SOME COMPANIONS OF GRÜSS INEQUALITY IN 2-INNER PRODUCT SPACES AND APPLICATIONS FOR DETERMINANTAL INTEGRAL INEQUALITIES

R.W. FREESE, S.S. DRAGOMIR, Y.J. CHO★, AND S.S. KIM♦

Abstract. Some companions of Grüss inequality in 2-inner product spaces and applications for determinantal integral inequalities are given.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot,\cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x,x|z) \geq 0$ and $(x,x|z) = 0$ if and only if x and z are linearly dependent,
($2\mathrm{I}_2$) $(x,x|z) = (z,z|x)$,
($2I_3$) $(y,x|z) = \overline{(x,y|z)}$,
($2I_4$) $(\alpha x,y|z) = \alpha(x,y|z)$ for any scalar $\alpha \in \mathbb{K}$,
($2I_5$) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

$(\cdot,\cdot|\cdot)$ is called a 2-*inner product* on X and $(X,(\cdot,\cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot,\cdot|\cdot)$ can be immediately obtained as follows [2]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y,x|z) = (x,y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0,y|z) = 0, \quad (x,0|z) = 0$$

and also

$$(x,\alpha y|z) = \bar{\alpha}(x,y|z). \tag{1.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

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★, ♦ Corresponding authors.

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{1.2}$$

In the real case, (1.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{1.3}$$

and, using this formula, it is easy to see, for any $\alpha \in \mathbb{R}$, that

$$(x, y|\alpha z) = \alpha^2 (x, y|z). \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (1.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbb{C}$, that

$$(x, y|\alpha z) = |\alpha|^2 (x, y|z). \tag{1.6}$$

However, for $\alpha \in \mathbb{R}$, (1.6) reduces to (1.4).
Also, from (1.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\| \cdot | \cdot \|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$.

It is easy to see that, this function satisfies the following conditions:

($2\mathrm{N}_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

($2\mathrm{N}_2$) $\|z|x\| = \|x|z\|$,

($2\mathrm{N}_3$) $\|\alpha x|z\| = |\alpha| \|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

($2\mathrm{N}_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\| \cdot | \cdot \|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2-*norm* on X and $(X, \| \cdot | \cdot \|)$ is called a *linear* 2-*normed space* [6]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \| \cdot | \cdot \|)$ with the 2-norm defined by (1.10).

In the recent paper [5], the authors have established the following Grüss' type inequality holding in 2-inner product spaces.

Theorem 1. *Let $(X, (.,.|.))$ be a 2-inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e, z \in X$, with $\|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in X such that the conditions*

$$\operatorname{Re}(\Phi e - x, x - \varphi e|z) \geq 0, \quad \operatorname{Re}(\Gamma e - y, y - \gamma e|z) \geq 0 \tag{1.11}$$

hold, or equivalently, the following assumptions

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Phi - \varphi|, \left\| y - \frac{\gamma + \Gamma}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.12}$$

are valid, then one has the inequality

$$|(x, y|z) - (x, e|z)(e, y|z)| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|. \tag{1.13}$$

The constant $\frac{1}{4}$ is best possible.

The following result improving (1.13) holds (cf. [5, Theorem 1]).

Theorem 2. *Let $(X, (.,.|.))$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e, z \in X$, with $\|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in X such that the conditions* (1.11)*, or equivalently,* (1.12) *hold, then we have the inequality*

$$\begin{aligned}
&|(x, y|z) - (x, e|z)(e, y|z)| \\
&\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re}(\Phi e - x, x - \varphi e|z)]^{\frac{1}{2}} [\operatorname{Re}(\Gamma e - y, y - \gamma e|z)]^{\frac{1}{2}} \\
&\left(\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right).
\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

The following companion of Grüss inequality in inner product spaces also holds (see [5, Theorem 4]).

