NORM ESTIMATES FOR THE DIFFERENCE BETWEEN BOCHNER'S INTEGRAL AND THE CONVEX COMBINATION OF FUNCTION'S VALUES

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Abstract. Norm estimates between the Bochner integral of a vector-valued function in Banach spaces having the Radon-Nikodym property and the convex combination of function's values taken on a division of the interval $[a,b]$, are given.

1. Introduction

A Banach space X with the property that every absolutely continuous $X-$valued function is almost everywhere differentiable is said to be a *Radon-Nikodym* space [6, pp. 217–219] or [2, 12] (see also [3]). For example, every reflexive Banach space (in particular, every Hilbert space) is a Radon-Nikodym space, but the space $L_\infty[0,1]$ of all $\mathbb{K}-$valued, essentially bounded functions defined on the interval $[0,1]$, endowed with the norm

$$\|g\|_\infty := ess \sup_{t\in[0,1]} |g(t)|,$$

is a Banach space which is not a Radon-Nikodym space.

A function $f:[a,b]\to X$ is *measurable* if there exists a sequence of simple functions (f_n) (with $f_n:[a,b]\to X$) which converges punctually a.e. on $[a,b]$.

It is well-known that a measurable function $f:[a,b]\to X$ is Bochner integrable if and only if its norm, i.e., the function $t\longmapsto \|f\|(t) := \|f(t)\| : [a,b]\to X$ is Lebesgue integrable on $[a,b]$, (see for example [11]).

Further, we use the integration by parts formula. This holds under the following general conditions:

Let $-\infty<a<b<\infty$ and f,g be two mappings defined on $[a,b]$ such that f is $\mathbb{C}$-valued and g is X-valued, where X is a real or complex Banach space. If f,g are differentiable on $[a,b]$ and their derivatives are Bochner integrable on $[a,b]$, then

$$(B)\int_a^b f'g = f(b)g(b) - f(a)g(b) - (B)\int_a^b fg'.$$

For some results on the Ostrowski inequality for real-valued functions, see [1], [5], [9] and [10], and the references therein.

The following theorem concerning a version of Ostrowski's inequality for vector-valued functions has been obtained in [3].

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Theorem 1. *Let $(X;\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f:[a,b]\to X$ an absolutely continuous function on $[a,b]$ with the property that $f'\in L_\infty([a,b];X)$, i.e.,*

$$|||f'|||_{[a,b],\infty}:=ess\sup_{t\in[a,b]}\|f'(t)\|<\infty.$$

Then we have the inequalities:

$$\begin{aligned}(1.1)\qquad &\left\|f(s)-\frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\|\\ \le\;& \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt+\int_s^b (b-t)\|f'(t)\|\,dt\right]\\ \le\;& \frac{1}{2(b-a)}\left[(s-a)^2|||f'|||_{[a,s],\infty}+(b-s)^2|||f'|||_{[s,b],\infty}\right]\\ \le\;& \left[\frac{1}{4}+\left(\frac{s-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)|||f'|||_{[a,b],\infty}\\ \le\;& \frac{1}{2}(b-a)|||f'|||_{[a,b],\infty};\end{aligned}$$

for any $s\in[a,b]$, where $(B)\int_a^b f(t)\,dt$ is the Bochner integral of f.

Bounds involving the $p-$norms, $p\in[1,\infty)$, of the derivative f', are embodied in the following theorem [3].

Theorem 2. *Let $(X,\|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ with the property that $f'\in L_p([a,b];X)$, $p\in[1,\infty)$, i.e.,*

$$(1.2)\qquad |||f'|||_{[a,b],p}:=\left(\int_a^b\|f'(t)\|^p\,dt\right)^{\frac{1}{p}}<\infty.$$

Then we have the inequalities

$$\begin{aligned}(1.3)\qquad &\left\|f(s)-\frac{1}{b-a}(B)\int_a^b f(t)\,dt\right\|\\ \le\;& \frac{1}{b-a}\left[\int_a^s (t-a)\|f'(t)\|\,dt+\int_s^b (b-t)\|f'(t)\|\,dt\right]\end{aligned}$$

$$\leq \begin{cases} \dfrac{1}{b-a}\left[(s-a)\,|||f'|||_{[a,s],1}+(b-s)\,|||f'|||_{[s,b],1}\right] \\ \hfill \textit{if } f'\in L_1([a,b];X)\,; \\ \dfrac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left[(s-a)^{\frac{1}{q}+1}\,|||f'|||_{[a,s],p}+(b-s)^{\frac{1}{q}+1}\,|||f'|||_{[s,b],p}\right] \\ \hfill \textit{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ and } f'\in L_p([a,b];X)\,; \end{cases}$$

$$\leq \begin{cases} \left[\dfrac{1}{2}+\left|\dfrac{s-\frac{a+b}{2}}{b-a}\right|\right]|||f'|||_{[a,b],1} \qquad \textit{if } f'\in L_1([a,b];X)\,; \\ \dfrac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\dfrac{s-a}{b-a}\right)^{q+1}+\left(\dfrac{b-s}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\,|||f'|||_{[a,b],p} \\ \hfill \textit{if } f'\in L_p([a,b];X)\,. \end{cases}$$

