# SOME BOMBIERI, SELBERG AND HEILBRONN TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract. Some results related to the Bombieri type generalisation of Bessel's inequality in 2-inner product spaces are given. The corresponding versions for Selberg and Heilbronn inequalities for 2-inner products and applications for determinantal integral inequalities are also pointed out.

## 1. Introduction

Let $(X;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space $X$, i.e., $(f_i,f_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where $\delta_{ij}$ is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [10, p. 391]):

$$\sum_{i=1}^{n}|(x,f_i)|^2\leq\|x\|^2, \tag{1.1}$$

for any $x\in X$.

For other results related to Bessel's inequality, see [6] – [8] and Chapter XV in the book [10].

In 1971, E. Bombieri [3] (see also [10, p. 394]) gave the following generalisation of Bessel's inequality.

**Theorem 1.** *If $x,y_1,\dots,y_n$ are vectors in the inner product space $(X;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}, \tag{1.2}$$

*holds.*

It is obvious that if $(y_i)_{1\leq i\leq n}$ are supposed to be orthonormal, then from (1.2) one would deduce Bessel's inequality (1.1).

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [10, p. 394]):

**Theorem 2.** *Let $x,y_1,\dots,y_n$ be vectors in $X$ with $y_i\neq0$ $(i=1,\dots,n)$, then*

$$\sum_{i=1}^{n}\frac{|(x,y_i)|^2}{\sum_{j=1}^{n}|(y_i,y_j)|}\leq\|x\|^2. \tag{1.3}$$

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In this case, also, if $(y_i)_{1\leq i\leq n}$ are orthonormal, then one may deduce Bessel's inequality.

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [9] (see also [10, p. 395]).

**Theorem 3.** *With the assumptions of Theorem 1,*

$$\sum_{i=1}^{n}|(x,y_i)|\leq\|x\|\left(\sum_{i,j=1}^{n}|(y_i,y_j)|\right)^{\frac{1}{2}}. \tag{1.4}$$

If in (1.4) one chooses $y_i=f_i$ $(i=1,\ldots,n)$, where $(f_i)_{1\leq i\leq n}$ are orthonormal vectors in $X$, then

$$\sum_{i=1}^{n}|(x,f_i)|\leq\sqrt{n}\,\|x\|, \quad \text{for any} \quad x\in X. \tag{1.5}$$

In 1992 J.E. Pečarić [12] (see also [10, p. 394]) proved the following general inequality in inner product spaces.

**Theorem 4.** *Let $x,y_1,\ldots,y_n\in X$ and $c_1,\ldots,c_n\in\mathbb{K}$. Then*

$$\begin{aligned}\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2 &\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)\\ &\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}.\end{aligned} \tag{1.6}$$

He showed that the Bombieri inequality (1.2) may be obtained from (1.6) for the choice $c_i=\overline{(x,y_i)}$ (using the second inequality), the Selberg inequality (1.3) may be obtained from the first part of (1.6) for the choice

$$c_i=\frac{\overline{(x,y_i)}}{\sum_{j=1}^{n}|(y_i,y_j)|}, i\in\{1,\ldots,n\};$$

while the Heilbronn inequality (1.4) may be obtained from the first part of (1.6) if one chooses $c_i=\frac{\overline{(x,y_i)}}{|(x,y_i)|}$, for any $i\in\{1,\ldots,n\}$.

For other results connected with the above, see [7] and [8].

It is the main aim of the present paper to point out the corresponding versions of Bombieri, Selberg and Heilbronn inequalities in 2-inner product spaces. Some natural generalizations and related results are also given. Applications for determinantal integral inequalities are provided.

For a comprehensive list of fundamental results on 2-inner product spaces and linear 2-normed spaces, see the recent books [4] and [11] where further references are given.

## 2. Bessel's Inequality in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in [4]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let $X$ be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if $x$ and $z$ are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on $X$ and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product spaces can be immediately obtained as follows [5]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{2.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{2.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (2.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{2.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (2.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha \in \mathbb{C}$,

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{2.6}$$

However, for $\alpha \in \mathbb{R}$, (2.6) reduces to (2.4).