Theorem 3. *Let* $(X, (.,.|.))$ *be a 2-inner product space over* $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ *and* $e, z \in X$, $\|e|z\| = 1$. *If* $\gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in X$ *are so that either*

$$\operatorname{Re}\left(\Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e|z\right) \geq 0, \tag{1.14}$$

or equivalently,

$$\left\|\frac{x+y}{2} - \frac{\gamma+\Gamma}{2}\cdot e|z\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.15}$$

holds, then we have the inequality

$$\operatorname{Re}\left[(x,y|z) - (x,e|z)\,(e,y|z)\right] \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2. \tag{1.16}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

The following corollary can be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}\left[(x,y|z) - (x,e|z)\,(e,y|z)\right]$$

(see for details [5]).

Corollary 1. *Let* $(X, (.,.|.))$ *be a 2-inner product space over* $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ *and* $e, z \in X$, $\|e, z\| = 1$. *If* $\gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in X$ *are so that*

$$\operatorname{Re}\left(\Gamma e - \frac{x\pm y}{2}, \frac{x\pm y}{2} - \gamma e|z\right) \geq 0 \tag{1.17}$$

or equivalently,

$$\left\|\frac{x\pm y}{2} - \frac{\gamma+\Gamma}{2}\cdot e|z\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.18}$$

then we have the inequality

$$\left|\operatorname{Re}\left[(x,y|z) - (x,e|z)\,(e,y|z)\right]\right| \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2. \tag{1.19}$$

If the inner product space X *is real, then, for* $m, M \in \mathbb{R}$, *with* $M > m$,

$$\left(Me - \frac{x\pm y}{2}, \frac{x\pm y}{2} - me|z\right) \geq 0 \tag{1.20}$$

or equivalently,

$$\left\|\frac{x\pm y}{2} - \frac{m+M}{2}\cdot e|z\right\| \leq \frac{1}{2}\left(M-m\right), \tag{1.21}$$

implies

$$\left|(x,y|z) - (x,e|z)\,(e,y|z)\right| \leq \frac{1}{4}\left(M-m\right)^2. \tag{1.22}$$

In both inequalities (1.19) *and* (1.22), *the constant* $\frac{1}{4}$ *is best possible.*

It is the main aim of this paper to provide other inequalities of Grüss type in the general setting of 2-inner product spaces over the real or complex number field $\mathbb{K}$. Applications for determinantal integral inequalities are pointed out as well.

2. A Grüss Type Inequality for 2-Inner Products

The following Grüss type inequality in inner product spaces holds.

Theorem 4. *Let $x, y, z, e \in X$ with $\|e, z\| = 1$ and the scalars $a, A, b, B \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ such that $\operatorname{Re}(\bar{a}A) > 0$ and $\operatorname{Re}(\bar{b}B) > 0$. If*

$$\operatorname{Re}(Ae - x, x - ae|z) \geq 0 \quad \text{and} \quad \operatorname{Re}(Be - y, y - be|z) \geq 0, \tag{2.1}$$

or equivalently,

$$\left\| x - \frac{a + A}{2} e|z \right\| \leq \frac{1}{2} |A - a| \quad \text{and} \quad \left\| y - \frac{b + B}{2} e|z \right\| \leq \frac{1}{2} |B - b|, \tag{2.2}$$

then we have the inequality

$$|(x, y|z) - (x, e|z)(e, y|z)| \leq \frac{1}{4} \frac{|A - a| |B - a|}{\sqrt{\operatorname{Re}(\bar{a}A) \operatorname{Re}(\bar{b}B)}} |(x, e|z)(e, y|z)|. \tag{2.3}$$

The constant $\frac{1}{4}$ in (2.3) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Let u, x, z, U be vectors in the 2-inner product space $(X, (.,.|))$ over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ with $u \neq U$. We claim (see also [5]), that

$$\operatorname{Re}(U - x, x - u|z) \geq 0$$

if and only if

$$\left\| x - \frac{u + U}{2} |z \right\| \leq \frac{1}{2} \|U - u|z\|.$$

Define

$$I_1 := \operatorname{Re}(U - x, x - u|z), I_2 := \frac{1}{4} \|U - u|z\|^2 - \left\| x - \frac{u + U}{2} |z \right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}[(x, u|z) + (U, x|z)] - \operatorname{Re}(U, u|z) - \|x|z\|^2$$

and thus, obviously, $I_1 \geq 0$ if and only if $I_2 \geq 0$ showing, for $U := Ae, u := ae$ $(U := Be, u := be)$, that the required equivalence holds true.

