SOME BOAS-BELLMAN TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract. Some inequalities in 2-inner product spaces generalizing Bessel's result that are similar to the Boas-Bellman inequality from inner product spaces, are given. Applications for determinantal integral inequalities are also provided.

1. Introduction

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space H, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [9, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\leq\|x\|^2,$$

for any $x\in H$.

For other results related to Bessel's inequality, see [5] – [7] and Chapter XV in the book [9].

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [9, p. 392]).

Theorem 1. *If $x,y_1,\dots,y_n$ are elements of an inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right],$$

holds.

It is the main aim of the present paper to point out the corresponding version of Boas-Bellman inequality in 2-inner product spaces. Some natural generalizations and related results are also pointed out. Applications for determinantal integral inequalities are provided.

For a comprehensive list of fundamental results on 2-inner product spaces and linear 2-normed spaces, see the recent books [3] and [8] where further references are given.

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2. Bessel's Inequality in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows [4]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{2.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{2.2}$$

In the real case, (2.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{2.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (2.2), yields

$$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]+\frac{i}{4}[(z,z|x+iy)-(z,z|x-iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha\in\mathbb{C}$,

$$(x,y|\alpha z)=|\alpha|^2(x,y|z). \tag{2.6}$$

However, for $\alpha\in\mathbb{R}$, (2.6) reduces to (2.4).

Also, from (2.6) it follows that

$$(x,y|0)=0.$$

(4) For any three given vectors $x,y,z\in X$, consider the vector $u=(y,y|z)x-(x,y|z)y$. By $(2I_1)$, we know that $(u,u|z)\geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u,u|z)\geq 0$ can be rewritten as

$$(y,y|z)[(x,x|z)(y,y|z)-|(x,y|z)|^2]\geq 0. \tag{2.7}$$

For $x=z$, (2.7) becomes

$$-(y,y|z)|(z,y|z)|^2\geq 0,$$

which implies that

$$(z,y|z)=(y,z|z)=0 \tag{2.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y,z\in X$. Now, if y and z are linearly independent, then $(y,y|z)>0$ and, from (2.7), it follows that

$$|(x,y|z)|^2\leq (x,x|z)(y,y|z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x,y,z\in X$ and is strict unless the vectors $u=(y,y|z)x-(x,y|z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x,y and z are linearly dependent.

In any given 2-inner product space $(X,(\cdot,\cdot|\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X\times X$ by

$$\|x|z\|=\sqrt{(x,x|z)}, \tag{2.10}$$

for all $x,z\in X$.

It is easy to see that this function satisfies the following conditions:

$(2\mathrm{N}_1)$ $\|x|z\|\geq 0$ and $\|x|z\|=0$ if and only if x and z are linearly dependent,

$(2\mathrm{N}_2)$ $\|z|x\|=\|x|z\|$,

$(2\mathrm{N}_3)$ $\|\alpha x|z\|=|\alpha|\|x|z\|$ for any scalar $\alpha\in\mathbb{K}$,

$(2\mathrm{N}_4)$ $\|x+x'|z\|\leq\|x|z\|+\|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X\times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2-*norm* on X and $(X,\|\cdot|\cdot\|)$ is called a *linear* 2-*normed space* [8].

Whenever a 2-inner product space $(X, (\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot|\cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot,\cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\le i\le n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X, (e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1,\dots,n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1\le i\le n}$ is $z-$orthonormal), then the following inequality is the corresponding Bessel's inequality (see for example [4]) for $z-$orthonormal family $(e_i)_{1\le i\le n}$ in the 2-inner product space $(X; (\cdot,\cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \le \|x|z\|^2 , \tag{2.11}$$

for any $x \in X$. For more details on this inequality, see the recent paper [4] and the references therein.

3. Some Inequalities for 2-Norms

We start with the following lemma which is also interesting in itself.

