

A LOWER BOUND FOR RATIO OF POWER MEANS

FENG QI, BAI-NI GUO, AND CHAO-PING CHEN

ABSTRACT. Let n and m be natural numbers. Suppose $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by $P_n(r) = (\frac{1}{n} \sum_{i=1}^n a_i^r)^{1/r}$. Then $P_n(r)/P_{n+m}(r) \geq a_n/a_{n+m}$. The lower bound is the best possible.

1. INTRODUCTION

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n} \sum_{i=1}^n i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \quad (1)$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [1], and the right-hand side Martins' inequality [8].

Let $\{a_i\}_{i \in \mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leq a_i^2$ for $i \geq 2$, we call $\{a_i\}_{i \in \mathbb{N}}$ a logarithmically concave sequence.

In [2], Martins' inequality was generalized as follows: Let $\{a_i\}_{i \in \mathbb{N}}$ be an increasing, logarithmically concave, positive, and nonconstant sequence satisfying $(a_{\ell+1}/a_\ell)^\ell \geq (a_\ell/a_{\ell-1})^{\ell-1}$ for any positive integer $\ell > 1$, then

$$\left(\frac{\frac{1}{n} \sum_{i=1}^n a_i^r}{\frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r} \right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \quad (2)$$

where r is a positive number, $n, m \in \mathbb{N}$, and $a_i!$ denotes the sequence factorial $\prod_{i=1}^n a_i$. The upper bound is best possible.

2000 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Inequality, logarithmically convex sequence, ratio, power means, lower bound.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), Doctor Fund of Jiaozuo Institute of Technology, China.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

Recently, in [14], another generalization of Martins' inequality was obtained: Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\{i[\frac{a_{i+1}}{a_i} - 1]\}_{i=1}^{n+m-1}$ is increasing. Then the inequality (2) between ratios of the power means and of the geometric means holds. The upper bound is the best possible.

Alzer's inequality has invoked the interest of several mathematicians including, for examples, P. Cerone [3], Ch.-P. Chen [3], S. S. Dragomir [3], N. Elezović [4], B.-N. Guo [5, 16, 17], J.-Ch. Kuang [6], L. Debnath [15], Zh. Liu [7], Q.-M. Luo [18], N. Ozeki [9], J. Pečarić [4], J. Sándor [19, 20], J. S. Ume [21], the first author [10]–[13] of this paper, and so on.

In [22], a general form of Alzer's inequality was obtained: Let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing positive sequence, and let m be a natural number. If $\{a_i\}_{i=1}^{\infty}$ is logarithmically concave and the sequence $\{(\frac{a_{n+1}}{a_n})^n\}_{i=1}^{\infty}$ is increasing, then

$$\frac{a_n}{a_{n+m}} < \left(\frac{\frac{1}{n} \sum_{i=1}^n a_i^r}{\frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r} \right)^{1/r}. \quad (3)$$

In this short note, utilizing the mathematical induction, we obtain the following

Theorem 1. *Let n and m be natural numbers. Suppose $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by*

$$P_n(r) = \left(\frac{1}{n} \sum_{i=1}^n a_i^r \right)^{\frac{1}{r}}. \quad (4)$$

Then the sequence $\left\{ \frac{P_i(r)}{a_i} \right\}_{i=1}^{n+m}$ is decreasing for any given positive real number r , that is,

$$\frac{P_n(r)}{P_{n+m}(r)} \geq \frac{a_n}{a_{n+m}}. \quad (5)$$

The lower bound in (5) is the best possible.

Considering that the exponential functions a^{x^α} and a^{α^x} for given constants $\alpha \geq 1$ and $a > 1$ is logarithmically convex on $[0, \infty)$, as a corollary of Theorem 1, we have

Corollary 1. *Let $\alpha \geq 1$ and $a > 1$ be two constants. For any given real number r , the following inequalities hold:*

$$\frac{a^{(n+k)^\alpha}}{a^{(n+m+k)^\alpha}} \leq \left(\frac{\frac{1}{n} \sum_{i=k+1}^{n+k} a^{i^\alpha r}}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} a^{i^\alpha r}} \right)^{1/r}, \quad (6)$$

$$\frac{a^{\alpha^{n+k}}}{a^{\alpha^{n+m+k}}} \leq \left(\frac{\frac{1}{n} \sum_{i=k+1}^{n+k} a^{\alpha^i r}}{\frac{1}{n+m} \sum_{i=k+1}^{n+m+k} a^{\alpha^i r}} \right)^{1/r}, \quad (7)$$

where n and m are natural numbers, and k is a nonnegative integer. The lower bounds above are the best possible.

2. PROOF OF THEOREM 1

The inequality (5) is equivalent to

$$\frac{\frac{1}{n} \sum_{i=1}^n a_i^r}{\frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r} \geq \frac{a_n^r}{a_{n+m}^r}, \quad (8)$$

that is,

$$\frac{1}{(n+m)a_{n+m}^r} \sum_{i=1}^{n+m} a_i^r \leq \frac{1}{na_n^r} \sum_{i=1}^n a_i^r. \quad (9)$$

This is also equivalent to

$$\frac{1}{(n+1)a_{n+1}^r} \sum_{i=1}^{n+1} a_i^r \leq \frac{1}{na_n^r} \sum_{i=1}^n a_i^r. \quad (10)$$

Since

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^n a_i^r + a_{n+1}^r, \quad (11)$$

inequality (10) reduces to

$$\sum_{i=1}^n a_i^r \geq \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}. \quad (12)$$

It is easy to see that inequality (12) holds for $n = 1$.