Apply Schwarz's inequality in $(X; (\cdot, \cdot|\cdot))$ for the vectors $x - (x, e|z) e$ and $y - (y, e|z) e$, to get (see also [3]) that

$$|(x, y|z) - (x, e|z)(e, y|z)|^2 \leq \left(\|x|z\|^2 - |(x, e|z)|^2 \right) \left(\|y|z\|^2 - |(y, e|z)|^2 \right). \tag{2.4}$$

Now, assume that $u, v \in X$, and $c, C \in \mathbb{K}$ with $\operatorname{Re}(\bar{c}C) > 0$ and $\operatorname{Re}(Cv - u, u - cv|z) \geq 0$. This last inequality is equivalent to

$$\|u|z\|^2 + \operatorname{Re}(\bar{c}C) \|v|z\|^2 \leq \operatorname{Re}\left[C\overline{(u, v|z)} + \bar{c}(u, v|z) \right] \tag{2.5}$$

$$= \operatorname{Re}\left[\left(\bar{C} + \bar{c} \right) (u, v|z) \right], \tag{2.6}$$

since it is obvious that

$$\operatorname{Re}\left[C\overline{(u, v|z)} \right] = \operatorname{Re}\left[\bar{C}(u, v|z) \right].$$

Dividing this inequality by $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}} > 0$, we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u|z\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v|z\|^2 \leq \frac{\operatorname{Re}\left[\left(\bar{C} + \bar{c} \right) (u, v|z) \right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}. \tag{2.7}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0,\ p, q \geq 0,$$

we deduce

$$2 \|u|z\| \|v|z\| \leq \frac{1}{[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u|z\|^2 + [\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}} \|v|z\|^2 . \tag{2.8}$$

Making use of (2.7) and (2.8) and the fact that, for any $z \in \mathbb{C}$, $\mathrm{Re}(z) \leq |z|$, we get

$$\|u|z\| \|v|z\| \leq \frac{\mathrm{Re}\left[\left(\bar{C} + \bar{c}\right)(u, v|z)\right]}{2[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|c + C|}{2[\mathrm{Re}(\bar{c}C)]^{\frac{1}{2}}} |(u, v|z)| .$$

Consequently, we have

$$\begin{aligned} \|u|z\|^2 \|v|z\|^2 - |(u, v|z)|^2 &\leq \left[\frac{|c + C|^2}{4[\mathrm{Re}(\bar{c}C)]} - 1\right] |(u, v|z)|^2 \\ &= \frac{1}{4} \frac{|C - c|^2}{\mathrm{Re}(\bar{c}C)} |(u, v|z)|^2 . \end{aligned} \tag{2.9}$$

Now, if we write (2.9) for the choices $u = x$, $v = e$ and $u = y$, $v = e$ respectively and use (2.4), we deduce the desired result (2.2). The sharpness of the constant will be proved in the case where X is a real 2-inner product space. ∎

The following corollary which provides a simpler Grüss type inequality for real constants (and, in particular, for real 2-inner product spaces) holds.

Corollary 2. *With the assumptions of Theorem 4 and if* $m, M, n, N \in \mathbb{R}$ *are such that* $M > m > 0$, $N > n > 0$ *and either*

$$(Me - x, x - me|z) \geq 0 \text{ and } (Ne - y, y - ne|z) \geq 0$$

or equivalently,

$$\left\|x - \frac{m + M}{2} e|z\right\| \leq \frac{1}{2}(M - m) \quad \text{and} \quad \left\|y - \frac{n + N}{2} e|z\right\| \leq \frac{1}{2}(N - n), \tag{2.10}$$

holds, then we have the inequality

$$|(x, y|z) - (x, e|z)(e, y|z)| \leq \frac{1}{4} \cdot \frac{(M - m)(N - n)}{\sqrt{mnMN}} |(x, e|z)(e, y|z)| . \tag{2.11}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. We will prove the best constant in (2.11) for $x = y$. To do that, let assume there is a constant $k > 0$ so that

$$\|x|z\|^2 - (x, e|z)^2 \leq k \cdot \frac{(M - m)^2}{mM} (x, e|z)^2 \tag{2.12}$$

provided

$$(Me - x, x - me|z) \geq 0, \tag{2.13}$$

where $M > m > 0$, $e, z \in X$ with $\|e|z\| = 1$ and $x \in X$.

