ON A MEAN VALUE ON INTERVAL $[a, b]$ IN THE CONTEXT OF COMPLEMENTARY AND RECIPROCAL MEANS

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Abstract. **In introduction - statement 1^0, besides "parallelogram of the means" from article [1], we also state the means (5) from the article [2]. In the statement 2^0, we give the features of the means (5) on the graph of the function $y = M_x(a,b)$ together with calculating their corresponding indexes on the graph. In the statement 3^0, the mean $K[\overline{M}(a,b)]$ is considered, as well as its reciprocal mean.**

1^0. In the paper [1] by figure 2 there are the graphs of the functions given

$$(1) \qquad y = M_x(a,b) \qquad \text{and} \qquad y_1 = R[M_x(a,b)] = M_{1-x}(a,b),$$

where the graph of the reciprocal function $R[M_x(a,b)]$ is axis symmetrical to the graph of the function $M_x(a,b)$ in relation to the line $x = \frac{1}{2}\big(R[M_{\frac{1}{2}+x}(a,b)] = M_{\frac{1}{2}-x}(a,b)\big)$ and to the middle points marked on them:

$$(2) \qquad H\Big(0, \frac{2ab}{a+b}\Big),\ G\Big(\frac{1}{2}, \sqrt{ab}\Big),\ A\Big(1, \frac{a+b}{2}\Big),\ L\Big(\frac{3}{2}, \frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\Big),\ M\Big(2, \frac{a^2+b^2}{a+b}\Big),$$

where the quadrable $HGML$ is a parallelogram. Namely, from the complementarity:

$$(3) \qquad M(a,b) + H(a,b) = G(a,b) + L(a,b) = a + b,$$

follows

$$(4) \qquad G(a,b) - H(a,b) = M(a,b) - L(a,b) \ \text{ and } \ L(a,b) - H(a,b) = M(a,b) - G(a,b).$$

Next, in the paper [2] the mean is introduced, which is marked here as $\overline{M}(a,b)$:

$$(5) \qquad \overline{M}(a,b) = \frac{4ab}{a^2 + 6ab + b^2} \quad \text{and} \quad R[\overline{M}(a,b)] = \frac{A(a,b) + H(a,b)}{2}.$$

2^0. At this point we give the graphic construction of the means $\overline{M}(a,b)$ and $R[\overline{M}(a,b)]$ on the graph of the function $y = M_x(a,b)$, as well as the verification of the relations:

$$(6) \qquad G(a,b) < R[\overline{M}(a,b)] < A(a,b) \qquad \text{and} \qquad H(a,b) < \overline{M}(a,b) < G(a,b);$$

and the calculation of their corresponding indexes on the graph of the function.

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According to the downward convexity of the graph of the function $y = M_x(a,b)$ on the interval $[0,1]$ is true (figure):

$$(7) \qquad R[\overline{M}(a,b)] = \frac{A(a,b) + H(a,b)}{2} > G(a,b) \qquad \text{and} \qquad R[\overline{M}(a,b)] < A(a,b),$$

i.e.

$$(8) \qquad R[\overline{M}(a,b)] \in \big(G(a,b), A(a,b)\big) - \text{ the point } \overline{A}.$$

Next, taking into account that $y = M_x(a,b)$ strictly increases

$$(9) \qquad R[\overline{M}(a,b)] \in M_x(a,b)$$

for the unique $x = x_0 \in \big(\frac{1}{2}, 1\big)$ - the point B on the graph of the function $y = M_x(a,b)$ was created by the intersection of the line from point $\overline{A}$, being parallel to x-axis and the mentioned graph.