It is the main aim of this paper to point out estimates between the Bochner integral of a vector-valued function with values in Banach spaces having the Radon-Nikodym property and the convex combination of values taken on a given division of the interval $[a,b]$. The obtained results naturally extend Ostrowski's type inequalities mentioned above. Some particular cases for two and three points rules are also given.

2. The Results

Let $a\leq b$ and $c\in\mathbb{R}$. Define the mapping

$$\mu_p(a,c,b):=\int_a^b |t-c|^p\,dt \quad \text{if} \quad p\in[1,\infty);$$
$$\mu_\infty(a,c,b):=\max_{t\in[a,b]}|t-c| \quad \text{if} \quad p=\infty.$$

We observe that

(1) If $c<a$, then

$$\mu_p(a,c,b)=\int_a^b (t-c)^p\,dt = \frac{1}{p+1}\left[(b-c)^{p+1}-(a-c)^{p+1}\right], \quad \text{if} \quad p\in[1,\infty)$$

and

$$\mu_\infty(a,c,b)=b-c.$$

(2) If $c\in[a,b]$, then

$$\mu_p(a,c,b)=\int_a^c (c-t)^p\,dt+\int_c^b (t-c)^p\,dt = \frac{1}{p+1}\left[(c-a)^{p+1}+(b-c)^{p+1}\right]$$

if $p\in[1,\infty)$ and

$$\mu_\infty(a,c,b)=\max(c-a,b-c)=\frac{1}{2}(b-a)+\left|c-\frac{a+b}{2}\right|.$$

(3) If $b < c$, then

$$\mu_p(a,c,b) = \int_a^b (c-t)^p\,dt = \frac{1}{p+1}\left[(c-a)^{p+1} - (c-b)^{p+1}\right], \quad \text{if} \quad p \in [1,\infty)$$

and

$$\mu_\infty(a,c,b) = c - a.$$

Consequently, we may conclude that

$$\mu_p(a,c,b) = \begin{cases} \frac{1}{p+1}\left[(b-c)^{p+1} - (a-c)^{p+1}\right] & \text{if } c < a; \\ \frac{1}{p+1}\left[(c-a)^{p+1} + (b-c)^{p+1}\right] & \text{if } c \in [a,b]; \\ \frac{1}{p+1}\left[(c-a)^{p+1} - (c-b)^{p+1}\right] & \text{if } b < c; \end{cases}$$

if $p \in [1,\infty)$ and

$$\mu_\infty(a,c,b) = \begin{cases} b - c & \text{if } c < a; \\ \frac{1}{2}(b-a) + \left|c - \frac{a+b}{2}\right| & \text{if } c \in [a,b]; \\ c - a & \text{if } b < c. \end{cases}$$

The following integral identity is interesting in itself as well.

Lemma 1. *Let $f : [a,b] \to X$ be an absolutely continuous function on the Banach space X, X is with the property of Radon-Nikodym, $a \le x_1 \le \cdots \le x_{n-1} \le x_n \le b$ and $p_i > 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^n p_i = 1$. Then we have the equality:*

$$\sum_{i=1}^n p_i f(x_i) - \frac{1}{b-a}(B)\int_a^b f(t)\,dt = \frac{1}{b-a}(B)\int_a^{x_1}(t-a)f'(t)\,dt + \frac{1}{b-a}\sum_{i=1}^{n-1}(B)\int_{x_i}^{x_{i+1}}\left[t - \left(P_i b + \bar{P}_i a\right)\right]f'(t)\,dt + \frac{1}{b-a}(B)\int_{x_n}^b (t-b)f'(t)\,dt, \tag{2.1}$$

where $(B)\int_a^b f(t)\,dt$ is the Bochner integral, $P_i := \sum_{k=1}^i p_k$ and $\bar{P}_i = 1 - P_i$.

The sum in the middle is assumed to be zero when $n = 1$.