Lemma 1. *Let $z_1,\dots,z_n, z \in X$ and $\mu_1,\dots,\mu_n \in \mathbb{K}$. Then one has the inequality:*

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} \mu_i z_i|z\right\|^2 \\
&\le \begin{cases}
\max\limits_{1\le i\le n} |\mu_i|^2 \sum\limits_{i=1}^{n} \|z_i|z\|^2 ; \\[2ex]
\left(\sum\limits_{i=1}^{n} |\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i|z\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\sum\limits_{i=1}^{n} |\mu_i|^2 \max\limits_{1\le i\le n} \|z_i|z\|^2 ,
\end{cases} \\
&\quad + \begin{cases}
\max\limits_{1\le i\ne j\le n} \{|\mu_i\mu_j|\} \sum\limits_{1\le i\ne j\le n} |(z_i, z_j|z)| ; \\[2ex]
\left[\left(\sum\limits_{i=1}^{n} |\mu_i|^{\gamma}\right)^2 - \left(\sum\limits_{i=1}^{n} |\mu_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\le i\ne j\le n} |(z_i, z_j|z)|^{\delta}\right)^{\frac{1}{\delta}}, \\
\qquad \text{where } \gamma>1, \quad \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\left[\left(\sum\limits_{i=1}^{n} |\mu_i|\right)^2 - \sum\limits_{i=1}^{n} |\mu_i|^2\right] \max\limits_{1\le i\ne j\le n} |(z_i, z_j|z)| .
\end{cases}
\end{aligned} \tag{3.1}$$

Proof. We observe that

$$
\begin{aligned}
(3.2)\qquad \left\| \sum_{i=1}^{n} \mu_i z_i | z \right\|^2 &= \left(\sum_{i=1}^{n} \mu_i z_i, \sum_{j=1}^{n} \mu_j z_j | z \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \overline{\mu_j} \left(z_i, z_j | z \right) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \overline{\mu_j} \left(z_i, z_j | z \right) \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \mu_i \right| \left| \overline{\mu_j} \right| \left| \left(z_i, z_j | z \right) \right| \\
&= \sum_{i=1}^{n} \left| \mu_i \right|^2 \left\| z_i | z \right\|^2 + \sum_{1 \leq i \neq j \leq n} \left| \mu_i \right| \left| \mu_j \right| \left| \left(z_i, z_j | z \right) \right| .
\end{aligned}
$$

Using Hölder's inequality, we may write that

$$
(3.3)\qquad \sum_{i=1}^{n} \left| \mu_i \right|^2 \left\| z_i | z \right\|^2
\leq \begin{cases}
\max\limits_{1 \leq i \leq n} \left| \mu_i \right|^2 \sum\limits_{i=1}^{n} \left\| z_i | z \right\|^2 ; \\[2ex]
\left(\sum\limits_{i=1}^{n} \left| \mu_i \right|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \left\| z_i | z \right\|^{2\beta} \right)^{\frac{1}{\beta}}, \quad \text{where } \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex]
\sum\limits_{i=1}^{n} \left| \mu_i \right|^2 \max\limits_{1 \leq i \leq n} \left\| z_i | z \right\|^2 .
\end{cases}
$$

By Hölder's inequality for double sums, we also have

$$
(3.4)\qquad \sum_{1 \leq i \neq j \leq n} \left| \mu_i \right| \left| \mu_j \right| \left| \left(z_i, z_j | z \right) \right|
$$

$$\leq \begin{cases} \max\limits_{1\leq i\neq j\leq n} \left|\mu_i\mu_j\right| \sum\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|; \\ \left(\sum\limits_{1\leq i\neq j\leq n} \left|\mu_i\right|^\gamma \left|\mu_j\right|^\gamma\right)^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|^\delta\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \sum\limits_{1\leq i\neq j\leq n} \left|\mu_i\right|\left|\mu_j\right| \max\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|, \end{cases}$$

$$= \begin{cases} \max\limits_{1\leq i\neq j\leq n} \left\{\left|\mu_i\mu_j\right|\right\} \sum\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|; \\ \left[\left(\sum\limits_{i=1}^n \left|\mu_i\right|^\gamma\right)^2 - \left(\sum\limits_{i=1}^n \left|\mu_i\right|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|^\delta\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left[\left(\sum\limits_{i=1}^n \left|\mu_i\right|\right)^2 - \sum\limits_{i=1}^n \left|\mu_i\right|^2\right] \max\limits_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|. \end{cases}$$

Utilizing (3.3) and (3.4) in (3.2), we may deduce the desired result (3.1). ∎

Remark 1. *Inequality (3.1) contains in fact 9 different inequalities which may be obtained combining the first 3 ones with the last 3 ones.*

A particular result is embodied in the following inequality.