Also, from (2.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if $u$ and $z$ are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as,

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{2.7}$$

For $x = z$, (2.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{2.8}$$

provided $y$ and $z$ are linearly independent. Obviously, when $y$ and $z$ are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y, z \in X$. Now, if $y$ and $z$ are linearly independent, then $(y, y|z) > 0$ and, from (2.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when $y$ and $z$ are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and $z$ are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors $x, y$ and $z$ are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot\,|\,\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{2.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

$(2\mathrm{N}_1)$ $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if $x$ and $z$ are linearly dependent,

$(2\mathrm{N}_2)$ $\|z|x\| = \|x|z\|$,

$(2\mathrm{N}_3)$ $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

$(2\mathrm{N}_4)$ $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2-*norm* on $X$ and $(X, \|\cdot|\cdot\|)$ is called a *linear* 2-*normed space* [11]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot|\cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space $X$, and, for a given $z \in X, (f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ where $\delta_{ij}$ is the Kronecker delta (we say that the family $(f_i)_{1 \leq i \leq n}$ is $z-$orthonormal), then the following inequality is the corresponding Bessel's inequality (see for example [5]) for the $z-$orthonormal family $(f_i)_{1\leq i\leq n}$ in the 2-inner product space $(X;(\cdot,\cdot|\cdot))$:

$$\sum_{i=1}^{n}|(x,f_i|z)|^2\leq\|x|z\|^2\,, \tag{2.11}$$

for any $x\in X$. For more details on this inequality, see the recent paper [5] and the references therein.

## 3. Some Inequalities for 2-Norms

We start with the following lemma which is also interesting in itself.

**Lemma 1.** *Let $(X,(\cdot,\cdot|\cdot))$ be a 2-inner product space over $\mathbb{K}$ and $z_1,\dots,z_n,z\in X$, $a_1,\dots,a_n\in\mathbb{K}$, then*

$$\left\|\sum_{i=1}^{n}a_iz_i|z\right\|^2 \tag{3.1}$$

$$\leq\begin{cases}\max\limits_{1\leq k\leq n}|a_k|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|;\\[2mm]
\max\limits_{1\leq k\leq n}|a_k|\left(\sum\limits_{i=1}^{n}|a_i|^r\right)^{\frac1r}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^s\right)^{\frac1s}, & r>1,\ \frac1r+\frac1s=1;\\[2mm]
\max\limits_{1\leq k\leq n}|a_k|\sum\limits_{k=1}^{n}|a_k|\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right);\\[2mm]
\left(\sum\limits_{k=1}^{n}|a_k|^p\right)^{\frac1p}\max\limits_{1\leq i\leq n}|a_i|\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^q\right)^{\frac1q}, & p>1,\ \frac1p+\frac1q=1;\\[2mm]
\left(\sum\limits_{k=1}^{n}|a_k|^p\right)^{\frac1p}\left(\sum\limits_{i=1}^{n}|a_i|^t\right)^{\frac1t}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|^q\right)^{\frac uq}\right]^{\frac1u}, & p>1,\ \frac1p+\frac1q=1;\\
 & t>1,\ \frac1t+\frac1u=1;\\[2mm]
\left(\sum\limits_{k=1}^{n}|a_k|^p\right)^{\frac1p}\sum\limits_{i=1}^{n}|a_i|\max\limits_{1\leq i\leq n}\left\{\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|^q\right)^{\frac1q}\right\}, & p>1,\ \frac1p+\frac1q=1;\\[2mm]
\sum\limits_{k=1}^{n}|a_k|\max\limits_{1\leq i\leq n}|a_i|\sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n}|(z_i,z_j|z)|\right];\\[2mm]
\sum\limits_{k=1}^{n}|a_k|\left(\sum\limits_{i=1}^{n}|a_i|^m\right)^{\frac1m}\left(\sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n}|(z_i,z_j|z)|\right]^l\right)^{\frac1l}, & m>1,\ \frac1m+\frac1l=1;\\[2mm]
\left(\sum\limits_{k=1}^{n}|a_k|\right)^2\max\limits_{i,1\leq j\leq n}|(z_i,z_j|z)|.\end{cases}$$

*Proof.* We observe that

$$\left\|\sum_{i=1}^{n} a_i z_i | z\right\|^2 = \left(\sum_{i=1}^{n} a_i z_i, \sum_{j=1}^{n} a_j z_j | z\right) \tag{3.2}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a_j}\left(z_i, z_j | z\right) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{a_j}\left(z_i, z_j | z\right)\right|$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_i|\,|a_j|\,|(z_i, z_j|z)| = \sum_{i=1}^{n} |a_i| \left(\sum_{j=1}^{n} |a_j|\,|(z_i, z_j|z)|\right)$$
$$:= M.$$