Proof. We know that, on utilizing the integration by parts formula, for any $x \in [a,b]$, we have the representation (see for example [3])

$$f(x) = \frac{1}{b-a}(B)\int_a^b f(t)\,dt + \frac{1}{b-a}(B)\int_a^b k(x,t)f'(t)\,dt, \tag{2.2}$$

where

$$k(x,t) = \begin{cases} t - a & \text{if } a \le t \le x \le b, \\ t - b & \text{if } a \le x < t \le b. \end{cases}$$

Putting in (2.2) $x = x_i$ $(i = 1, \dots, n)$, multiplying by $p_i \geq 0$ and summing over i from 1 to n, we deduce

$$\sum_{i=1}^{n} p_i f(x_i) = \frac{1}{b-a} (B) \int_a^b f(t)\, dt + \frac{1}{b-a} (B) \int_a^b \left[\sum_{i=1}^{n} p_i k(x_i, t) \right] f'(t)\, dt. \tag{2.3}$$

However,

$$k(x_1, t) = \begin{cases} t-a & \text{if } a \leq t \leq x_1 \leq b, \\ \\ t-b & \text{if } a \leq x_1 < t \leq b, \end{cases}$$

$$\dots\dots\dots\dots\dots\dots$$

$$k(x_n, t) = \begin{cases} t-a & \text{if } a \leq t \leq x_n \leq b, \\ \\ t-b & \text{if } a \leq x_n < t \leq b, \end{cases}$$

then

$$S(\bar{x}, \bar{p}, t) := \sum_{i=1}^{n} p_i k(x_i, t) \tag{2.4}$$

$$= \begin{cases} p_1(t-a) + p_2(t-a) + \cdots + p_{n-1}(t-a) + p_n(t-a), & a \leq t \leq x_1 \leq b, \\ p_1(t-b) + p_2(t-a) + \cdots + p_{n-1}(t-a) + p_n(t-a), & a \leq x_1 < t \leq x_2 \leq b, \\ \\ \dots\dots\dots\dots\dots\dots \\ \\ p_1(t-b) + p_2(t-b) + \cdots + p_{n-1}(t-b) + p_n(t-a), & a \leq x_{n-1} \leq t \leq x_n \leq b, \\ p_1(t-b) + p_2(t-b) + \cdots + p_{n-1}(t-b) + p_n(t-b), & a \leq x_n < t \leq b, \end{cases}$$

$$= \begin{cases} t-a, & a \leq t \leq x_1 \leq b, \\ p_1(t-b) + (p_2 + \cdots + p_n)(t-a), & a \leq x_1 < t \leq x_2 \leq b, \\ \\ \dots\dots\dots\dots\dots\dots \\ \\ (p_1 + \cdots + p_{n-1})(t-b) + p_n(t-a), & a \leq x_{n-1} \leq t \leq x_n \leq b, \\ t-b, & a \leq x_n < t \leq b, \end{cases}$$

$$= \begin{cases} t-a, & a \leq t \leq x_1 \leq b, \\ t - [p_1 b + (p_2 + \cdots + p_n) a], & a \leq x_1 < t \leq x_2 \leq b, \\ \\ \dots\dots\dots\dots\dots\dots \\ \\ t - [(p_1 + \cdots + p_{n-1}) b + p_n a], & a \leq x_{n-1} \leq t \leq x_n \leq b, \\ t-b, & a \leq x_n < t \leq b, \end{cases}$$

$$= \begin{cases} t-a, & a \leq t \leq x_1 \leq b, \\ t - \left(P_1 b + \bar{P}_1 a\right), & a \leq x_1 < t \leq x_2 \leq b, \\ \dots\dots\dots\dots\dots\dots \\ t - \left(P_i b + \bar{P}_i a\right) & a \leq x_i \leq t \leq x_{i+1} \leq b, \\ \dots\dots\dots\dots\dots\dots \\ t - \left(P_{n-1} b + \bar{P}_{n-1} a\right), & a \leq x_{n-1} \leq t \leq x_n \leq b, \\ t-b, & a \leq x_n < t \leq b. \end{cases}$$

Consequently, by (2.3), we have

$$\begin{aligned}
(2.5)\qquad & \sum_{i=1}^{n} p_i f(x_i) \\
&= \frac{1}{b-a}(B)\int_a^b f(t)\,dt + \frac{1}{b-a}(B)\int_a^b S(\bar{x},\bar{p},t)\,f'(t)\,dt \quad \text{(by (2.4))} \\
&= \frac{1}{b-a}(B)\int_a^b f(t)\,dt + \frac{1}{b-a}(B)\int_a^{x_1}(t-a)\,f'(t)\,dt \\
&\qquad + \frac{1}{b-a}\sum_{i=1}^{n-1}(B)\int_a^b \left[t-\left(P_i b+\bar{P}_i a\right)\right] f'(t)\,dt \\
&\qquad + \frac{1}{b-a}(B)\int_{x_n}^b (t-b)\,f'(t)\,dt,
\end{aligned}$$

and the representation (2.1) is proved. ∎

The following result in approximating the Bochner integral $(B)\int_a^b f(t)\,dt$ in terms of the convex combination of $(f(x_i))_{i=\overline{1,n}}$ with the weights $(p_i)_{i=\overline{1,n}}$ holds.

