{R}$, defined by $f(t) = t - \frac{a+b}{2}$, $u(t) = t$ and

$$g(t) = \begin{cases} -1 & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ 1 & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then both f and u are Lipschitzian with the constant $L = K = 1$ and g is Riemann integrable on $[a,b]$.

We obviously have

$$\begin{aligned} |T(f,g;u)| &= \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \\ &= \frac{b-a}{4}, \end{aligned}$$

$$\int_a^b \left|t-\frac{a+b}{2}\right| \left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt = \frac{(b-a)^2}{4}$$

$$\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty = \|g\|_\infty = 1$$

and

$$\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_p = \|g\|_p = (b-a)^{\frac{1}{p}}.$$

Consequently, by (2.28), one has

$$\frac{b-a}{4} \le \frac{C}{b-a}\frac{(b-a)^2}{4} \le \begin{cases} \dfrac{D(b-a)^2}{b-a}\cdot 1 \\[2ex] \dfrac{E(b-a)^2}{(q+1)^{\frac{1}{q}}(b-a)} \end{cases}$$

giving

$$\frac{1}{4} \leq \frac{C}{4} \leq \begin{cases} D \\ \\ \dfrac{E}{(q+1)^{\frac{1}{q}}}, & q > 1. \end{cases}$$

From the first inequality we obtain $C \geq 1$. Also, we get $D \geq \frac{1}{4}$ and $E \geq \frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q \to 1+$, we deduce $E \geq \frac{1}{2}$ and the corollary is proved. ∎

3. A Quadrature Formula

Let us consider the partition of the interval $[a,b]$ given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \tag{3.1}$$

Denote $v(I_n) := \max\left\{h_i | i = \overline{0,n-1}\right\}$ where $h_i := x_{i+1} - x_i$, $i = \overline{0,n-1}$.

If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and if we define

$$M_i := \sup_{t \in [x_i, x_{i+1}]} f(t), \quad m_i := \inf_{t \in [x_i, x_{i+1}]} f(t), \text{ and}$$
$$v(f, I_n) = \max_{i=\overline{0,n-1}} (M_i - m_i),$$

then, obviously, by the continuity of f on $[a,b]$, for any $\varepsilon > 0$, we may find a division I_n with norm $v(I_n) < \delta$ such that $v(f, I_n) < \varepsilon$.

Consider now the quadrature rule

$$S_n(f,g;u,I_n) := \sum_{i=0}^{n-1} \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\,du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\,du(t) \tag{3.2}$$

provided $f, g \in C[a,b]$, $u \in BV[a,b]$ and $u(x_{i+1}) \neq u(x_i)$, $i = 0, \dots, n-1$.

We may now state the following result in approximating the Stieltjes integral

$$\int_a^b f(t)\,g(t)\,du(t).$$

Theorem 7. *Let $f, g \in C[a,b]$ and $u \in BV[a,b]$. If I_n is a division of the interval $[a,b]$ and $u(x_{i+1}) \neq u(x_i)$, $i = 0, \dots, n-1$, then we have:*

$$\int_a^b f(t)\,g(t)\,du(t) = S_n(f,g;u,I_n) + R_n(f,g;u,I_n), \tag{3.3}$$

where $S_n(f,g;u,I_n)$ is as defined in (3.2) and the remainder $R_n(f,g;u,I_n)$ satisfies the estimate

$$|R_n(f,g;u,I_n)| \leq \frac{1}{2} v(f, I_n) \times \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\,du(s) \right\|_{[x_i, x_{i+1}],\infty} \bigvee_a^b (u). \tag{3.4}$$

The constant $\frac{1}{2}$ is sharp in (3.4) in the sense that it cannot be replaced by a smaller constant.

Proof. Applying the inequality (2.2) on the intervals $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, we have

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(t)\, g(t)\, du(t) \right. \\ &\qquad \left. - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\, du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\, du(t) \right| \\ &\leq \frac{1}{2}(M_i - m_i) \sup_{t \in [x_i, x_{i+1}]} \left| g(t) - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right| \bigvee_{x_i}^{x_{i+1}} (u). \end{aligned} \tag{3.5}$$