Corollary 1. *With the assumptions in Lemma 1, we have*

$$\left\|\sum_{i=1}^n \mu_i z_i | z\right\|^2 \tag{3.5}$$

$$\leq \sum_{i=1}^n \left|\mu_i\right|^2 \left\{\max_{1\leq i\leq n} \left\|z_i|z\right\|^2 + \frac{\left[\left(\sum_{i=1}^n \left|\mu_i\right|^2\right)^2 - \sum_{i=1}^n \left|\mu_i\right|^4\right]^{\frac{1}{2}}}{\sum_{i=1}^n \left|\mu_i\right|^2} \left(\sum_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|^2\right)^{\frac{1}{2}}\right\}$$

$$\leq \sum_{i=1}^n \left|\mu_i\right|^2 \left\{\max_{1\leq i\leq n} \left\|z_i|z\right\|^2 + \left(\sum_{1\leq i\neq j\leq n} \left|(z_i, z_j|z)\right|^2\right)^{\frac{1}{2}}\right\}.$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma = \delta = 2$.

The second inequality in (3.5) follows by the fact that

$$\left[\left(\sum_{i=1}^n \left|\mu_i\right|^2\right)^2 - \sum_{i=1}^n \left|\mu_i\right|^4\right]^{\frac{1}{2}} \leq \sum_{i=1}^n \left|\mu_i\right|^2.$$

Applying the following Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{n} a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq n, \tag{3.6}$$

we may write that

$$\left(\sum_{i=1}^{n} |\mu_i|^{\gamma}\right)^2 - \sum_{i=1}^{n} |\mu_i|^{2\gamma} \leq (n-1) \sum_{i=1}^{n} |\mu_i|^{2\gamma} \qquad (n \geq 1) \tag{3.7}$$

and

$$\left(\sum_{i=1}^{n} |\mu_i|\right)^2 - \sum_{i=1}^{n} |\mu_i|^2 \leq (n-1) \sum_{i=1}^{n} |\mu_i|^2 \qquad (n \geq 1). \tag{3.8}$$

Also, it is obvious that:

$$\max_{1 \leq i \neq j \leq n} \left\{ |\mu_i \mu_j| \right\} \leq \max_{1 \leq i \leq n} |\mu_i|^2. \tag{3.9}$$

Consequently, we may state the following coarser upper bounds for $\|\sum_{i=1}^{n} \mu_i z_i|z\|^2$ that may be useful in applications.

Corollary 2. *With the assumptions in Lemma 1, we have the inequalities:*

$$\left\|\sum_{i=1}^{n} \mu_i z_i|z\right\|^2 \tag{3.10}$$

$$\leq \begin{cases} \max\limits_{1 \leq i \leq n} |\mu_i|^2 \sum\limits_{i=1}^{n} \|z_i|z\|^2; \\[2ex] \left(\sum\limits_{i=1}^{n} |\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum\limits_{i=1}^{n} \|z_i|z\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \sum\limits_{i=1}^{n} |\mu_i|^2 \max\limits_{1 \leq i \leq n} \|z_i|z\|^2, \end{cases}$$

$$+ \begin{cases} \max\limits_{1 \leq i \leq n} |\mu_i|^2 \sum\limits_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|; \\[2ex] (n-1)^{\frac{1}{\gamma}} \left(\sum\limits_{i=1}^{n} |\mu_i|^{2\gamma}\right)^{\frac{1}{\gamma}} \left(\sum\limits_{1 \leq i \neq j \leq n} |z\left(_i, z_j|z\right)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\[2ex] (n-1) \sum\limits_{i=1}^{n} |\mu_i|^2 \max\limits_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|. \end{cases}$$

The proof is obvious by Lemma 1 on applying the inequalities (3.7) – (3.9).