Theorem 3. *Assume that $f:[a,b]\to X$, $(x_i)_{i=\overline{1,n}}$ and $(p_i)_{i=\overline{1,n}}$ are as in Lemma 1. Then we have the inequality:*

$$\begin{aligned}
(2.6)\qquad & \left\|(B)\int_a^b f(t)\,dt-(b-a)\sum_{i=1}^{n}p_i f(x_i)\right\| \\
&\le \int_a^{x_1}(t-a)\,\|f'(t)\|\,dt+\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}\left|t-\left(P_i b+\bar{P}_i a\right)\right|\,\|f'(t)\|\,dt \\
&\qquad\qquad + \int_{x_n}^{b}(b-t)\,\|f'(t)\|\,dt
\end{aligned}$$

$$\le\begin{cases}
(x_1-a)\,|||f'|||_{[a,x_1],1}+\sum\limits_{i=1}^{n-1}\mu_\infty\left(x_i,P_i b+\bar{P}_i a,x_{i+1}\right)|||f'|||_{[x_i,x_{i+1}],1} \\
\qquad\qquad +(b-x_n)\,|||f'|||_{[x_n,b],1} \\[2ex]
\frac{(x_1-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\,|||f'|||_{[a,x_1],p}+\sum\limits_{i=1}^{n-1}\left[\mu_q\left(x_i,P_i b+\bar{P}_i a,x_{i+1}\right)\right]^{\frac{1}{q}}|||f'|||_{[x_i,x_{i+1}],p} \\
\qquad +\frac{(b-x_n)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\,|||f'|||_{[x_n,b],p} \quad \textit{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
\qquad\qquad \textit{and if } f'\in L_p([a,b];X); \\[2ex]
\frac{(x_1-a)^2}{2}\,|||f'|||_{[a,x_1],\infty}+\sum\limits_{i=1}^{n-1}\mu_1\left(x_i,P_i b+\bar{P}_i a,x_{i+1}\right)|||f'|||_{[x_i,x_{i+1}],\infty} \\
\qquad +\frac{(b-x_n)^2}{2}\,|||f'|||_{[x_n,b],\infty} \quad \textit{if } f'\in L_\infty([a,b];X);
\end{cases}$$

$$\leq \begin{cases} \max\left(x_1 - a, \max_{i=\overline{1,n-1}} \left\{\mu_\infty\left(x_i, P_i b + \bar{P}_i a, x_{i+1}\right)\right\}, b - x_n\right) |||f'|||_{[a,b],1} \\ \hfill \text{if } f' \in L_1([a,b];X); \\ \left[\frac{(x_1-a)^{q+1}}{q+1} + \sum_{i=1}^{n-1} \mu_q\left(x_i, P_i b + \bar{P}_i a, x_{i+1}\right) + \frac{(b-x_n)^{q+1}}{q+1}\right]^{\frac{1}{q}} |||f'|||_{[a,b],p} \\ \hfill \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ and if } f' \in L_p([a,b];X); \\ \left[\frac{(x_1-a)^2}{2} + \sum_{i=1}^{n-1} \mu_1\left(x_i, P_i b + \bar{P}_i a, x_{i+1}\right) + \frac{(b-x_n)^2}{2}\right] |||f'|||_{[a,b],\infty} \\ \hfill \text{if } f' \in L_\infty([a,b];X); \end{cases}$$

where $L_p([a,b];X)$, $p \in [1,\infty]$ *are the usual vector-valued Lebesgue spaces and*

$$|||h|||_{[\alpha,\beta],\infty} := ess \sup_{t\in[\alpha,\beta]} \|h(t)\|,$$

$$|||h|||_{[\alpha,\beta],p} := \left(\int_\alpha^\beta \|h(t)\|^p \, dt\right)^{\frac{1}{p}},\ p \geq 1,$$

and the functions $\mu_q(\cdot,\cdot,\cdot)$, $q \in [1,\infty]$ *were defined above.*

Proof. Using the properties of the norm, we have, by (2.1), that

$$\begin{aligned} &\left\|(B)\int_a^b f(t)\,dt - (b-a)\sum_{i=1}^n p_i f(x_i)\right\| \\ &\qquad \leq \int_a^{x_1} (t-a)\|f'(t)\|\,dt + \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} \left|t - \left(P_i b + \bar{P}_i a\right)\right| \|f'(t)\|\,dt \\ &\qquad\qquad + \int_{x_n}^b (b-t)\|f'(t)\|\,dt \end{aligned}$$

and the first inequality in (2.6) is proved.

