

A REFINEMENT OF JENSEN'S DISCRETE INEQUALITY FOR DIFFERENTIABLE CONVEX FUNCTIONS

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ABSTRACT. A refinement of Jensen's discrete inequality and applications for the celebrated Arithmetic Mean – Geometric Mean – Harmonic Mean inequality and Cauchy-Schwartz-Bunikowski inequality are pointed out.

1. INTRODUCTION

The following inequality is well known in literature as Jensen's inequality:

$$(1.1) \quad f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i),$$

provided $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, $x_i \in [a, b]$, and $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$.

Its central role in Analytic Inequality Theory is determined by the fact that many other fundamental results such as: the Arithmetic Mean – Geometric Mean – Harmonic Mean inequality, or the Hölder and Minkowski inequalities, or even the Ky Fan inequality may be obtained from Jensen's inequality by appropriate choices of the function f .

There is an extensive literature devoted to Jensen's inequality concerning different generalizations, refinements, counterparts and converse results, see, for example [1] – [21].

The main aim of this paper is to point out a new refinement of this classical result. Two applications in connection with the celebrated $A - G - H$ -means inequality and the Cauchy-Bunyakowski-Schwartz inequality are mentioned as well.

2. A REFINEMENT OF JENSEN'S INEQUALITY

The following refinement of Jensen's inequality holds.

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Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable convex function on (a, b) and $x_i \in (a, b)$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$. Then one has the inequality

$$(2.1) \quad \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \\ \geq \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \right. \\ \left. - \left| f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \cdot \frac{1}{P_n} \sum_{i=1}^n p_i \left| x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right| \right| \geq 0.$$

Proof. Since f is differentiable convex on (a, b) , then for each $x, y \in (a, b)$, one has the inequality

$$(2.2) \quad f(x) - f(y) \geq (x - y) f'(y).$$

Using the properties of the modulus, we have

$$(2.3) \quad f(x) - f(y) - (x - y) f'(y) = |f(x) - f(y) - (x - y) f'(y)| \\ \geq ||f(x) - f(y)| - |x - y| |f'(y)||$$

for each $x, y \in (a, b)$.

If we choose $y = \frac{1}{P_n} \sum_{j=1}^n p_j x_j$ and $x = x_i$, $i \in \{1, \dots, n\}$, then we have

$$(2.4) \quad f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) - \left(x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \\ \geq \left\| \left| f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| - \left| x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right| \left| f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \right\|$$

for any $i \in \{1, \dots, n\}$.

If we multiply (2.4) by $p_i \geq 0$, sum over i from 1 to n , and divide by $P_n > 0$, we deduce

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \\ - \frac{1}{P_n} \sum_{i=1}^n p_i \left(x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right)$$

$$\begin{aligned}
 &\geq \frac{1}{P_n} \sum_{i=1}^n p_i \left\| \left| f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| - \left| x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right| \left| f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \right\| \\
 &\geq \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| f(x_i) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \right. \\
 &\quad \left. - \frac{1}{P_n} \sum_{i=1}^n p_i \left| x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right| \cdot \left| f'\left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j\right) \right| \right|.
 \end{aligned}$$

Since

$$\frac{1}{P_n} \sum_{i=1}^n p_i \left(x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right) = 0,$$

the inequality (2.1) is proved. ■

In particular, we have the following result for unweighted means.

Corollary 1. *With the above assumptions for f and x_i , one has the inequality*

$$\begin{aligned}
 (2.5) \quad &\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \\
 &\geq \left| \frac{1}{n} \sum_{i=1}^n \left| x_i - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \right| \right. \\
 &\quad \left. - \left| f'\left(\frac{x_1 + \cdots + x_n}{n}\right) \right| \cdot \frac{1}{n} \sum_{i=1}^n \left| x_i - \frac{1}{n} \sum_{j=1}^n x_j \right| \right| \geq 0.
 \end{aligned}$$

Remark 1. *Similar integral inequalities may be stated as well. We omit the details.*

3. A REFINEMENT OF $A - G - H$ INEQUALITY

For a positive n -tuple $\bar{x} = (x_1, \dots, x_n)$ and $\bar{p} = (p_1, \dots, p_n)$ with $p_i \geq 0$ and $\sum_{i=1}^n p_i =: P_n > 0$, define

$$A_n(\bar{p}, \bar{x}) := \frac{1}{P_n} \sum_{i=1}^n p_i x_i \quad (\text{the weighted arithmetic mean}),$$

$$G_n(\bar{p}, \bar{x}) := \left(\prod_{i=1}^n x_i^{p_i} \right)^{\frac{1}{P_n}} \quad (\text{the weighted geometric mean}),$$

$$H_n(\bar{p}, \bar{x}) := \frac{P_n}{\sum_{i=1}^n \frac{p_i}{x_i}} = \left[A_n\left(\bar{p}, \frac{1}{\bar{x}}\right) \right]^{-1} \quad (\text{the weighted harmonic mean}).$$

The following inequality

$$(3.1) \quad A_n(\bar{p}, \bar{x}) \geq G_n(\bar{p}, \bar{x}) \geq H_n(\bar{p}, \bar{x})$$

is well known in the literature as the Arithmetic Mean – Geometric Mean – Harmonic Mean ($A - G - H$)-means inequality.