For $\varepsilon > 0$, consider $M := 2 + \varepsilon, m := \varepsilon > 0, y \in X$ with $\|y|z\| = 1$ and $(e, y|z) = 0$. Define $x := (1 + \varepsilon) e + y$. With these choices, we have

$$\begin{aligned} (Me - x, x - me|z) &= ((2 + \varepsilon) e - (1 + \varepsilon) e - y, (1 + \varepsilon) e + y - \varepsilon e|z) \\ &= (e - y, e + y|z) = \|e|z\|^2 - \|y|z\|^2 = 0 \end{aligned}$$

and thus the condition (2.13) is obviously satisfied.

On the other hand,

$$\|x|z\|^2 = \|(1+\varepsilon)\,e + y|z\|^2 = (1+\varepsilon)^2 + 1$$

and

$$(x, e|z)^2 = ((1+\varepsilon)\,e + y, e|z)^2 = (1+\varepsilon)^2 .$$

Replacing the above values in the inequality (2.12), we get

$$1 \le \frac{4k\,(\varepsilon+1)^2}{\varepsilon\,(\varepsilon+2)} \tag{2.14}$$

for any $\varepsilon > 0$. Letting $\varepsilon \to +\infty$, we deduce $k \ge \frac{1}{4}$, proving the fact that $\frac{1}{4}$ is the best constant in (2.11). ∎

Remark 1. *If $(x, e|z)\,, (y, e|z)$ are assumed not to be zero, then the inequality (2.3) is equivalent to*

$$\left|\frac{(x, y|z)}{(x, e|z)\,(e, y|z)} - 1\right| \le \frac{1}{4}\frac{|A-a|\,|B-a|}{\sqrt{\operatorname{Re}(\bar{a}A)\operatorname{Re}\left(\bar{b}B\right)}}, \tag{2.15}$$

while the inequality (2.11) is equivalent to

$$\left|\frac{(x, y|z)}{(x, e|z)\,(e, y|z)} - 1\right| \le \frac{1}{4}\cdot\frac{(M-m)\,(N-n)}{\sqrt{mnMN}}. \tag{2.16}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

3. Some Related Results

The following result which provides a generalisation of Theorem 3 holds.

Theorem 5. *Let $(X; (\cdot,\cdot|\cdot))$ be a 2-inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$. If $\gamma, \Gamma \in \mathbb{K}$, $e, x, y, z \in X$ with $\|e, z\| = 1$ and $\lambda \in (0,1)$ are such that*

$$\operatorname{Re}\left(\Gamma e - (\lambda x + (1-\lambda)\,y)\,, (\lambda x + (1-\lambda)\,y) - \gamma e|z\right) \ge 0, \tag{3.1}$$

or equivalently,

$$\left\|\lambda x + (1-\lambda)\,y - \frac{\gamma+\Gamma}{2}e|z\right\| \le \frac{1}{2}\,|\Gamma-\gamma|\,, \tag{3.2}$$

then we have the inequality

$$\operatorname{Re}\left[(x, y|z) - (x, e|z)\,(e, y|z)\right] \le \frac{1}{16}\cdot\frac{1}{\lambda\,(1-\lambda)}\,|\Gamma-\gamma|^2 . \tag{3.3}$$

The constant $\frac{1}{16}$ is the best possible constant in (3.3) in the sense that it cannot be replaced by a smaller one.