Now, observe that

$$\begin{aligned} &\int_a^{x_1} (t-a)\|f'(t)\|\,dt \\ &\qquad \leq \begin{cases} (x_1-a)\,|||f'|||_{[a,x_1],1}; \\ \frac{(x_1-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} |||f'|||_{[a,x_1],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p([a,b];X); \\ \frac{(x_1-a)^2}{2} |||f'|||_{[a,x_1],\infty} & \text{if } f' \in L_\infty([a,b];X); \end{cases} \end{aligned}$$

and

$$\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}\left|t-\left(P_ib+\bar{P}_ia\right)\right|\left\|f'(t)\right\|dt$$

$$\leq\begin{cases}\sup_{t\in[x_i,x_{i+1}]}\left|t-\left(P_ib+\bar{P}_ia\right)\right|\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],1}; \\ \\ \left(\int_{x_i}^{x_{i+1}}\left|t-\left(P_ib+\bar{P}_ia\right)\right|^q dt\right)^{\frac{1}{q}}\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],p} \\ \quad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f'\in L_p([a,b];X); \\ \\ \int_{x_i}^{x_{i+1}}\left|t-\left(P_ib+\bar{P}_ia\right)\right|dt\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],\infty} \quad \text{if } f'\in L_\infty([a,b];X);\end{cases}$$

$$=\begin{cases}\mu_\infty\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],1}; \\ \\ \left[\mu_q\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\right]^{\frac{1}{q}}\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],p} \\ \quad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f'\in L_p([a,b];X); \\ \\ \mu_1\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],\infty} \quad \text{if } f'\in L_\infty([a,b];X);\end{cases}$$

and

$$\int_{x_n}^{b}(b-t)\left\|f'(t)\right\|dt$$

$$\leq\begin{cases}(b-x_n)\left\|\left|f'\right|\right\|_{[x_n,b],1}; \\ \\ \frac{(b-x_n)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|\left|f'\right|\right\|_{[x_n,b],p} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p([a,b];X); \\ \\ \frac{(b-x_n)^2}{2}\left\|\left|f'\right|\right\|_{[x_n,b],\infty} \quad \text{if } f'\in L_\infty([a,b];X);\end{cases}$$

giving the second inequality in (2.6).

Finally, observe that

$$\begin{aligned}&(x_1-a)\left\|\left|f'\right|\right\|_{[a,x_1],1}+\sum_{i=1}^{n-1}\mu_\infty\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],1} \\ &+(b-x_n)\left\|\left|f'\right|\right\|_{[x_n,b],1} \\ \leq\ &\max\left\{x_1-a,\max_{i=\overline{1,n-1}}\left\{\mu_\infty\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\right\},b-x_n\right\} \\ &\times\left[\left\|\left|f'\right|\right\|_{[a,x_1],1}+\sum_{i=1}^{n-1}\left\|\left|f'\right|\right\|_{[x_i,x_{i+1}],1}+\left\|\left|f'\right|\right\|_{[x_n,b],1}\right] \\ \leq\ &\max\left\{x_1-a,\max_{i=\overline{1,n-1}}\left\{\mu_\infty\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\right\},b-x_n\right\}\left\|\left|f'\right|\right\|_{[a,b],1},\end{aligned}$$