Summing the inequalities (3.5) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned} &|R_n(f, g; u, I_n)| \\ &\leq \frac{1}{2} \sum_{i=0}^{n-1} (M_i - m_i) \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ &\qquad \times \bigvee_{x_i}^{x_{i+1}} (u) \\ &\leq \frac{1}{2} v(f, I_n) \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ &\qquad \times \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (u) \\ &= \frac{1}{2} v(f, I_n) \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\, du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ &\qquad \times \bigvee_a^b (u), \end{aligned} \tag{3.6}$$

and the estimate (3.4) is obtained. ∎

Remark 1. *Similar results may be stated for either u monotonic or Lipschitzian. We omit the details.*

We may now state another result in approximating the Stieltjes integral

$$\int_a^b f(t)\, g(t)\, du(t).$$

Theorem 8. *Let $f, g : [a, b] \to \mathbb{R}$ be such that f is of $r-H-$Hölder type on $[a, b]$ (see Theorem 4), g is continuous on $[a, b]$, I_n is as above and $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ with $u(x_{i+1}) \neq u(x_i)$, $i = 0, \ldots, n-1$. Then we have the representation*

$$\int_a^b f(t)\, g(t)\, du(t) = S_n(f, g; u, I_n) + R_n(f, g; u, I_n), \tag{3.7}$$

where the quadrature $S_n(f,g;u,I_n)$ *is as defined in (3.2) and the remainder* $R_n(f,g;u,I_n)$ *satisfies the estimate*

$$|R_n(f,g;u,I_n)| \le \frac{H}{2^r}[v(I_n)]^r \times \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\,du(s) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_a^b (u), \tag{3.8}$$

where $v(I_n) := \max\left\{h_i \mid i=\overline{0,n-1}\right\}$.

Proof. Applying the inequality (2.14) on the interval $[x_i,x_{i+1}]$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(t)\,g(t)\,du(t) - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} f(t)\,du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right| \le \frac{Hh_i^r}{2^r} \left\| g - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_{x_i}^{x_{i+1}} (u), \tag{3.9}$$

for each $i \in \{0,\dots,n-1\}$.

Summing the inequalities (3.9) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned} |R_n(f,g;u,I_n)| &\le \frac{H}{2^r} \sum_{i=0}^{n-1} h_i^r \left\| g - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_{x_i}^{x_{i+1}} (u) \\ &\le \frac{H}{2^r}[v(f)]^n \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} g(t)\,du(t) \right\|_{[x_i,x_{i+1}],\infty} \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (u) \\ &= \frac{H}{2^r}[v(f)]^n \max_{i=\overline{0,n-1}} \left\| g - \frac{1}{u(x_{i+1})-u(x_i)} \int_{x_i}^{x_{i+1}} g(s)\,du(s) \right\|_{[x_i,x_{i+1}],\infty} \bigvee_a^b (u), \end{aligned} \tag{3.10}$$

and the inequality (3.8) is obtained. ∎

Remark 2. *Similar results may be stated if one uses Theorem 5 and Theorem 6. We omit the details.*

4. Some Particular Cases

For $f,g,w:[a,b]\to\mathbb{R}$, integrable and with the property that $\int_a^b w(t)\,dt \ne 0$, reconsider the weighted Čebyšev functional

$$T_w(f,g) := \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,g(t)\,dt - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,dt \cdot \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,g(t)\,dt. \tag{4.1}$$

1. If $f,g,w:[a,b]\to\mathbb{R}$ are continuous and there exists the real constants m,M such that

$$m\leq f(t)\leq M \text{ for each } t\in[a,b], \tag{4.2}$$

then one has the inequality

$$|T_w(f,g)|\leq\frac{1}{2}(M-m)\frac{1}{\left|\int_a^b w(s)\,ds\right|}\times\left\|g-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],\infty}\int_a^b|w(s)|\,ds. \tag{4.3}$$

The proof follows by Theorem 1 on choosing $u(t)=\int_a^t w(s)\,ds$.

2. If f,g,w are as in **1** and $w(s)\geq 0$ for $s\in[a,b]$, then one has the inequality

$$|T_w(f,g)|\leq\frac{1}{2}(M-m)\frac{1}{\int_a^b w(s)\,ds}\times\int_a^b\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|w(s)\,ds. \tag{4.4}$$

The proof follows by Theorem 2 on choosing $u(t)=\int_a^t w(s)\,ds$.