Remark 2. *The following inequalities which are incorporated in (3.10) are of special interest:*

$$\left\|\sum_{i=1}^{n} \mu_i z_i|z\right\|^2 \leq \max_{1 \leq i \leq n} |\mu_i|^2 \left[\sum_{i=1}^{n} \|z_i|z\|^2 + \sum_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|\right]; \tag{3.11}$$

$$\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2 \tag{3.12}$$
$$\leq\left(\sum_{i=1}^{n}|\mu_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\|z_i|z\|^{2q}\right)^{\frac{1}{q}}+(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j|z)|^{q}\right)^{\frac{1}{q}}\right],$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$; *and*

$$\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2\leq\sum_{i=1}^{n}|\mu_i|^2\left[\max_{1\leq i\leq n}\|z_i|z\|^2+(n-1)\max_{1\leq i\neq j\leq n}|(z_i,z_j|z)|\right]. \tag{3.13}$$

4. Some Inequalities for Fourier Coefficients

The following results holds

Theorem 2. *Let* $x,y_1,\dots,y_n,z$ *be vectors of a 2-inner product space* $(X;(\cdot,\cdot|\cdot))$ *and* $c_1,\dots,c_n\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$. *Then one has the inequalities:*

$$\left|\sum_{i=1}^{n}c_i\,(x,y_i|z)\right|^2 \tag{4.1}$$
$$\leq\|x|z\|^2\times\begin{cases}\max\limits_{1\leq i\leq n}|c_i|^2\sum\limits_{i=1}^{n}\|y_i|z\|^2;\\[2ex] \left(\sum\limits_{i=1}^{n}|c_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\|y_i|z\|^{2\beta}\right)^{\frac{1}{\beta}}, \quad \text{where } \ \alpha>1,\frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \sum\limits_{i=1}^{n}|c_i|^2\max\limits_{1\leq i\leq n}\|y_i|z\|^2;\end{cases}$$
$$+\|x|z\|^2\times\begin{cases}\max\limits_{1\leq i\neq j\leq n}\{|c_ic_j|\}\sum\limits_{1\leq i\neq j\leq n}|(y_i,y_j|z)|;\\[2ex] \left[\left(\sum\limits_{i=1}^{n}|c_i|^{\gamma}\right)^2-\left(\sum\limits_{i=1}^{n}|c_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}}\left(\sum\limits_{1\leq i\neq j\leq n}|(y_i,y_j|z)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \text{where } \ \gamma>1, \quad \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \left[\left(\sum\limits_{i=1}^{n}|c_i|\right)^2-\sum\limits_{i=1}^{n}|c_i|^2\right]\max\limits_{1\leq i\neq j\leq n}|(y_i,y_j|z)|.\end{cases}$$

Proof. We note that

$$\sum_{i=1}^{n}c_i\,(x,y_i|z)=\left(x,\sum_{i=1}^{n}\overline{c_i}y_i|z\right).$$

Using Schwarz's inequality in 2-inner product spaces, we have

$$\left|\sum_{i=1}^{n}c_i\,(x,y_i|z)\right|^2\leq\|x|z\|^2\left\|\sum_{i=1}^{n}\overline{c_i}y_i|z\right\|^2.$$

Now using Lemma 1 with $\mu_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (4.1). ∎

The following particular inequalities that may be obtained by the Corollaries 1, 2, and Remark 2, hold.

Corollary 3. *With the assumptions in Theorem 2, one has the inequalities:*

$$(4.2)\qquad \left|\sum_{i=1}^{n} c_i \left(x, y_i|z\right)\right|^2$$
$$\leq \|x|z\|^2 \times \begin{cases} \sum\limits_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}; \\ \max\limits_{1\leq i\leq n} |c_i|^2 \left\{ \sum\limits_{i=1}^{n} \|y_i|z\|^2 + \sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)| \right\}; \\ \left(\sum\limits_{i=1}^{n} |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum\limits_{i=1}^{n} \|y_i|z\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^q \right)^{\frac{1}{q}} \right\}, \\ \text{where } \ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1\leq i\leq n} \|y_i|z\|^2 + (n-1) \max\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)| \right\}. \end{cases}$$

5. Some Boas-Bellman Type Inequalities in 2-Inner Product Spaces

If one chooses $c_i = \overline{(x, y_i|z)}$ $(i = 1, \dots, n)$ in (4.1), then it is possible to obtain 9 different inequalities between the Fourier coefficients $(x, y_i|z)$ and the 2-norms and 2-inner products of the vectors y_i $(i = 1, \dots, n)$. We restrict ourselves only to those inequalities that may be obtained from (4.2).