Using Hölder's inequality, we may write,

$$\sum_{j=1}^{n} |a_j|\,|(z_i, z_j|z)| \leq \begin{cases} \max\limits_{1\leq k\leq n} |a_k| \sum\limits_{j=1}^{n} |(z_i, z_j|z)| \\ \left(\sum\limits_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n} |a_k| \max\limits_{1\leq j\leq n} |(z_i, z_j|z)| \end{cases} \tag{3.3}$$

for any $i \in \{1, \ldots, n\}$, giving

$$M \leq \begin{cases} \max\limits_{1\leq k\leq n} |a_k| \sum\limits_{i=1}^{n} |a_i| \sum\limits_{j=1}^{n} |(z_i, z_j|z)| =: M_1; \\ \left(\sum\limits_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |a_i| \left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|^q\right)^{\frac{1}{q}} := M_p, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n} |a_k| \sum\limits_{i=1}^{n} |a_i| \max\limits_{1\leq j\leq n} |(z_i, z_j|z)| =: M_\infty. \end{cases} \tag{3.4}$$

By Hölder's inequality we also have:

$$\sum_{i=1}^{n} |a_i| \left(\sum_{j=1}^{n} |(z_i, z_j|z)|\right) \tag{3.5}$$
$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |a_i| \sum\limits_{i,j=1}^{n} |(z_i, z_j|z)|; \\ \left(\sum\limits_{i=1}^{n} |a_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum\limits_{i=1}^{n} |a_i| \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right); \end{cases}$$

and thus

$$M_1 \le \begin{cases} \max\limits_{1\le k\le n} |a_k|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j|z)| ; \\[2ex] \max\limits_{1\le k\le n} |a_k| \left(\sum\limits_{i=1}^{n} |a_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\[2ex] \max\limits_{1\le k\le n} |a_k| \sum\limits_{i=1}^{n} |a_i| \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|\right); \end{cases}$$

and the first 3 inequalities in (3.1) are obtained.

By Hölder's inequality we also have:

$$M_p \le \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\le i\le n} |a_i| \sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|^q\right)^{\frac{1}{q}} ; \\[2ex] \left(\sum\limits_{i=1}^{n} |a_i|^t\right)^{\frac{1}{t}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|^q\right)^{\frac{u}{q}}\right)^{\frac{1}{u}}, \quad t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\[2ex] \sum\limits_{i=1}^{n} |a_i| \max\limits_{1\le i\le n}\left\{\left(\sum\limits_{j=1}^{n} |(z_i, z_j|z)|^q\right)^{\frac{1}{q}}\right\}; \end{cases}$$

and the next 3 inequalities in (3.1) are proved.

Finally, by the same Hölder inequality we may state that:

$$M_\infty \le \sum_{k=1}^{n} |a_k| \times \begin{cases} \max\limits_{1\le i\le n} |a_i| \sum\limits_{i=1}^{n}\left(\max\limits_{1\le j\le n} |(z_i, z_j|z)|\right); \\[2ex] \left(\sum\limits_{i=1}^{n} |a_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n}\left(\max\limits_{1\le j\le n} |(z_i, z_j|z)|\right)^l\right)^{\frac{1}{l}}, \quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\[2ex] \sum\limits_{i=1}^{n} |a_i| \max\limits_{1\le i,j\le n} |(z_i, z_j|z)| ; \end{cases}$$

and the last 3 inequalities in (3.1) are proved. ∎

To obtain some bounds for $\left\|\sum_{i=1}^{n} a_i z_i|z\right\|^2$ in terms of $\sum_{i=1}^{n} |a_i|^2$, then the following corollaries may be used.