3. If f,g are Riemann integrable on $[a,b]$ and f satisfies (4.2), and w is continuous on $[a,b]$, then one has the inequality

$$|T_w(f,g)|\leq\frac{1}{2}\|w\|_{[a,b],\infty}(M-m)\frac{1}{\left|\int_a^b w(s)\,ds\right|}\times\int_a^b\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|ds. \tag{4.5}$$

The proof follows by Theorem 5 on choosing $u(t)=\int_a^t w(s)\,ds$.

4. If $f,g,w:[a,b]\to\mathbb{R}$ are continuous and f is of $r-H-$Hölder type (see Theorem 4), then one has the inequality

$$|T_w(f,g)|\leq\frac{H|b-a|^r}{2^r}\cdot\frac{1}{\left|\int_a^b w(s)\,ds\right|}\times\left\|g-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right\|_{[a,b],\infty}\int_a^b|w(s)|\,ds.$$

The proof follows by Theorem 4 on choosing $u(t)=\int_a^t w(s)\,ds$.

5. If f,g,w are as in **4** and $w(s)\geq 0$ for $s\in[a,b]$, then one has the inequality

$$\begin{aligned}|T_w(f,g)| &\leq\frac{H}{\int_a^b w(s)\,ds}\int_a^b\left|t-\frac{a+b}{2}\right|^r\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|w(s)\,ds\\ &\leq\frac{H(b-a)^r}{2^r\int_a^b w(s)\,ds}\int_a^b\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)w(s)\,ds\right|w(s)\,ds.\end{aligned} \tag{4.6}$$

The proof follows by Theorem 5 on choosing $u(t)=\int_a^t w(s)\,ds$.

6. If f is of $r-H-$Hölder type, g are Riemann integrable on $[a,b]$ and w is continuous on $[a,b]$, then one has the inequality

$$
\begin{aligned}
(4.7)\qquad & |T_w(f,g)| \\
&\leq \frac{H\|w\|_{[a,b],\infty}}{\left|\int_a^b w(s)\,ds\right|}\int_a^b \left|t-\frac{a+b}{2}\right|^r\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds\right|dt
\end{aligned}
$$

$$
\leq \begin{cases}
\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r+1}}{2^r(r+1)\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds\right\|_{[a,b],\infty}; \\[3ex]
\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds\right\|_{[a,b],p}, \\
\qquad\qquad\qquad\qquad\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[3ex]
\dfrac{H\|w\|_{[a,b],\infty}(b-a)^{r}}{2^r\left|\int_a^b w(s)\,ds\right|}\left\|g-\dfrac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds\right\|_{[a,b],1}.
\end{cases}
$$

The proof follows by Theorem 6 on choosing $u(t)=\int_a^t w(s)\,ds$.

5. Other Inequalities for Stieltjes Integral

In [11], the authors have considered the following functional

$$D(f;u):=\int_a^b f(x)\,du(x)-[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt,$$

provided that the involved integrals exist.

In the same paper, the following result in estimating the above functional has been obtained.

Theorem 9. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)|\leq L|x-y|\quad \text{for any } x,y\in[a,b]\quad (L>0) \tag{5.1}$$

and f is Riemann integrable on $[a,b]$. If $m,M\in\mathbb{R}$ are such that

$$m\leq f(x)\leq M\quad \text{for any } x,y\in[a,b], \tag{5.2}$$

then we have the inequality

$$|D(f;u)|\leq\frac{1}{2}L(M-m)(b-a). \tag{5.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [12], the following result complementing the above one was obtained.

Theorem 10. *Let $f,u:[a,b]\to\mathbb{R}$ be such that $u:[a,b]\to\mathbb{R}$ is of bounded variation in $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is $K-$Lipschitzian $(K>0)$. Then we have the inequality*

$$|D(f;u)|\leq\frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{5.4}$$

The constant $\frac{1}{2}$ *is sharp in the above sense.*

In this section further similar results will be pointed out.

The following identity is interesting in itself.