Proof. We know that for any $z, u \in X$ one has

$$\operatorname{Re}(z, u|z) \le \frac{1}{4}\,\|z + u|z\|^2 .$$

Then, for any $a, b \in X$ and $\lambda \in (0,1)$, one has

$$\operatorname{Re}(a, b|z) \le \frac{1}{4\lambda\,(1-\lambda)}\,\|\lambda a + (1-\lambda)\,b|z\|^2 . \tag{3.4}$$

Since

$$(x, y|z) - (x, e|z)\,(e, y|z) = (x - (x, e|z)\,e, y - (y, e|z)\,e|z) \qquad (\text{as } \|e|z\| = 1),$$

using (3.4), we have

$$\begin{aligned} (3.5) \qquad & \operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right] \\ &= \operatorname{Re}\left[(x-(x,e|z)\,e, y-(y,e|z)\,e|z)\right] \\ &\le \frac{1}{4\lambda(1-\lambda)}\left\|\lambda(x-(x,e|z)\,e)+(1-\lambda)(y-(y,e|z)\,e)\,|z\right\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)}\left\|\lambda x+(1-\lambda)\,y-(\lambda x+(1-\lambda)\,y,e|z)\,e|z\right\|^2. \end{aligned}$$

Since, for $m,e,z\in X$ with $\|e|z\|=1$, one has the equality

$$\|m-(m,e|z)\,e|z\|^2=\|m|z\|^2-|(m,e|z)|^2, \tag{3.6}$$

then, by (3.5), we deduce the inequality

$$\begin{aligned} (3.7) \qquad & \operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right] \\ &\le \frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)\,y|z\|^2-|(\lambda x+(1-\lambda)\,y,e|z)|^2\right]. \end{aligned}$$

Now, if we apply Grüss' inequality

$$0\le\|a|z\|^2-|(a,e|z)|^2\le\frac{1}{4}|\Gamma-\gamma|^2$$

provided

$$\operatorname{Re}(\Gamma e-a,a-\gamma e|z)\ge 0,$$

for $a=\lambda x+(1-\lambda)\,y$, then we have

$$\|\lambda x+(1-\lambda)\,y|z\|^2-|(\lambda x+(1-\lambda)\,y,e|z)|^2\le\frac{1}{4}|\Gamma-\gamma|^2. \tag{3.8}$$

Utilising (3.7) and (3.8) we deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C>0$, provided (3.1) is valid, i.e.,

$$\operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right]\le C\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \tag{3.9}$$

If in (3.9) we choose $x=y$, provided (3.1) holds with $x=y$ and $\lambda\in(0,1)$, then

$$\|x|z\|^2-|(x,e|z)|^2\le C\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2, \tag{3.10}$$

provided

$$\operatorname{Re}(\Gamma e-x,x-\gamma e|z)\ge 0.$$

Since we know, in Grüss' inequality, that the constant $\frac{1}{4}$ is best possible, then by (3.10), one has

$$\frac{1}{4}\le\frac{C}{\lambda(1-\lambda)}$$

for $\lambda\in(0,1)$, which gives, for $\lambda=\frac{1}{2}$, that $C\ge\frac{1}{16}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above result:

Corollary 3. *Assume that γ, Γ, e, x, y and λ are as in Theorem 5. If*

$$\operatorname{Re}\left(\Gamma e-(\lambda x \pm(1-\lambda) y),(\lambda x \pm(1-\lambda) y)-\gamma e \mid z\right) \geq 0, \tag{3.11}$$

or equivalently,

$$\left\|\lambda x \pm(1-\lambda) y-\frac{\gamma+\Gamma}{2} e \middle| z\right\| \leq \frac{1}{2}|\Gamma-\gamma|^{2}, \tag{3.12}$$

then we have the inequality

$$|\operatorname{Re}[(x, y \mid z)-(x, e \mid z)(e, y \mid z)]| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^{2}. \tag{3.13}$$

The constant $\frac{1}{16}$ is best possible in (3.13).

Proof. Using Theorem 5 for $(-y)$ instead of y, we have that

$$\operatorname{Re}\left(\Gamma e-(\lambda x-(1-\lambda) y),(\lambda x-(1-\lambda) y)-\gamma e \mid z\right) \geq 0,$$

which implies that

$$\operatorname{Re}[-(x, y \mid z)+(x, e \mid z)(e, y \mid z)] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^{2}.$$

This last inequality shows that

$$\operatorname{Re}[(x, y \mid z)-(x, e \mid z)(e, y \mid z)] \geq-\frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^{2}. \tag{3.14}$$