Using Theorem 1, we may improve this result as follows.

Proposition 1. *Suppose that \bar{x}, \bar{p} are as above. Then we have the inequality*

$$(3.2) \quad \frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \geq \exp \left[\left| A_n \left(\bar{p}, \left| \ln \left(\frac{\bar{x}}{A_n(\bar{p}, \bar{x})} \right) \right| \right) - A_n \left(\bar{p}, \left| \frac{x - A_n(\bar{p}, \bar{x})}{A_n(\bar{p}, \bar{x})} \right| \right) \right] \geq 1,$$

where for a function h , we denote $h(\bar{x}) := (h(x_1), \dots, h(x_n))$.

Proof. Applying the inequality (2.1) for $f(x) = -\ln x$, we get

$$\begin{aligned} & \ln \left[\frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \right] \\ & \geq \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| \ln \left(\frac{x_i}{A_n(p, x)} \right) \right| - A_n^{-1}(p, x) \cdot \frac{1}{P_n} \sum_{i=1}^n p_i |x_i - A_n(p, x)| \right| \geq 0, \end{aligned}$$

from where we get the desired inequality (3.2). ■

The following proposition also holds.

Proposition 2. *Suppose that \bar{x}, \bar{p} are as above. Then we have the inequality:*

$$(3.3) \quad \frac{G_n(\bar{p}, \bar{x})}{H_n(\bar{p}, \bar{x})} \geq \exp \left[\left| A_n \left(\bar{p}, \left| \ln \left(\frac{H_n(\bar{p}, \bar{x})}{\bar{x}} \right) \right| \right) - A_n \left(\bar{p}, \left| \frac{H_n(\bar{p}, \bar{x}) - \bar{x}}{\bar{x}} \right| \right) \right] \geq 1,$$

Proof. Follows by Proposition 1 on choosing $\frac{1}{\bar{x}}$ instead of \bar{x} . ■

4. A REFINEMENT OF CAUCHY-BUNIAKOWSKI-SCHWARTZ'S INEQUALITY

The following inequality is well known in the literature as the Cauchy-Buniakowski-Schwartz inequality:

$$(4.1) \quad \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left(\sum_{i=1}^n a_i b_i \right)^2,$$

for any $a_i, b_i \in \mathbb{R}$ ($i \in \{1, \dots, n\}$).

The following refinement of (4.1) holds.

Proposition 3. *If $a_i, b_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, then one has the inequality:*

$$(4.2) \quad \begin{aligned} \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 & \geq \frac{1}{\sum_{i=1}^n b_i^2} \left\| \sum_{i=1}^n \begin{pmatrix} a_i^2 & b_i^2 \\ \left(\sum_{j=1}^n a_j b_j \right)^2 & \left(\sum_{j=1}^n b_j^2 \right)^2 \end{pmatrix} \right\| \\ & - 2 \left\| \sum_{k=1}^n a_k b_k \right\| \cdot \sum_{i=1}^n |b_i| \left\| \sum_{j=1}^n b_j \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \right\| \geq 0. \end{aligned}$$

Proof. If we apply Theorem 1 for $f(x) = x^2$, we get

$$(4.3) \quad \frac{1}{P_n} \sum_{i=1}^n p_i x_i^2 - \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right)^2 \geq \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left| x_i^2 - \left(\frac{1}{P_n} \sum_{j=1}^n p_j x_j \right)^2 \right| \right. \\ \left. - 2 \left| \frac{1}{P_n} \sum_{k=1}^n p_k x_k \right| \cdot \frac{1}{P_n} \sum_{i=1}^n p_i \left| x_i - \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right| \right| \geq 0.$$

If in (4.3), we choose $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, $i \in \{1, \dots, n\}$, we get

$$(4.4) \quad \frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n b_i^2} - \frac{(\sum_{i=1}^n a_i b_i)^2}{(\sum_{i=1}^n b_i^2)^2} \geq \left| \frac{1}{\sum_{i=1}^n b_i^2} \sum_{i=1}^n b_i^2 \cdot \left| \frac{a_i^2}{b_i^2} - \left(\frac{\sum_{j=1}^n a_j b_j}{\sum_{j=1}^n b_j^2} \right)^2 \right| \right. \\ \left. - 2 \left| \frac{\sum_{k=1}^n a_k b_k}{\sum_{i=1}^n b_i^2} \right| \cdot \frac{\sum_{i=1}^n b_i^2 \left| \frac{a_i}{b_i} - \frac{\sum_{j=1}^n a_j b_j}{\sum_{j=1}^n b_j^2} \right|}{\sum_{i=1}^n b_i^2} \right|,$$

which is clearly equivalent to (4.2). ■

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