Lemma 1. *Let* $f,u:[a,b]\to\mathbb{R}$ *be such that the Stieltjes integral* $\int_a^b f(t)\,du(t)$ *and the Riemann integral* $\int_a^b f(t)\,dt$ *exist. Then we have the identity*

$$\begin{aligned} D(f;u) &= \frac{1}{b-a}\int_a^b \Phi(t)\,df(t) = \frac{1}{b-a}\int_a^b \Gamma(t)\,df(t) \\ &= \frac{1}{b-a}\int_a^b (t-a)(b-t)\,\Delta(t)\,df(t), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \Phi(t) &: = \frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-t} - u(t), \quad t\in[a,b), \\ \Gamma(t) &: = (t-a)\,[u(b)-u(t)] - (b-t)\,[u(t)-u(a)], \quad t\in[a,b], \end{aligned}$$

and

$$\Delta(t) := [u;b,t] - [u;t,a], \quad t\in(a,b),$$

where $[u;\alpha,\beta]$ *is the divided difference, i.e., we recall it*

$$[u;\alpha,\beta] := \frac{u(\alpha)-u(\beta)}{\alpha-\beta}.$$

Proof. We observe that

$$\begin{aligned} \int_a^b \Phi(t)\,df(t) &= \int_a^b \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-t} - u(t)\right] df(t) \\ &= \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-t} - u(t)\right] f(t)\Big|_a^b \\ &\quad - \int_a^b f(t)\,d\left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-t} - u(t)\right] \\ &= [u(b)-u(b)] - [u(a)-u(a)] - \int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}dt - du(t)\right] \\ &= \int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \end{aligned}$$

and the first identity in (5.5) is proved.

The second and third identities are obvious. ∎

Remark 3. *If* u *is an integral, i.e.,* $u(t)=\int_a^t g(s)\,ds$*, then from (5.5) we deduce Cerone's result in* [1]

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \Psi(t)\,df(t), \tag{5.6}$$

where

$$\Psi(t) = \frac{t-a}{b-t}\int_a^b g(s)\,ds - \int_a^t g(s)\,ds \quad (t\in[a,b))$$
$$= (t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds \quad (t\in[a,b])$$
$$= (t-a)(b-t)\left[\frac{\int_t^b g(s)\,ds}{b-t} - \frac{\int_a^t g(s)\,ds}{t-a}\right] \quad (t\in(a,b)).$$

If $w:[a,b]\to\mathbb{R}$ *is integrable and* $\int_a^b w(t)\,dt \neq 0$, *then the choice*

$$u(t) := \frac{\int_a^t w(s)g(s)\,ds}{\int_a^t w(s)\,ds}, \quad t\in[a,b], \tag{5.7}$$

will produce

$$D(f;u) = \frac{\int_a^b w(s)f(s)g(s)\,ds}{\int_a^b w(s)\,ds} - \frac{\int_a^b w(s)g(s)\,ds}{\int_a^b w(s)\,ds}\cdot\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=: E(f,g;w).$$

The following corollary is thus a natural application of the above Lemma 1.

Corollary 4. *If* w,f,g *are Riemann integrable on* $[a,b]$ *and* $\int_a^b w(t)\,dt\neq 0$, *then*

$$E(f,g;w) = \int_a^b \Phi_w(t)\,df(t) = \frac{1}{b-a}\int_a^b \Gamma_w(t)\,df(t) \tag{5.8}$$
$$= \frac{1}{b-a}\int_a^b (t-a)(b-t)\,\Delta_w(t)\,df(t),$$

where

$$\Phi_w(t) = \left(\frac{t-a}{b-t}\right)\cdot\frac{\int_a^b w(s)g(s)\,ds}{\int_a^b w(s)\,ds} - \frac{\int_a^t w(s)g(s)\,ds}{\int_a^b w(s)\,ds},$$
$$\Gamma_w(t) = (t-a)\frac{\int_a^b w(s)g(s)\,ds}{\int_a^b w(s)\,ds} - (b-t)\frac{\int_a^t w(s)g(s)\,ds}{\int_a^b w(s)\,ds},$$
$$\Delta_w(t) = \frac{\int_t^b w(s)g(s)\,ds}{(b-t)\int_a^b w(s)\,ds} - \frac{\int_a^t w(s)g(s)\,ds}{(t-a)\int_a^b w(s)\,ds}.$$

The following general result in bounding the functional $D(f;u)$ may be stated.

Theorem 11. *Let* $f,u:[a,b]\to\mathbb{R}$.