Consequently, by (3.3) and (3.14) we deduce the desired inequality (3.13). ∎

Remark 2. *If $M, m \in \mathbb{R}$ with $M>m$ and, for $\lambda \in(0,1)$,*

$$\left\|\lambda x+(1-\lambda) y-\frac{M+m}{2} e \middle| z\right\| \leq \frac{1}{2}(M-m) \tag{3.15}$$

then

$$(x, y \mid z)-(x, e \mid z)(e, y \mid z) \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)}(M-m)^{2}.$$

If (3.15) holds with "±" instead of "+", then

$$|(x, y \mid z)-(x, e \mid z)(e, y \mid z)| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)}(M-m)^{2}. \tag{3.16}$$

Remark 3. *If $\lambda=\frac{1}{2}$ in (3.1) or (3.2), then we obtain the result from* [5] *mentioned in Theorem 3.*

For $\lambda=\frac{1}{2}$, Corollary 3 and Remark 2 will produce the corresponding results obtained in [5]. We omit the details.

The following similar result with the one incorporated in Theorem 5 may be stated as well.

Theorem 6. *Assume that γ, Γ, e, x, y and λ are as in Theorem 5. If* $\operatorname{Re}(\Gamma \bar{\gamma})>0$, *then we have the inequality:*

$$\begin{aligned} & \operatorname{Re}[(x, y \mid z)-(x, e \mid z)(e, y \mid z)] \\ \leq & \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \cdot \frac{|\Gamma-\gamma|^{2}}{\operatorname{Re}(\Gamma \bar{\gamma})}|(\lambda x+(1-\lambda) y, e \mid z)|^{2}. \end{aligned} \tag{3.17}$$

The constant $\frac{1}{16}$ is best possible in (3.17).

Proof. If we apply the companion of Grüss inequality incorporated in Theorem 4, then we may state that

$$\begin{aligned} (3.18) \qquad & \|\lambda x+(1-\lambda)y|z\|^2-|(\lambda x+(1-\lambda)y,e|z)|^2 \\ \leq\; & \frac{|\Gamma-\gamma|^2}{4\operatorname{Re}(\Gamma\overline{\gamma})}|(\lambda x+(1-\lambda)y,e|z)|^2. \end{aligned}$$

Utilising (3.7) and (3.18) we deduce the desired inequality (3.17). The sharpness of the constant may be shown in a similar way to the one used in Theorem 5 and we omit the details. ∎

Finally, we may state the following

Corollary 4. *Assume that* γ,Γ,e,x,y *and* λ *are as in Corollary 3. If* $\operatorname{Re}(\Gamma\overline{\gamma})>0$, *then we have the inequality:*

$$\begin{aligned} (3.19) \qquad & |\operatorname{Re}[(x,y|z)-(x,e|z)(e,y|z)]| \\ \leq\; & \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|(\lambda x+(1-\lambda)y,e|z)|^2. \end{aligned}$$

The constant $\frac{1}{16}$ *is best possible in (3.19).*

Remark 4. *The particular case* $\lambda=\frac{1}{2}$ *may produce some particular inequalities of interest, but we omit the details.*

4. Determinantal Integral Inequalities

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s)|f(s)|^2 d\mu(s)<\infty$, where $\rho:\Omega\to[0,\infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(4.1) \qquad (f,g|h)_\rho := \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}\begin{vmatrix} g(s) & g(t) \\ h(s) & h(t)\end{vmatrix} d\mu(s)\,d\mu(t),$$

where by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t)\end{bmatrix},$$

generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$(4.2) \qquad \|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}^2 d\mu(s)\,d\mu(t)\right)^{1/2}.$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{4.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2}, \tag{4.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

We recall that the pair of functions $(q,p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for *a.e.* $x, y \in \Omega$.

We note that, if $\Omega = [a,b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L^2_\rho(\Omega)$ is such that $h(x) \neq 0$ for *a.e.* $x \in \Omega$. Then, by the definition of 2-inner product $(f,g|h)_\rho$, we have

$$\begin{aligned} (f,g|h)_\rho \\ = \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) h^2(s) h^2(t) \left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)} \right) \left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)} \right) d\mu(s) d\mu(t) \end{aligned} \tag{4.5}$$

and thus a *sufficient condition* for the inequality

$$(f,g|h)_\rho \geq 0 \tag{4.6}$$

to hold, is that, the functions $\left(\frac{f}{h}, \frac{g}{h}\right)$ are synchronous. It is obvious that, this condition is not necessary.