From the first inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2 \right)^2 \leq \|x|z\|^2 \sum_{i=1}^{n} |(x, y_i|z)|^2 \left\{ \max_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\},$$

which is clearly equivalent to the following *Boas-Bellman type inequality* for 2-inner products:

$$(5.1)\qquad \sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\|^2 \left\{ \max_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}.$$

From the second inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \le \|x|z\|^2 \max_{1\le i\le n} |(x, y_i|z)|^2 \left\{\sum_{i=1}^{n} \|y_i|z\|^2 + \sum_{1\le i\ne j\le n} |(y_i, y_j|z)|\right\}.$$

Taking the square root in this inequality, we obtain

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\| \max_{1\le i\le n} |(x, y_i|z)| \left\{\sum_{i=1}^{n} \|y_i|z\|^2 + \sum_{1\le i\ne j\le n} |(y_i, y_j|z)|\right\}^{\frac{1}{2}}, \tag{5.2}$$

for any $x, y_1, \dots, y_n, z$ vectors in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$.

If we assume that $(e_i)_{1\le i\le n}$ is an orthonormal family in X with respect with the vector z, i.e., $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$, then by (5.1) we deduce Bessel's inequality (2.11), while from (5.2) we have

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \le \sqrt{n}\, \|x|z\| \max_{1\le i\le n} |(x, e_i|z)|, \quad x \in X. \tag{5.3}$$

From the third inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we deduce

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \le \|x|z\|^2 \left(\sum_{i=1}^{n} |(x, y_i|z)|^{2p}\right)^{\frac{1}{p}} \times \left\{\left(\sum_{i=1}^{n} \|y_i|z\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\le i\ne j\le n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}}\right\},$$

for $p > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$. Taking the square root in this inequality, we get

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\| \left(\sum_{i=1}^{n} |(x, y_i|z)|^{2p}\right)^{\frac{1}{2p}} \times \left\{\left(\sum_{i=1}^{n} \|y_i|z\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\le i\ne j\le n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}}, \tag{5.4}$$

for any $x, y_1, \dots, y_n, z \in X$, and $p > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (5.4) becomes, for an orthornormal family $(e_i)_{1\le i\le n}$ with respect of the vector z,

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \le n^{\frac{1}{q}}\, \|x|z\| \left(\sum_{i=1}^{n} |(x, e_i|z)|^{2p}\right)^{\frac{1}{2p}}, \quad x \in X. \tag{5.5}$$

Finally, the choice $c_i = \overline{(x, y_i|z)}$ $(i = 1, \ldots, n)$ will produce in the last inequality in (4.2)

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \leq \|x|z\|^2 \sum_{i=1}^{n} |(x, y_i|z)|^2 \left\{\max_{1\leq i\leq n} \|y_i|z\|^2 + (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j|z)|\right\},$$

which gives the following inequality

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\|^2 \left\{\max_{1\leq i\leq n} \|y_i|z\|^2 + (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j|z)|\right\}, \tag{5.6}$$

for any $x, y_1, \ldots, y_n, z \in X$.

It is obvious that (5.6) will give for $z-$orthonormal families, the Bessel inequality mentioned in (2.11) from Introduction.

Remark 3. *Observe that, both Boas-Bellman type inequality for 2-inner products incorporated in (5.1) and the inequality (5.6) become in the particular case of $z-$orthonormal families, the regular Bessel's inequality. Consequently, a comparison of the upper bounds is necessary.*

It suffices to consider the quantities

$$A_n := \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}}$$

and

$$B_n := (n-1) \max_{1\leq i\neq j\leq n} |(y_i, y_j|z)|,$$

where $n \geq 1$, and $y_1, \ldots, y_n, z \in X$.

If we choose $n = 3$, we have

$$A_3 = \sqrt{2}\left((y_1, y_2|z)^2 + (y_2, y_3|z)^2 + (y_{3,} y_1|z)^2\right)^{1/2}$$

and

$$B_3 = 2 \max\{|(y_1, y_2|z)|, |(y_2, y_3|z)|, |(y_{3,} y_1|z)|\}$$

where $y_1, y_2, y_3, z \in X$.