**Corollary 1.** *Let $z_1, \ldots, z_n, z$ and $a_1, \ldots, a_n$ be as in Lemma 1. If $1 < p \leq 2$, $1 < t \leq 2$, then one has the inequality*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{n} |a_k|^2 \left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |(z_i, z_j|z)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}} \tag{3.6}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

*Proof.* By the monotonicity of power means, we may write,

$$\left( \frac{\sum_{k=1}^{n} |a_k|^p}{n} \right)^{\frac{1}{p}} \leq \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}}; \quad 1 < p \leq 2,$$

$$\left( \frac{\sum_{k=1}^{n} |a_k|^t}{n} \right)^{\frac{1}{t}} \leq \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}}; \quad 1 < t \leq 2,$$

from which we get

$$\left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}},$$

$$\left( \sum_{k=1}^{n} |a_k|^t \right)^{\frac{1}{t}} \leq n^{\frac{1}{t}-\frac{1}{2}} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}}.$$

Using the fifth inequality in (3.1), we deduce (3.6). ∎

**Remark 1.** *An interesting particular case is the one for $p = q = t = u = 2$, giving*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq \sum_{k=1}^{n} |a_k|^2 \left( \sum_{i,j=1}^{n} |(z_i, z_j|z)|^2 \right)^{\frac{1}{2}}. \tag{3.7}$$

**Corollary 2.** *With the assumptions of Lemma 1 and if $1 < p \leq 2$, then*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq n^{\frac{1}{p}} \sum_{k=1}^{n} |a_k|^2 \max_{1 \leq i \leq n} \left[ \left( \sum_{j=1}^{n} |(z_i, z_j|z)|^q \right)^{\frac{1}{q}} \right], \tag{3.8}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* Since

$$\left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{n} |a_k| \leq n^{\frac{1}{2}} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}},$$

then by the sixth inequality in (3.1) we deduce (3.8). ∎

In a similar fashion, one may prove the following two corollaries.

**Corollary 3.** *With the assumptions of Lemma 1 and if* $1 < m \leq 2$, *then*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq n^{\frac{1}{m}} \sum_{k=1}^{n} |a_k|^2 \left( \sum_{i=1}^{n} \left[ \max_{1 \leq j \leq n} |(z_i, z_j | z)| \right]^l \right)^{\frac{1}{l}}, \tag{3.9}$$

*where* $\frac{1}{m} + \frac{1}{l} = 1$.

**Corollary 4.** *With the assumptions of Lemma 1, we have:*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq n \sum_{k=1}^{n} |a_k|^2 \max_{1 \leq i,j \leq n} |(z_i, z_j | z)| . \tag{3.10}$$

The following lemma may be of interest as well.

**Lemma 2.** *With the assumptions of Lemma 1, one has the inequalities,*

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{j=1}^{n} |(z_i, z_j | z)| \tag{3.11}$$

$$\leq \begin{cases} \sum_{i=1}^{n} |a_i|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^{n} |(z_i, z_j | z)| \right]; \\ \left( \sum_{i=1}^{n} |a_i|^{2p} \right)^{\frac{1}{p}} \left( \left( \sum_{j=1}^{n} |(z_i, z_j | z)| \right)^q \right)^{\frac{1}{q}}, \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |a_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j | z)| . \end{cases}$$

*Proof.* As in Lemma 1, we know that,

$$\left\| \sum_{i=1}^{n} a_i z_i | z \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i| \, |a_j| \, |(z_i, z_j | z)| . \tag{3.12}$$

Using the simple observation that (see also [5, p. 394])

$$|a_i| \, |a_j| \leq \frac{1}{2} \left( |a_i|^2 + |a_j|^2 \right), \quad i, j \in \{1, \dots, n\}$$

we have,

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i| \, |a_j| \, |(z_i, z_j | z)| &\leq \frac{1}{2} \sum_{i,j=1}^{n} \left( |a_i|^2 + |a_j|^2 \right) |(z_i, z_j | z)| \\ &= \frac{1}{2} \left[ \sum_{i,j=1}^{n} |a_i|^2 |(z_i, z_j | z)| + \sum_{i,j=1}^{n} |a_j|^2 |(z_i, z_j | z)| \right] \\ &= \sum_{i,j=1}^{n} |a_i|^2 |(z_i, z_j | z)| , \end{aligned}$$

which proves the first inequality in (3.11).

The second part follows by Hölder's inequality and we omit the details. ∎

## 4. Some Inequalities for Fourier Coefficients in 2-Inner Product Spaces

We are now able to point out the following result.