by the discrete Hölder's inequality we have that

$$\begin{aligned}
&\frac{(x_1-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}|||f'|||_{[a,x_1],p}+\sum_{i=1}^{n-1}\left[\mu_q\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\right]^{\frac{1}{q}}|||f'|||_{[x_i,x_{i+1}],p}\\
&+\frac{(b-x_n)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}|||f'|||_{[x_n,b],p}\\
\leq\;&\left\{\left[\frac{(x_1-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\right]^q+\sum_{i=1}^{n-1}\left(\left[\mu_q\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)\right]^{\frac{1}{q}}\right)^q\right.\\
&\left.+\left[\frac{(b-x_n)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\right]^q\right\}^{\frac{1}{q}}\times\left[|||f'|||^p_{[x_n,b],p}+|||f'|||^p_{[a,x_1],p}+\sum_{i=1}^{n-1}|||f'|||^p_{[x_i,x_{i+1}],p}\right]^{\frac{1}{p}}\\
=\;&\left[\frac{(x_1-a)^{q+1}}{q+1}+\sum_{i=1}^{n-1}\mu_q\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)+\frac{(b-x_n)^{q+1}}{q+1}\right]^{\frac{1}{q}}|||f'|||_{[a,b],p}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(x_1-a)^2}{2}|||f'|||_{[a,x_1],\infty}+\sum_{i=1}^{n-1}\mu_1\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)|||f'|||_{[x_i,x_{i+1}],\infty}\\
&+\frac{(b-x_n)^2}{2}|||f'|||_{[x_n,b],\infty}\\
\leq\;&\left[\frac{(x_1-a)^2}{2}+\sum_{i=1}^{n-1}\mu_1\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)+\frac{(b-x_n)^2}{2}\right]\\
&\times\max\left\{|||f'|||_{[a,x_1],\infty},\max_{i=\overline{1,n-1}}|||f'|||_{[x_i,x_{i+1}],\infty},|||f'|||_{[x_n,b],\infty}\right\}\\
=\;&\left[\frac{(x_1-a)^2}{2}+\sum_{i=1}^{n-1}\mu_1\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)+\frac{(b-x_n)^2}{2}\right]|||f'|||_{[a,b],\infty}
\end{aligned}$$

and the theorem is completely proved. ∎

It is a natural assumption to consider the weights $p_i>0$ $(i=1,\dots,n)$ for which $\xi_i:=P_ib+\bar{P}_ia$ $(\in[a,b])$ will be in the interval $[x_i,x_{i+1}]$ $(i=1,\dots,n)$. In this case we have:

$$\mu_\infty\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)=\frac{1}{2}h_i+\left|P_ib+\bar{P}_ia-\frac{x_i+x_{i+1}}{2}\right|,$$

where $h_i:=x_{i+1}-x_i$, and for $p\in[1,\infty)$

$$\mu_p\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)=\frac{1}{p+1}\left[\left(P_ib+\bar{P}_ia-x_i\right)^{p+1}+\left(x_{i+1}-P_ib-\bar{P}_ia\right)^{p+1}\right].$$

Note that for $p=1$, we have

$$\mu_1\left(x_i,P_ib+\bar{P}_ia,x_{i+1}\right)=\frac{1}{4}h_i^2+\left(P_ib+\bar{P}_ia-\frac{x_i+x_{i+1}}{2}\right)^2.$$

The following corollary is important for applications.

Corollary 1. *With the assumptions of Lemma 1 and if $x_i \leq P_i b + \bar{P}_i a \leq x_{i+1}$ for each $i = 1, \ldots, n-1$, then we have the inequalities:*

$$\left\| (B) \int_a^b f(t)\, dt - (b-a) \sum_{i=1}^{n} p_i f(x_i) \right\| \tag{2.7}$$

$$\leq \begin{cases} (x_1 - a) \left|\left|\left| f' \right|\right|\right|_{[a,x_1],1} + \sum_{i=1}^{n-1} \left[\frac{1}{2} h_i + \left| P_i b + \bar{P}_i a - \frac{x_i + x_{i+1}}{2} \right| \right] \left|\left|\left| f' \right|\right|\right|_{[x_i,x_{i+1}],1} \\ \qquad + (b - x_n) \left|\left|\left| f' \right|\right|\right|_{[x_n,b],1}; \\[2ex] \frac{(x_1-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left|\left|\left| f' \right|\right|\right|_{[a,x_1],p} \\ \quad + \frac{1}{(q+1)^{\frac{1}{q}}} \sum_{i=1}^{n-1} \left[\left(P_i b + \bar{P}_i a - x_i \right)^{q+1} + \left(x_{i+1} - P_i b - \bar{P}_i a \right)^{q+1} \right]^{\frac{1}{q}} \left|\left|\left| f' \right|\right|\right|_{[x_i,x_{i+1}],p} \\ \qquad + \frac{(b-x_n)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left|\left|\left| f' \right|\right|\right|_{[x_n,b],p} \quad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \qquad \text{and if } f' \in L_p([a,b]; X); \\[2ex] \frac{(x_1-a)^2}{2} \left|\left|\left| f' \right|\right|\right|_{[a,x_1],\infty} + \sum_{i=1}^{n-1} \left[\frac{1}{4} h_i^2 + \left(P_i b + \bar{P}_i a - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left|\left|\left| f' \right|\right|\right|_{[x_i,x_{i+1}],\infty} \\ \qquad + \frac{(b-x_n)^2}{2} \left|\left|\left| f' \right|\right|\right|_{[x_n,b],\infty} \quad \text{if } f' \in L_\infty([a,b]; X); \end{cases}$$