(*i*) *If* f *is of bounded variation and* u *is continuous on* $[a,b]$, *then*

$$|D(f;u)| \leq \begin{cases} \sup\limits_{t\in[a,b)} |\Phi(t)| \bigvee_a^b(f), \\ \\ \frac{1}{b-a}\sup\limits_{t\in[a,b]} |\Gamma(t)| \bigvee_a^b(f), \\ \\ \frac{1}{b-a}\sup\limits_{t\in(a,b)} \left[(t-a)(b-t)|\Delta(t)|\right] \bigvee_a^b(f). \end{cases} \tag{5.9}$$

(ii) *If f is $L-$Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$|D(f;u)| \leq \begin{cases} L\int_a^b |\Phi(t)|\,dt, \\ \frac{L}{b-a}\int_a^b |\Gamma(t)|\,dt, \\ \frac{L}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\,dt. \end{cases} \tag{5.10}$$

(iii) *If f is monotonic nondecreasing on $[a,b]$ and u is continuous on $[a,b]$, then*

$$|D(f;u)| \leq \begin{cases} \int_a^b |\Phi(t)|\,df(t), \\ \frac{1}{b-a}\int_a^b |\Gamma(t)|\,df(t), \\ \frac{1}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\,df(t). \end{cases} \tag{5.11}$$

Proof. Follows by Lemma 1 on taking into account that

$$\left|\int_c^d p(t)\,dv(t)\right| \leq \sup_{t\in[a,b]}|p(t)|\bigvee_c^d(t)$$

if p is continuous on $[c,d]$ and v is of bounded variation,

$$\left|\int_c^d p(t)\,dv(t)\right| \leq L\int_c^d |p(t)|\,dt;$$

if v is $L-$Lipschitzian on $[c,d]$ and p is Riemann integrable on $[c,d]$ and

$$\left|\int_c^d p(t)\,dv(t)\right| \leq \int_c^d |p(t)|\,dt$$

if p is continuous on $[c,d]$ and v is monotonic nondecreasing on $[c,d]$. ∎

It is natural to consider the following corollaries, since they provide simpler bounds for the functional $D(f;u)$ in terms of Δ defined in Lemma 1.

Corollary 5. *If f is of bounded variation and u is continuous on $[a,b]$, then*

$$\begin{aligned} |D(f;u)| &\leq \frac{1}{b-a}\sup_{t\in[a,b]}\left[(t-a)(b-t)\Delta(t)\right]\bigvee_a^b(f) \\ &\leq \frac{b-a}{4}\|\Delta\|_\infty\bigvee_a^b(f). \end{aligned} \tag{5.12}$$

Corollary 6. *If f is $L-$Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$\begin{aligned} (5.13)\quad & |D(f;u)| \\ (5.14)\quad & \leq \frac{L}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\,dt \\ & \leq \begin{cases} \frac{1}{6}L(b-a)^2\|\Delta\|_\infty, \\ L(b-a)^{1+\frac{1}{q}}[B(q+1,q+1)]^{\frac{1}{q}}\|\Delta\|_p,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}L(b-a)\|\Delta\|_1, \end{cases} \end{aligned}$$

and

Corollary 7. *If f is monotonic nondecreasing and g is continuous, then*

$$\begin{aligned} (5.15)\quad & |D(f;u)| \\ & \leq \frac{1}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\,dt \\ & \leq \begin{cases} \frac{1}{4}(b-a)\int_a^b |\Delta(t)|\,df(t), \\ \frac{1}{b-a}\left(\int_a^b [(b-t)(t-a)]^q\,df(t)\right)^{\frac{1}{q}}\left(\int_a^b |\Delta(t)|^p\,df(t)\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{b-a}\|\Delta\|_\infty\int_a^b (t-a)(b-t)\,df(t). \end{cases} \end{aligned}$$

Remark 4. *If one chooses in Corollaries 5 – 7, $u(t)=\int_a^t g(s)\,ds$, then the result incorporated in Theorems 4 – 6 of* [1] *are recaptured.*

Finally, the following result on the positivity of the functional $D(f;u)$ holds.

Theorem 12. *Let f be a monotonic nondecreasing function on $[a,b]$. If u is such that*

$$\Delta(t)=\Delta(u;a,t,b):=[u;b,t]-[u;t,a]\geq 0 \tag{5.16}$$

for any $t\in(a,b)$, then we have the inequality

$$D(f;u)\geq\frac{1}{b-a}\left|\int_a^b (t-a)(b-t)\left[|[u;b,t]|-|[u;t,a]|\right]df(t)\right|\geq 0. \tag{5.17}$$

The proof is similar to the case in Theorem 3 of [2] and we omit the details.