**Theorem 5.** *Let $x, y_1, \ldots, y_n, z$ be vectors of a 2-inner product space $(X; (\cdot, \cdot|\cdot))$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \tag{4.1}$$

$$\leq \|x|z\|^2 \times \begin{cases} \max\limits_{1\le k\le n} |c_k|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|; \\ \max\limits_{1\le k\le n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^r\right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^s\right]^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ \max\limits_{1\le k\le n} |c_k| \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right); \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^q\right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \qquad\qquad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |c_i| \max\limits_{1\le i\le n} \left\{\left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}}\right\}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\le i\le n} |c_i| \sum\limits_{i=1}^{n} \left[\max\limits_{1\le j\le n} |(y_i, y_j|z)|\right]; \\ \sum\limits_{k=1}^{n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n} \left[\max\limits_{1\le j\le n} |(y_i, y_j|z)|\right]^l\right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|\right)^2 \max\limits_{i,1\le j\le n} |(y_i, y_j|z)|. \end{cases}$$

*Proof.* We note that

$$\sum_{i=1}^{n} c_i (x, y_i|z) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i|z\right).$$

Using Schwarz's inequality in 2-inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i|z\right\|^2. \tag{4.2}$$

Finally, using Lemma 1 with $a_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (4.1). We omit the details. ∎

The following corollaries may be useful if one needs bounds in terms of $\sum_{i=1}^{n} |c_i|^2$.

**Corollary 5.** *With the assumptions in Theorem 5 and if* $1 < p \le 2$, $1 < t \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, *one has the inequality:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{i=1}^{n} |c_i|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j|z)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{4.3}$$

*and, in particular, for* $p = q = t = u = 2$,

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}}. \tag{4.4}$$

The proof is similar to that of Corollary 1.

**Corollary 6.** *With the assumptions in Theorem 5 and if* $1 < p \le 2$, *then,*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 n^{\frac{1}{p}} \sum_{k=1}^{n} |c_k|^2 \max_{1\le i\le n} \left[\sum_{j=1}^{n} |(y_i, y_j|z)|^q\right]^{\frac{1}{q}}, \tag{4.5}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is similar to that of Corollary 2.
The following two inequalities also hold.

**Corollary 7.** *With the above assumptions for* $X, y_i, c_i$ *and if* $1 < m \le 2$, *then,*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 n^{\frac{1}{m}} \sum_{k=1}^{n} |c_k|^2 \left(\sum_{i=1}^{n} \left[\max_{1\le j\le n} |(y_i, y_j|z)|\right]^l\right)^{\frac{1}{l}}, \tag{4.6}$$

*where* $\frac{1}{m} + \frac{1}{l} = 1$.

**Corollary 8.** *With the above assumptions for* $X, y_i, c_i$, *one has*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 n \sum_{k=1}^{n} |c_k|^2 \max_{i,1\le j\le n} |(y_i, y_j|z)|. \tag{4.7}$$

Using Lemma 2, we may state the following result as well.

**Remark 2.** *With the assumptions of Theorem 5,*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 \sum_{i=1}^{n} |c_i|^2 \sum_{j=1}^{n} |(y_i, y_j|z)| \tag{4.8}$$

$$\leq \|x|z\|^2 \times \begin{cases} \sum\limits_{i=1}^{n} |c_i|^2 \max\limits_{1\leq i\leq n} \left[ \sum\limits_{j=1}^{n} |(y_i, y_j|z)| \right]; \\ \left( \sum\limits_{i=1}^{n} |c_i|^{2p} \right)^{\frac{1}{p}} \left( \sum\limits_{i=1}^{n} \left( \sum\limits_{j=1}^{n} |(y_i, y_j|z)| \right)^q \right)^{\frac{1}{q}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j|z)| . \end{cases}$$

## 5. Bombieri, Selberg and Heilbronn Inequalities in 2-Inner Product Spaces

We first note the following Bombieri type inequality for 2-inner products as an important consequence of the second part of (4.8),

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\|^2 \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} |(y_i, y_j|z)| \right\}. \tag{5.1}$$

This result can be easily derived from the first branch of that inequality for the choice $c_i = \overline{(x, y_i|z)}$ $(i = 1, \ldots, n)$.

It is obvious that if $(y_i)_{1\leq i\leq n}$ is a $z-$orthonormal family in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$, then (5.1) will produce Bessel's inequality (2.11).