Assume that inequality (12) holds for some $n > 1$. Using the principle of mathematical induction, considering equality (11) and the inductive hypothesis, it is easy to show that the induction for inequality (12) on $n + 1$ can be written as

$$\frac{(n+2)a_{n+2}^r - (n+1)a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r} \geq \left(\frac{a_{n+2}}{a_{n+1}} \right)^r, \quad (13)$$

which can be rearranged as

$$k \left[\left(\frac{a_{n+1}}{a_{n+2}} \right)^r - \left(\frac{a_n}{a_{n+1}} \right)^r \right] + \left(\frac{a_{n+1}}{a_{n+2}} \right)^r \leq 1. \quad (14)$$

Since the sequence $\{a_i\}_{i=1}^{n+m}$ is increasing, we have $\frac{a_{n+1}}{a_{n+2}} \leq 1$ and $\left(\frac{a_{n+1}}{a_{n+2}} \right)^r \leq 1$. From the logarithmical convexity of the sequence $\{a_i\}_{i=1}^{n+m}$, it follows that $\frac{a_{n+1}}{a_{n+2}} \leq \frac{a_n}{a_{n+1}}$ and $\left(\frac{a_{n+1}}{a_{n+2}} \right)^r - \left(\frac{a_n}{a_{n+1}} \right)^r \leq 0$. Therefore, inequality (14) is valid. Thus, the inequality (5) holds.

It can easily be shown by L'Hospital rule that

$$\lim_{r \rightarrow \infty} \frac{P_n(r)}{P_{n+m}(r)} = \frac{a_n}{a_{n+m}}. \quad (15)$$

Hence, the lower bound in (5) is the best possible. The proof is complete.

REFERENCES

- [1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.
- [2] T. H. Chan, P. Gao and F. Qi, *On a generalization of Martins' inequality*, Monatsh. Math. (2002), accepted. RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 12, 93–101. Available online at <http://rgmia.vu.edu.au/v4n1.html>.
- [3] Ch.-P. Chen, F. Qi, P. Cerone, and S. S. Dragomir, *Monotonicity of sequences involving convex and concave functions*, RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 1. Available online at <http://rgmia.vu.edu.au/v5n1.html>.
- [4] N. Elezović and J. Pečarić, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366–369.
- [5] B.-N. Guo and F. Qi, *An algebraic inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 8, 55–61. Available online at <http://rgmia.vu.edu.au/v4n1.html>.
- [6] J.-Ch. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83** (1999), 123–127.
- [7] Zh. Liu, *New generalization of H. Alzer's inequality*, Tamkang J. Math. **34** (2003), accepted.
- [8] J. S. Martins, *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. **139** (1988), 281–288.
- [9] N. Ozeki, *On some inequalities*, J. College Arts Sci. Chiba Univ. **4** (1965), no. 3, 211–214. (Japanese)
- [10] F. Qi, *An algebraic inequality*, J. Inequal. Pure and Appl. Math. **2** (2001), no. 1, Art. 13. Available online at http://jipam.vu.edu.au/v2n1/006_00.html. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Art. 8, 81–83. Available online at <http://rgmia.vu.edu.au/v2n1.html>.
- [11] F. Qi, *Generalizations of Alzer's and Kuang's inequality*, Tamkang J. Math. **31** (2000), no. 3, 223–227. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 12, 891–895. Available online at <http://rgmia.vu.edu.au/v2n6.html>.
- [12] F. Qi, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), 294–297.
- [13] F. Qi, *Inequalities and monotonicity of sequences involving $\sqrt[n]{(n+k)!/k!}$* , RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 8, 685–692. Available online at <http://rgmia.vu.edu.au/v2n5.html>.
- [14] F. Qi, *On a new generalization of Martins' inequality*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 13. Available online at <http://rgmia.vu.edu.au/v5n3.html>.
- [15] F. Qi and L. Debnath, *On a new generalization of Alzer's inequality*, Internat. J. Math. Math. Sci. **23** (2000), no. 12, 815–818.

- [16] F. Qi and B.-N. Guo, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese J. Math. **7** (2003), no. 2, in press.
- [17] F. Qi and B.-N. Guo, *Monotonicity of sequences involving convex function and sequence*, RGMIA Res. Rep. Coll. **3** (2000), no. 2, Art. 14, 321–329. Available online at <http://rgmia.vu.edu.au/v3n2.html>.
- [18] F. Qi and Q.-M. Luo, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145–148. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 14, 909–912. Available online at <http://rgmia.vu.edu.au/v2n6.html>.
- [19] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.
- [20] J. Sándor, *Comments on an inequality for the sum of powers of positive numbers*, RGMIA Res. Rep. Coll. **2** (1999), no. 2, 259–261. Available online at <http://rgmia.vu.edu.au/v2n2.html>.
- [21] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.
- [22] Z. Xu and D. Xu, *A general form of Alzer's inequality*, Comput. Math. Appl. **44** (2002), no. 3-4, 365–373.

(Qi, Guo, and Chen) DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, #142, MID-JIEFANG ROAD, JIAOZUO CITY, HENAN 454000, CHINA
E-mail address, Qi: qifeng@jz.it.edu.cn
URL, Qi: <http://rgmia.vu.edu.au/qi.html>
E-mail address, Guo: guobaini@jz.it.edu.cn