If we consider $a := |(y_1, y_2|z)| \geq 0, b := |(y_2, y_3|z)| \geq 0$ and $c := |(y_{3,} y_1|z)| \geq 0$, then we have to compare

$$A := \sqrt{2}\left(a^2 + b^2 + c^2\right)^{1/2}$$

with

$$B_3 = 2 \max\{a, b, c\}.$$

If we assume that $b = c = 1$, then $A := \sqrt{2}\left(a^2 + 2\right)^{1/2}$, $B_3 = 2\max\{a, 1\}$. Finally, for $a = 1$, we get $A = \sqrt{6}, B = 2$ showing that $A > B$, while for $a = 2$ we have $A = \sqrt{12}, B = 4$ showing that $B > A$.

In conclusion, we may state that the bounds

$$M_1 := \|x|z\|^2 \left\{\max_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}}\right\}$$

and

$$M_2 := \|x|z\|^2 \left\{ \max_{1\le i\le n} \|y_i|z\|^2 + (n-1) \max_{1\le i\ne j\le n} |(y_i, y_j|z)| \right\}$$

for the Bessel's sum $\sum_{i=1}^{n} |(x, y_i|z)|^2$, *cannot be compared in general, meaning that some time one is better than the other.*

6. Applications for Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s)|f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s)\, d\mu(t), \tag{6.1}$$

where by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)\, d\mu(t) \right)^{1/2}. \tag{6.2}$$

A simple calculation with integrals reveals that

$$(f, g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{6.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2}, \tag{6.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

Using the representations (6.3), (6.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities, as follows.

Proposition 1. *Let* $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, *where* $\rho : \Omega \to [0, \infty)$ *is a measurable function on* Ω. *Then we have the inequality*

$$\sum_{i=1}^{n} \left| \begin{array}{cc} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|^2$$
$$\leq \left| \begin{array}{cc} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \times \left\{ \max_{1\leq i\leq n} \left| \begin{array}{cc} \int_\Omega \rho g_i^2 d\mu & \int_\Omega \rho g_i h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \right.$$
$$\left. + \left(\sum_{1\leq i\neq j\leq n}^{n} \left| \begin{array}{cc} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|^2 \right)^{1/2} \right\}.$$

The proof follows by the inequality (5.1) applied for the 2-inner product and 2-norm defined in (6.1) and (6.2), and utilizing the identities (6.3) and (6.4).

If one uses the inequality (5.6), then that one may state the following result as well

Proposition 2. *Let* $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, *where* $\rho : \Omega \to [0, \infty)$ *is a measurable function on* Ω. *Then we have the inequality*

$$\sum_{i=1}^{n} \left| \begin{array}{cc} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|^2$$
$$\leq \left| \begin{array}{cc} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \times \left\{ \max_{1\leq i\leq n} \left| \begin{array}{cc} \int_\Omega \rho g_i^2 d\mu & \int_\Omega \rho g_i h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \right.$$
$$\left. + (n-1) \max_{1\leq i\neq j\leq n} \left| \begin{array}{cc} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right| \right\}.$$

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References

[1] R. BELLMAN, Almost orthogonal series, *Bull. Amer. Math. Soc.,* **50** (1944), 517–519.

[2] R.P. BOAS, A general moment problem, *Amer. J. Math.,* **63** (1941), 361–370.

[3] Y.J. CHO, P.C.S. LIN, S.S. KIM and A. MISIAK, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2001

[4] Y.J. CHO, M. MATIĆ and J.E. PEČARIĆ, On Gram's determinant in 2-inner product spaces, *J. Korean Math. Soc.,* **38**(2001), No. 6, pp. 1125-1156.

[5] S.S. DRAGOMIR and J. SÁNDOR, On Bessel's and Gram's inequality in prehilbertian spaces, *Periodica Math. Hung.,* **29**(3) (1994), 197–205.

[6] S.S. DRAGOMIR and B. MOND, On the Boas-Bellman generalisation of Bessel's inequality in inner product spaces, *Italian J. of Pure & Appl. Math.,* **3** (1998), 29–35.

[7] S.S. DRAGOMIR, B. MOND and J.E. PEČARIĆ, Some remarks on Bessel's inequality in inner product spaces, *Studia Univ. Babeş-Bolyai, Mathematica,* **37**(4) (1992), 77–86.

[8] R.W. FREESE and Y.J. CHO, *Geometry of Linear 2-Normed Spaces*, Nova Science Publishers, Inc., New York, 2001.

[9] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Clasical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

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