Remark 5. *It is easy to see that a sufficient condition for (5.16) to hold is that $u:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$.*

Remark 6. *Similar results for composite rules in approximating the Stieltjes integral may be stated but we omit the details.*

6. Some Results for Monotonic Integrators

The following result holds.

Theorem 13. *Let $f : [a,b] \to \mathbb{R}$ be $L-$Lipschitzian on $[a,b]$ and u monotonic nondecreasing on $[a,b]$. Then we have he inequality*

$$|D(f;u)| \leq \frac{1}{2} L(b-a)[u(b)-u(a)-K(u)] \leq \frac{1}{2} L(b-a)[u(b)-u(a)], \tag{6.1}$$

where

$$K(u) := \frac{4}{(b-a)^2} \int_a^b u(x)\left(x-\frac{a+b}{2}\right)dx = \frac{4}{(b-a)^2} \int_a^b \left[u(x)-u\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)dx \geq 0. \tag{6.2}$$

The constant $\frac{1}{2}$ in both inequalities is sharp in the sense that it cannot be replaced by a smaller constat.

Proof. As u is monotonic nondecreasing on $[a,b]$, then

$$\left|\int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt\right| = \left|\int_a^b \left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)du(x)\right| \leq \int_a^b \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| du(x). \tag{6.3}$$

Taking into account that f is $L-$Lipschitzian, we have the following Ostrowski type inequality (see for example [7])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq L\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \tag{6.4}$$

for all $x \in [a,b]$, from where we deduce

$$\int_a^b \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| du(x) \leq L(b-a)\int_a^b \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]du(x). \tag{6.5}$$

Now, observe that, by the integration by parts formula for the Stieltjes integral, we have

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 du(x) = u(x)\left(x-\frac{a+b}{2}\right)^2\Big|_a^b - 2\int_a^b u(x)\left(x-\frac{a+b}{2}\right)dx = \frac{(b-a)^2}{4}[u(b)-u(a)] - 2\int_a^b u(x)\left(x-\frac{a+b}{2}\right)dx$$

and then

$$\int_a^b \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] du(x) = \frac{1}{2}\left[u(b)-u(a)\right]-\frac{2}{(b-a)^2}\int_a^b u(x)\left(x-\frac{a+b}{2}\right)dx. \tag{6.6}$$

Using (6.3) – (6.6) we deduce the first part of (6.1).

The second part is obvious by (6.2) which follows by the monotonicity of u on $[a,b]$.

To prove the sharpness of the constant $\frac{1}{2}$, assume that (6.7) holds with the constants $C,D>0$, i.e.,

$$|D(f;u)| \le CL(b-a)\left[u(b)-u(a)-K(u)\right] \le DL(b-a)\left[u(b)-u(a)\right]. \tag{6.7}$$

Consider the functions $f,u:[a,b]\to\mathbb{R}$ given by $f(x)=x-\frac{a+b}{2}$ and

$$u(x)=\begin{cases} 0 & \text{if } x\in[a,b) \\ 1 & \text{if } x=b. \end{cases}$$

Thus f is $L-$Lipschitzian with the constant $L=1$ and u is monotonic nondecreasing.

We observe that

$$D(f;u)=\int_a^b f(x)\,du(x)=f(x)\,u(x)\Big|_a^b-\int_a^b u(x)\,dx=\frac{b-a}{2},$$

$K(u)=0$ and $u(b)-u(a)=1$, giving in (6.7)

$$\frac{b-a}{2}\le C(b-a)\le D(b-a)$$

and thus $C,D\ge\frac{1}{2}$ proving the sharpness of the constant $\frac{1}{2}$ in (6.1). ∎

Another result of this type is the following one.

Theorem 14. *Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ be of bounded variation such that the Stieltjes integral $\int_a^b f(x)\,du(x)$ exists. Then we have the inequality*

$$|D(f;u)| \le \left[u(b)-u(a)-Q(u)\right]\bigvee_a^b(f) \le \left[u(b)-u(a)\right]\bigvee_a^b(f), \tag{6.8}$$

where

$$Q(u):=\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\,dx = \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(x-\frac{a+b}{2}\right)\left[u(x)-u\left(\frac{a+b}{2}\right)\right]dx\ge 0. \tag{6.9}$$

The first inequality in (6.8) is sharp in the sense that the constant $c=1$ cannot be replaced by a smaller constant.