If one chooses in the first inequality of (4.8),

$$c_i = \frac{\overline{(x, y_i|z)}}{\sum\limits_{j=1}^{n} |(y_i, y_j|z)|}, i = 1, \ldots, n$$

then one can state the following inequality,

$$\sum_{i=1}^{n} \frac{|(x, y_i|z)|^2}{\sum_{j=1}^{n} |(y_i, y_j|z)|} \leq \|x|z\|^2, z \in X, \tag{5.2}$$

provided that $\sum\limits_{j=1}^{n} |(y_i, y_j|z)| \neq 0.$

When $(y_i)_{1\leq i\leq n}$ is a $z-$orthonormal family in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$, then (5.1) will produce Bessel's inequality (2.11) as well.

The inequality (5.2) is the corresponding version for 2-inner product spaces of the Selberg inequality.

Finally, if one considers

$$c_i = \frac{\overline{(x, y_i|z)}}{|(y_i, y_j|z)|}, i = 1, \ldots, n$$

in the first inequality of (4.8), then after simple computation we deduce the following result,

$$\sum_{i=1}^{n} |(x, y_i|z)| \leq \|x|z\| \left( \sum_{i,j=1}^{n} |(y_i, y_j|z)| \right)^{\frac{1}{2}}, \tag{5.3}$$

which is the corresponding version for 2-inner products of Heilbronn's result.

## 6. More Inequalities of the Bombieri Type in 2-Inner Product Spaces

Further, we point out other inequalities of Bombieri type that may be obtained from (4.1) on choosing $c_i = \overline{(x, y_i|z)}$ $(i = 1, \dots, n)$.

If the above choice is made in the first inequality of (4.1), then one obtains:

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \leq \|x|z\|^2 \max_{1\leq i\leq n} |(x, y_i|z)|^2 \sum_{i,j=1}^{n} |(y_i, y_j|z)|$$

giving,

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\| \max_{1\leq i\leq n} |(x, y_i|z)| \left(\sum_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{2}}, \quad x \in X. \tag{6.1}$$

If the same choice for $c_i$ is made in the second inequality of (4.1), then

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2$$
$$\leq \|x|z\|^2 \max_{1\leq i\leq n} |(x, y_i|z)| \left(\sum_{i=1}^{n} |(x, y_i|z)|^r\right)^{\frac{1}{r}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)^s\right]^{\frac{1}{s}},$$

implying that,

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \tag{6.2}$$
$$\leq \|x|z\| \max_{1\leq i\leq n} |(x, y_i|z)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i|z)|^r\right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)^s\right]^{\frac{1}{2s}},$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

The other inequalities in (4.1) will produce the following results, respectively

$$\sum_{i=1}^{n} |(x, y_i|z)|^2$$
$$\leq \|x|z\| \max_{1\leq i\leq n} |(x, y_i|z)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i|z)|\right)^{\frac{1}{2}} \left[\max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right];$$

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \tag{6.3}$$
$$\leq \|x|z\| \max_{1\leq i\leq n} |(x, y_i|z)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i|z)|^p\right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}},$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\text{(6.4)}\quad \sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|\left(\sum_{i=1}^{n}\left|(x,y_i|z)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,y_i|z)\right|^t\right)^{\frac{1}{2t}}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(y_i,y_j|z)\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}},$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ t>1,\ \frac{1}{t}+\frac{1}{u}=1$;

$$\text{(6.5)}\quad \sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|\left(\sum_{i=1}^{n}\left|(x,y_i|z)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,y_i|z)\right|\right)^{\frac{1}{2}}\max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n}\left|(y_i,y_j|z)\right|^q\right)^{\frac{1}{2q}}\right\},$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|\left[\sum_{i=1}^{n}\left|(x,y_i|z)\right|\right]^{\frac{1}{2}}\max_{1\le i\le n}\left|(x,y_i|z)\right|^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left[\max_{1\le j\le n}\left|(y_i,y_j|z)\right|\right]\right)^{\frac{1}{2}};$$

$$\text{(6.6)}\quad \sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|\left[\sum_{i=1}^{n}\left|(x,y_i|z)\right|^m\right]^{\frac{1}{2m}}\left[\sum_{i=1}^{n}\left[\max_{1\le j\le n}\left|(y_i,y_j|z)\right|^l\right]\right]^{\frac{1}{2l}},$$

where $m>1,\ \frac{1}{m}+\frac{1}{l}=1$; and

$$\text{(6.7)}\quad \sum_{i=1}^{n}\left|(x,y_i|z)\right|^2 \le \|x|z\|\sum_{i=1}^{n}\left|(x,y_i|z)\right|\max_{i,1\le j\le n}\left|(y_i,y_j|z)\right|^{\frac{1}{2}}.$$