Using the representations (4.3), (4.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities as follows:

Proposition 1. *Let $f, g, h, u \in L^2_\rho(\Omega)$ with $h \neq 0$ a.e. and*

$$\int_\Omega \rho u^2 d\mu \int_\Omega \rho h^2 d\mu - \left(\int_\Omega \rho u h d\mu \right)^2 = 1. \tag{4.7}$$

If $M > m$ and $N > n$ are real numbers with the property that the functions

$$\left(M \cdot \frac{u}{h} - \frac{f}{h}, \frac{f}{h} - m \cdot \frac{u}{h} \right) \text{ and } \left(N \cdot \frac{u}{h} - \frac{g}{h}, \frac{g}{h} - n \cdot \frac{u}{h} \right) \tag{4.8}$$

are synchronous on Ω, then we have the following determinantal integral Grüss type inequality

$$\begin{aligned}&\Bigg|\det\begin{bmatrix}\int_\Omega \rho fgd\mu & \int_\Omega \rho fhd\mu\\ \int_\Omega \rho ghd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\\&-\det\begin{bmatrix}\int_\Omega \rho fud\mu & \int_\Omega \rho fhd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\cdot\det\begin{bmatrix}\int_\Omega \rho gud\mu & \int_\Omega \rho ghd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\Bigg|\\ \leq\ & \frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mMnM}}\\&\times\left|\det\begin{bmatrix}\int_\Omega \rho fud\mu & \int_\Omega \rho fhd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\cdot\det\begin{bmatrix}\int_\Omega \rho gud\mu & \int_\Omega \rho ghd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\right|\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 4 applied for the 2-inner product $(.,.|.)_\rho$ defined in (4.1).

Proposition 2. *Let $f,g,h,u\in L^2_\rho(\Omega)$ with $h\neq 0$ a.e. and*

$$\int_\Omega \rho u^2d\mu\int_\Omega \rho h^2d\mu-\left(\int_\Omega \rho uhd\mu\right)^2=1.$$

If $M>m$ and $N>n$ and $\lambda\in(0,1)$ are real numbers with the property that the functions

$$\left(M\cdot\frac{u}{h}-\left(\lambda\frac{f}{h}+(1-\lambda)\frac{g}{h}\right),\lambda\frac{f}{h}+(1-\lambda)\frac{g}{h}-m\cdot\frac{u}{h}\right)\tag{4.9}$$

are synchronous on Ω, then we have the following determinantal integral Grüss type inequality

$$\begin{aligned}J:=&\det\begin{bmatrix}\int_\Omega \rho fgd\mu & \int_\Omega \rho fhd\mu\\ \int_\Omega \rho ghd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\\&-\det\begin{bmatrix}\int_\Omega \rho fud\mu & \int_\Omega \rho fhd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\cdot\det\begin{bmatrix}\int_\Omega \rho gud\mu & \int_\Omega \rho ghd\mu\\ \int_\Omega \rho uhd\mu & \int_\Omega \rho h^2d\mu\end{bmatrix}\\ \leq\ & \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\cdot(M-m)^2.\end{aligned}$$

If (4.9) holds with "$\pm$" instead of "$+$", then

$$|J|\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\cdot(M-m)^2.$$

The proof is obvious by the inequality (3.15) and we omit the details.

Remark 5. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

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Department of Mathematics and Computer Science Saint Louis University Saint Louis, MO 63103, USA
E-mail address: freeserw@slu.edu

School of Computer Science & Mathematics, Victoria University, Melbourne, Victoria, Australia
E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au.SSDragomirWeb.html

Department of Mathematics Education, The Research Institute of Natural Sciences, College of Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: yjcho@nongae.gsnu.ac.kr

Department of Mathematics, Dongeui University, Pusan, 614-714, Korea
E-mail address: sskim@dongeui.ac.kr