Proof. Since u is monotonic nondecreasing, we have (see (6.3)) that

$$|D(f;u)| \leq \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| du(x). \tag{6.10}$$

Using the following Ostrowski type inequality obtained by the author in [8]

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{6.11}$$

for any $x \in [a,b]$, we have

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| du(x) \leq \bigvee_a^b (f) \int_a^b \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] du(x). \tag{6.12}$$

A simple calculation with the Stieltjes integral gives that

$$\begin{aligned}
&\int_a^b \left| x - \frac{a+b}{2} \right| du(x) \qquad (6.13)\\
&= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right) du(x) + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2} \right) du(x)\\
&= u(x)\left(\frac{a+b}{2} - x \right)\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} u(x)\,dx\\
&\quad + \left(x - \frac{a+b}{2} \right) u(x)\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b u(x)\,dx\\
&= \frac{1}{2}(b-a)[u(b) - u(a)] - \int_a^b \operatorname{sgn}\left(x - \frac{a+b}{2} \right) u(x)\,dx
\end{aligned}$$

and then by (6.10) – (6.13) we deduce the first inequality in (6.8).

The second part of (6.8) follows by (6.9) which holds by the monotonicity property of u.

Now, assume that the first inequality in (6.8) holds with a constant $E > 0$, i.e.,

$$|D(f;u)| \leq E \bigvee_a^b (f) [u(b) - u(a) - Q(u)]. \tag{6.14}$$

Consider the mappings $f, u : [a,b] \to \mathbb{R}$, $f(x) = x - \frac{a+b}{2}$, and

$$u(x) = \begin{cases} 0 & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ 1 & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then we have

$$\begin{aligned}
D(f;u) &= \int_a^b f(x)\,du(x) - \frac{u(b) - u(a)}{b-a}\int_a^b f(t)\,dt\\
&= \int_a^b \left(x - \frac{a+b}{2} \right) du(x) = \left(x - \frac{a+b}{2} \right) u(x)\Big|_a^b - \int_a^b u(x)\,dx\\
&= \frac{b-a}{2}[u(b) + u(a)] = \frac{b-a}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\bigvee_{a}^{b}(f)\left[u(b)-u(a)-Q(u)\right] \\
= &\ (b-a)\left[u(b)-u(a)-\left(\frac{1}{b-a}\int_{a}^{\frac{a+b}{2}}\operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\,dx\right.\right. \\
&\left.\left.+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\,dx\right)\right] \\
= &\ \frac{b-a}{2}.
\end{aligned}$$

Thus, by (6.14) we obtain

$$\frac{b-a}{2}\leq E\cdot\frac{b-a}{2},$$

showing that $E\geq 1$, and the theorem is proved. ∎

Remark 7. *Similar results for composite rules in approximating the Stieltjes integral may be stated, but we omit the details.*

For other inequalities of Grüss type, see [13]-[33].

References

[1] P. CERONE, On an identity for the Chebychev functional and some ramifications, *J. Ineq. Pure & Appl. Math.,* **3**(1) (2002), Article 4.

[2] P. CERONE and S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`].

[3] S.S. DRAGOMIR, On the Ostrowski's inequality for Riemann-Stieltjes integral and applications, *Korean J. Comput. & Appl. Math.*, **7**(3) (2000), 611-627.

[4] S.S. DRAGOMIR, Some inequalities for Riemann-Stieltjes integral and applications, *Optimisation and Related Topics*, Editor: A. Rubinov, Kluwer Academic Publishers, (2000), 197-235.

[5] S.S. DRAGOMIR, Some inequalities of midpoint and trapezoid type for the Riemann-Stieltjes integral, *Nonlinear Analysis*, **47**(4) (2001), 2333-2340

[6] S.S. DRAGOMIR, On the Ostrowski inequality for Riemann-Stieltjes integral where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, **5**(1) (2001), 35-45.

[7] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Computers and Mathematics with Applications*, **38** (1999), 33-37.

[8] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Mathematical Inequalities and Applications*, **4**(1) (2001), 59-66.

[9] S.S. DRAGOMIR, Sharp bounds of Čebyšev functional for Stieltjes integrals and applications, *RGMIA Res. Rep. Coll.,* **5** (2002), Supplement, Article 26. [ONLINE: `http://rgmia.vu.edu.au/v5(E).html`]

[10] S.S. DRAGOMIR, New estimates of the Čebyšev functional for Stieltjes integrals and applications, *RGMIA Res. Rep. Coll.,* **5** (2002), Supplement, Article 27. [ONLINE: `http://rgmia.vu.edu.au/v5(E).html`]

[11] S.S. DRAGOMIR and I. FEDOTOV, An inequalityof Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.