$$\leq \begin{cases} \max\left(x_1 - a, \frac{1}{2} \max_{i=\overline{1,n-1}} h_i + \max_{i=\overline{1,n-1}} \left| P_i b + \bar{P}_i a - \frac{x_i + x_{i+1}}{2} \right|, b - x_n \right) \left|\left|\left| f' \right|\right|\right|_{[a,b],1} \\ \qquad \text{if } f' \in L_1([a,b]; X); \\[2ex] \frac{1}{(q+1)^{\frac{1}{q}}} \left[(x_1 - a)^{q+1} + \sum_{i=1}^{n-1} \left[\left(P_i b + \bar{P}_i a - x_i \right)^{q+1} \right.\right. \\ \qquad \left.\left. + \left(x_{i+1} - P_i b - \bar{P}_i a \right)^{q+1} \right] + (b - x_n)^{q+1} \right]^{\frac{1}{q}} \left|\left|\left| f' \right|\right|\right|_{[a,b],p} \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ f' \in L_p([a,b]; X); \\[2ex] \left[\frac{(x_1-a)^2}{2} + \sum_{i=1}^{n-1} \left[\frac{1}{4} h_i^2 + \left(P_i b + \bar{P}_i a - \frac{x_i + x_{i+1}}{2} \right)^2 \right] + \frac{(b-x_n)^2}{2} \right] \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty} \\ \qquad \text{if } f' \in L_\infty([a,b]; X). \end{cases}$$

Remark 1. *For $n = 1$, we recapture from (2.7) the Ostrowski type inequalities incorporated in Theorems 1 and 2.*

3. The Case of Two Points

The following proposition is a particular case of Corollary 1 for $n = 2$ and will be considered with some details that are important for applications.

Proposition 1. *Let $(X, \|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a,b] \to X$ be an absolutely continuous function on $[a,b]$. If $a \leq x_1 \leq x_2 \leq b$ $(b > a)$ and $t \in [0,1]$ satisfies the condition*

$$(0 \leq) \frac{x_1 - a}{b - a} \leq t \leq \frac{x_2 - a}{b - a} (\leq 1),$$

then we have the inequalities

$$\left\| (B) \int_a^b f(t)\,dt - (b-a)\left[t f(x_2) + (1-t) f(x_1) \right] \right\| \tag{3.1}$$

$$\leq \begin{cases} (x_1 - a) \left|\left|\left| f' \right|\right|\right|_{[a,x_1],1} + \left[\frac{1}{2}(x_2 - x_1) + \left| tb + (1-t)a - \frac{x_1+x_2}{2} \right| \right] \left|\left|\left| f' \right|\right|\right|_{[x_1,x_2],1} \\ + (b - x_2) \left|\left|\left| f' \right|\right|\right|_{[x_2,b],1}; \\ \\ \frac{1}{(q+1)^{1/q}} \left\{ (x_1 - a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[a,x_1],p} \right. \\ + \left[(tb + (1-t)a - x_1)^{q+1} + (x_2 - tb - (1-t)a)^{q+1} \right]^{1/q} \left|\left|\left| f' \right|\right|\right|_{[x_1,x_2],p} \\ \left. + (b - x_2)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[x_2,b],p} \right\},\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, f' \in L_p([a,b];X); \\ \\ \frac{(x_1-a)^2}{2} \left|\left|\left| f' \right|\right|\right|_{[a,x_1],\infty} + \left[\frac{1}{4}(x_2 - x_1)^2 + \left[tb + (1-t)a - \frac{x_1+x_2}{2} \right]^2 \right] \left|\left|\left| f' \right|\right|\right|_{[x_1,x_2],\infty} \\ + \frac{(b-x_2)^2}{2} \left|\left|\left| f' \right|\right|\right|_{[x_2,b],\infty},\ f' \in L_\infty([a,b];X); \end{cases}$$

$$\leq \begin{cases} \max\left\{ x_1 - a, \frac{1}{2}(x_2 - x_1) + \left| tb + (1-t)a - \frac{x_1+x_2}{2} \right|, b - x_2 \right\} \left|\left|\left| f' \right|\right|\right|_{[a,b],1}; \\ \\ \frac{1}{(q+1)^{1/q}} \left\{ (x_1 - a)^{q+1} + (tb + (1-t)a - x_1)^{q+1} + (x_2 - tb - (1-t)a)^{q+1} \right. \\ \left. + (b - x_2)^{q+1} \right\}^{1/q} \left|\left|\left| f' \right|\right|\right|_{[a,b],p},\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, f' \in L_p([a,b];X); \\ \\ \left[\frac{(x_1-a)^2}{2} + \frac{1}{4}(x_2 - x_1)^2 + \left[tb + (1-t)a - \frac{x_1+x_2}{2} \right]^2 + \frac{(b-x_2)^2}{2} \right] \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty}, \\ f' \in L_\infty([a,b];X). \end{cases}$$

The following particular inequalities are of interest.