If in the above inequalities we assume that $(y_i)_{1\le i\le n}=(f_i)_{1\le i\le n}$, where $(f_i)_{1\le i\le n}$ are $z-$orthonormal vectors in the 2-inner product space $(X,(\cdot,\cdot|\cdot))$, then from (6.1) – (6.7) we may deduce the following inequalities similar, in a sense to Bessel's inequality:

$$\text{(6.8)}\quad \sum_{i=1}^{n}\left|(x,f_i|z)\right|^2 \le \sqrt{n}\,\|x|z\|\max_{1\le i\le n}\left\{\left|(x,f_i|z)\right|\right\};$$

$$\text{(6.9)}\quad \sum_{i=1}^{n}\left|(x,f_i|z)\right|^2 \le n^{\frac{1}{2s}}\,\|x|z\|\max_{1\le i\le n}\left\{\left|(x,f_i|z)\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}\left|(x,f_i|z)\right|^r\right)^{\frac{1}{2r}},$$

where $r>1,\ \frac{1}{r}+\frac{1}{s}=1$;

$$\text{(6.10)}\quad \sum_{i=1}^{n}\left|(x,f_i|z)\right|^2 \le \|x|z\|\max_{1\le i\le n}\left\{\left|(x,f_i|z)\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}\left|(x,f_i|z)\right|\right)^{\frac{1}{2}};$$

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le \sqrt{n}\,\|x|z\| \max_{1\le i\le n}\left\{|(x,f_i|z)|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}|(x,f_i|z)|^p\right)^{\frac{1}{2p}}, \tag{6.11}$$

where $p>1$;

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le n^{\frac{1}{2u}}\,\|x|z\|\left(\sum_{i=1}^{n}|(x,f_i|z)|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}|(x,f_i|z)|^t\right)^{\frac{1}{2t}}, \tag{6.12}$$

where $p>1$, $t>1$, $\frac{1}{t}+\frac{1}{u}=1$;

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le \|x|z\|\left(\sum_{i=1}^{n}|(x,f_i|z)|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}|(x,f_i|z)|\right)^{\frac{1}{2}}, \quad p>1; \tag{6.13}$$

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le \sqrt{n}\,\|x|z\|\left(\sum_{i=1}^{n}|(x,f_i|z)|\right)^{\frac{1}{2}}\max_{1\le i\le n}\left\{|(x,f_i|z)|^{\frac{1}{2}}\right\}; \tag{6.14}$$

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le n^{\frac{1}{2l}}\,\|x|z\|\left[\sum_{i=1}^{n}|(x,f_i|z)|^m\right]^{\frac{1}{m}}, \quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \tag{6.15}$$

$$\sum_{i=1}^{n}|(x,f_i|z)|^2 \le \|x|z\|\sum_{i=1}^{n}|(x,f_i|z)|\,. \tag{6.16}$$

Corollaries 5 – 8 will produce the following results which do not contain the Fourier coefficients in the right side of the inequality.

Indeed, if one chooses $c_i=\overline{(x,y_i|z)}$ in (4.3), then,

$$\left(\sum_{i=1}^{n}|(x,y_i|z)|^2\right)^2 \le \|x|z\|^2\,n^{\frac{1}{p}+\frac{1}{t}-1}\sum_{i=1}^{n}|(x,y_i|z)|^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j|z)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},$$

giving the following Bombieri type inequality:

$$\sum_{i=1}^{n}|(x,y_i|z)|^2 \le n^{\frac{1}{p}+\frac{1}{t}-1}\,\|x|z\|^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j|z)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{6.17}$$

where $1<p\le 2$, $1<t\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$.