[12] S.S. DRAGOMIR and I. FEDOTOV, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, *Non. Funct. Anal. & Appl.,* **6**(3) (2001), 425-437.

[13] B. G. PACHPATTE, On multidimensional Grüss type inequalities, *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 27, 6 pp.

[14] B. G. PACHPATTE, On Grüss type inequalities for double integrals. *J. Math. Anal. Appl.* **267** (2002), no. 2, 454–459.
[15] B. G. PACHPATTE, On Grüss type integral inequalities, *J. Inequal. Pure Appl. Math.* **3** (2002), no. 1, Article 11, 5 pp.
[16] X. LI, R. N. MOHAPATRA, R. S. RODRIGUEZ, Grüss-type inequalities. *J. Math. Anal. Appl.* **267** (2002), no. 2, 434–443.
[17] P. CERONE, S. S. DRAGOMIR, . Generalisations of the Grüss, Chebychev and Lupas inequalities for integrals over different intervals. *Int. J. Appl. Math.* (2001), no. 2, 117–128.
[18] P. F. RENAUD, A matrix formulation of Grüss inequality. *Linear Algebra Appl.* **335** (2001), 95–100.
[19] A. M. FINK, A treatise on Grüss' inequality. *Analytic and geometric inequalities and applications,* 93–113, Math. Appl., 478, Kluwer Acad. Publ., Dordrecht, 1999.
[20] S.S. DRAGOMIR, A generalization of Grüss's inequality in inner product spaces and applications. *J. Math. Anal. Appl.* **237** (1999), no. 1, 74–82.
[21] A. GNATEK, On some bilinear inequalities in the class of similarly ordered n-tuples. *J. Math. Anal. Appl.* **218** (1998), no. 1, 22–33.
[22] J. E. PEČARIĆ, S. S. DRAGOMIR, J. SANDOR, On some Grüss type inequalities for isotonic functionals. *Rad Hrvatske Akad. Znan. Umjet.* No. **467** (1994), 41–47.
[23] H.P. HEINIG, L. MALIGRANDA, Chebyshev inequality in function spaces. *Real Anal. Exchange* **17** (1991/92), no. 1, 211–247.
[24] S. S. DRAGOMIR, J. E. PEČARIĆ, J. SANDOR, On some Grüss type inequalities. *Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989)*, 155–158, Preprint, 89-6, Univ. "Babes-Bolyai", Cluj-Napoca, 1989.
[25] D. ANDRICA, C. BADEA, Grüss' inequality for positive linear functionals. *Period. Math. Hungar.* **19** (1988), no. 2, 155–167.
[26] J. E. PEČARIĆ, On an inequality of G. Grüss. *Mat. Vesnik* (1983), no. 1, 59–64.
[27] J. E. PEČARIĆ, On some inequalities analogous to Grüss' inequality. *Mat. Vesnik* **4**(17)(32) (1980), no. 2, 197–202.
[28] A. M. OSTROWSKI, On an integral inequality. *Aequationes Math.* **4** 1970 358–373.
[29] S. FEMPL, An inequality of Grüss. (Serbo-Croatian) *Mat. Vesnik* 2 (17) 1965 33–38.
[30] J. KARAMATA, Inégalités relatives aux quotients et à la différence de $\int fg$ et $\int f \cdot \int g$ (French) *Acad. Serbe Sci. Publ. Inst. Math.* **2**, (1948). 131–145.
[31] Y.T. LOU, A generalization of Schweitzer's inequality and Grüss's inequality and their relationship. (Chinese) *Qufu Shifan Daxue Xuebao Ziran Kexue Ban* **17** (1991), no. 4, 24–28.
[32] G.V. MILOVANOVIĆ, I.Z. MILOVANOVIĆ, On a generalization of certain results of A. Ostrowski and A. Lupas. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. **634-677** (1979), 62–69.
[33] A. LUPAS, The best constant in an integral inequality. *Mathematica (Cluj)* **15(38)** (1973), 219–222.

School of Communications and Informatics, Victoria University of Technology, P.O. Box 14428, MCMC 8001,, Victoria, Australia.
E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html