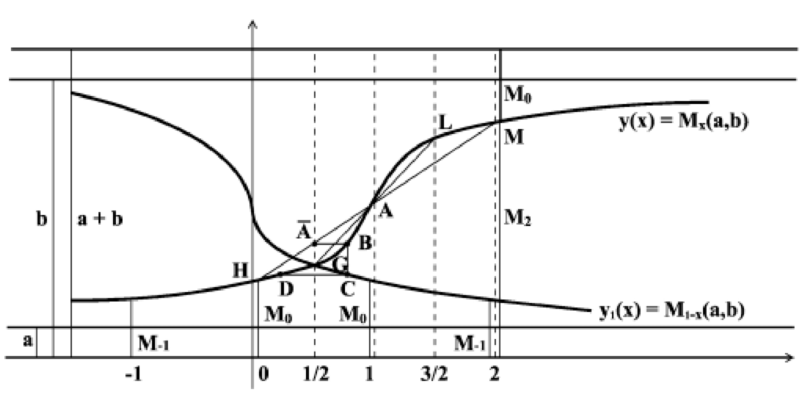
Next, the perpendicular from the point B to x-axis in the intersection with the graph of the function $R[M_x(a,b)] = M_{1-x}(a,b)$, brings us to the point C on the graph of function $y_1 = M_{1-x}(a,b)$. For $x_0 = \frac{1}{2} + t_0$, $t_0 \in \big(0, \frac{1}{2}\big)$, we obtain that [1]:

$$(10) \qquad R[M_{\frac{1}{2}+t_0}(a,b)] = M_{\frac{1}{2}-t_0} = M_{\overline{x}}(a,b),$$

where $\overline{x} = \frac{1}{2} - t_0 \in \big(0, \frac{1}{2}\big)$. Thus, the perpendicular from the point C onto y-axis in the intersection with the graph of the function $y = M_x(a,b)$, brings us to the value $\overline{M}(a,b)$ where

$$(11) \qquad \overline{M} = M_{\overline{x}}(a,b),\ \overline{x} \in \big(0, \frac{1}{2}\big) \quad \text{and} \quad H(a,b) < M_{\overline{x}}(a,b) < G(a,b)$$

- point D on the graph of function $y = M_x(a,b)$ (figure).



Figure

We are going to determine the numerical value for $\overline{x}$. The function

$$y=\frac{a^x+b^x}{a^{x-1}+b^{x-1}} \tag{12}$$

is an increasing one, so it has the inverse function which is also increasing. From (12) it follows that

$$\left(\frac{a}{b}\right)^{1-x}=\frac{b-y}{y-a}\,. \tag{13}$$

In our case

$$y=\frac{4ab(a+b)}{a^2+6ab+b^2}\,, \tag{14}$$

according to the relation (13), we obtain that

$$\overline{x}=\frac{\log(a+3b)-\log(b+3a)}{\log b-\log a}\,, \tag{15}$$

where $\overline{x}\in\left(0,\frac{1}{2}\right)$. Namely, the following inequality is true:

$$0<\frac{\log(a+3b)-\log(b+3a)}{\log b-\log a}<\frac{1}{2}\,,\qquad b>a>0. \tag{16}$$

Index x_0 is determined by the relation: $x_0=1-\overline{x}$.

3⁰. To estimate the complementary mean value for the mean $\overline{M}(a,b)$, and the reciprocal mean value for the mean $K[\overline{M}(a,b)]$ there is the following calculation:

$$\begin{aligned}K[\overline{M}(a,b)]=K[H(A,H)]&=K\left[\frac{2AH}{A+H}\right]=2A-\frac{2AH}{A+H}=2A\left(1-\frac{H}{A+H}\right)\\&=\frac{A}{H}\cdot\frac{2AH}{A+H}=\frac{A}{H}\cdot H(A,H)=\frac{A}{H}\cdot\overline{M}(a,b),\end{aligned}$$

and

$$R[K[\overline{M}(a,b)]]=R\left[\frac{A}{H}\cdot\overline{M}(a,b)\right]=\frac{H}{A}\cdot R[\overline{M}(a,b)]=\frac{H}{A}\cdot A(A,H)=\frac{H}{A}\cdot\overline{A}(a,b),$$

where $\overline{A}(a,b)=A(A,H)$. Finally

$$K[\overline{M}(a,b)]=\frac{A}{H}\cdot\overline{M}(a,b)\quad\text{and}\quad R[K[\overline{M}(a,b)]]=\frac{H}{A}\cdot\overline{A}(a,b). \tag{17}$$

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