If in this inequality we consider $p=q=t=u=2$, then

$$\sum_{i=1}^{n}|(x,y_i|z)|^2 \le \|x|z\|^2\left(\sum_{i,j=1}^{n}|(y_i,y_j|z)|^2\right)^{\frac{1}{2}}. \tag{6.18}$$

In a similar way, if $c_i=\overline{(x,y_i|z)}$ in (4.6), then,

$$\sum_{i=1}^{n}|(x,y_i|z)|^2 \le n^{\frac{1}{m}}\,\|x|z\|^2\left(\sum_{i=1}^{n}\left[\max_{1\le j\le n}|(y_i,y_j|z)|\right]^l\right)^{\frac{1}{l}}, \tag{6.19}$$

where $m>1$, $\frac{1}{m}+\frac{1}{l}=1$.

Finally, if $c_i = \overline{(x, y_i|z)}$ $(i = 1, \ldots, n)$, is taken in (4.7), then

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le n \|x|z\|^2 \max_{1\le i,j\le n} |(y_i, y_j|z)| . \tag{6.20}$$

**Remark 3.** *We now compare,*

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\|^2 \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |(y_i, y_j|z)| \right\} \tag{6.21}$$

*with,*

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\|^2 \left\{ \sum_{i,j=1}^{n} |(y_i, y_j|z)|^2 \right\}^{\frac{1}{2}} . \tag{6.22}$$

*Denote*

$$M_1 := \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |(y_i, y_j|z)| \right\}$$

*and*

$$M_2 := \left[ \sum_{i,j=1}^{n} |(y_i, y_j|z)|^2 \right]^{\frac{1}{2}} .$$

*If we choose* $n = 2$, *then for* $a := |(y_1, y_1|z)|, b := |(y_1, y_2|z)|, c := |(y_2, y_2|z)|, a, b, c > 0$, *we have*

$$M_1 = \max\{a + b, b + c\},$$
$$M_2 = \left(a^2 + 2b^2 + c^2\right)^{\frac{1}{2}} .$$

*Assume that* $a = 2, b = 1$ *and* $c = 3$. *Then* $M_1 = 4 > \sqrt{15} = M_2$, *showing that, in this case, the bound provided by (6.22) is better than the bound provided by (6.21). If* $(y_i)_{1\le i\le n}$ *are* $z-$*orthonormal vectors, then* $M_1 = 1$, $M_2 = \sqrt{n}$, *showing that in this case the Bombieri type inequality (which becomes Bessel's inequality) provides a better bound than (6.22).*

## 7. Applications for Determinantal Integral Inequalities

Let $(\Omega, \Sigma, \mu)$ be a measure space consisting of a set $\Omega$, a $\sigma-$algebra $\Sigma$ of subsets of $\Omega$ and a countably additive and positive measure $\mu$ on $\Sigma$ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions $f$ defined on $\Omega$ that are $2-\rho-$integrable on $\Omega$, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on $\Omega$.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s)\, d\mu(t), \tag{7.1}$$

where,

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)\,d\mu(t)\right)^{1/2}. \tag{7.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \begin{vmatrix} \int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{7.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2} \tag{7.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho fg d\mu$.

Using the representations (7.3), (7.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we have some interesting determinantal integral inequalities.

**Proposition 1.** *Let $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on $\Omega$, then we have the inequality,*

$$\sum_{i=1}^n \begin{vmatrix} \int_\Omega \rho fg_i d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2 \tag{7.5}$$

$$\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \max_{1\leq i\leq n}\left\{\sum_{j=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|\right\}.$$

The proof follows by the Bombieri type inequality for 2-inner products incorporated in 5.1.

**Proposition 2.** *Let $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on $\Omega$, then,*

$$\sum_{i=1}^n \frac{\begin{vmatrix} \int_\Omega \rho fg_i d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2}{\sum_{j=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right|} \tag{7.6}$$

$$\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}$$

*provided that*

$$\sum_{j=1}^n \left|\det\begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right| \neq 0$$

*for each $i \in \{1, ..., n\}$.*

This result follows by the Selberg type inequality (5.2).

Finally, by the use of the Heilbronn type inequality (5.3), we have:

**Proposition 3.** *With the above assumptions for $f, g_1, ..., g_n, h$,*

$$\sum_{i=1}^{n} \left| \det \begin{bmatrix} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \right| \tag{7.7}$$

$$\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2} \left\{ \sum_{i,j=1}^{n} \left| \det \begin{bmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \right| \right\}^{1/2}.$$

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