1. If $x_1 = a, x_2 = b$, then for any $t \in [0,1]$, we have the inequalities

$$\left\| (B) \int_a^b f(t)\,dt - (b-a)\left[t f(b) + (1-t) f(a) \right] \right\| \tag{3.2}$$

$$\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left| tb + (1-t)a - \frac{a+b}{2} \right| \right] \left|\left|\left| f' \right|\right|\right|_{[a,b],1}; \\ \\ \frac{1}{(q+1)^{1/q}} \left[t^{q+1} + (1-t)^{q+1} \right]^{1/q} (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[a,b],p}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, f' \in L_p([a,b];X); \\ \\ \left[\frac{1}{4}(b-a)^2 + \left(tb + (1-t)a - \frac{a+b}{2} \right)^2 \right] \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty}, f' \in L_\infty([a,b];X). \end{cases}$$

The best inequality one can get from (3.2) is for $t = \frac{1}{2}$, obtaining the trapezoidal rule

$$\left\| (B) \int_a^b f(t)\,dt - (b-a) \cdot \frac{f(b)+f(a)}{2} \right\| \tag{3.3}$$
$$\leq \begin{cases} \frac{1}{2}(b-a) \left|\left|\left| f' \right|\right|\right|_{[a,b],1}; \\ \frac{1}{2(q+1)^{1/q}} (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[a,b],p}, p>1, \frac{1}{p}+\frac{1}{q}=1, f' \in L_p([a,b];X); \\ \frac{1}{4}(b-a)^2 \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty}, f' \in L_\infty([a,b];X). \end{cases}$$

2. If $x_1 = \frac{3a+b}{4}, x_2 = \frac{a+3b}{4}$, then for any $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ we have the inequalities

$$\left\| (B) \int_a^b f(t)\,dt - (b-a) \left[tf\left(\frac{3a+b}{4}\right) + (1-t) f\left(\frac{a+3b}{4}\right) \right] \right\| \tag{3.4}$$

$$\leq \begin{cases} \frac{b-a}{4} \left|\left|\left| f' \right|\right|\right|_{\left[a,\frac{3a+b}{4}\right],1} + \left[\frac{1}{4}(b-a) + \left| tb + (1-t)a - \frac{a+b}{2} \right|\right] \left|\left|\left| f' \right|\right|\right|_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right],1} \\ +\frac{b-a}{4} \left|\left|\left| f' \right|\right|\right|_{\left[\frac{a+3b}{4},b\right],1}; \\ \frac{1}{4^{q+1}(q+1)^{1/q}} (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{\left[a,\frac{3a+b}{4}\right],p} \\ +\frac{1}{(q+1)^{1/q}} \left[\left(tb+(1-t)a - \frac{3a+b}{4}\right)^{q+1} \right. \\ \left. + \left(\frac{a+3b}{4} - tb - (1-t)a\right)^{q+1} \right]^{1/q} \left|\left|\left| f' \right|\right|\right|_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right],p} \\ +\frac{1}{4^{q+1}(q+1)^{1/q}} (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{\left[\frac{a+3b}{4},b\right],p}, \; p>1, \frac{1}{p}+\frac{1}{q}=1, f' \in L_p([a,b];X); \\ \frac{(b-a)^2}{8} \left|\left|\left| f' \right|\right|\right|_{\left[a,\frac{3a+b}{4}\right],\infty} + \left[\frac{1}{16}(b-a)^2 + \left[tb+(1-t)a-\frac{a+b}{2}\right]^2\right] \left|\left|\left| f' \right|\right|\right|_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right],\infty} \\ +\frac{(b-a)^2}{8} \left|\left|\left| f' \right|\right|\right|_{\left[\frac{a+3b}{4},b\right],\infty}, \; f' \in L_\infty([a,b];X); \end{cases}$$

$$\leq \begin{cases} \left[\frac{1}{4}(b-a) + \left| tb+(1-t)a-\frac{a+b}{2} \right|\right] \left|\left|\left| f' \right|\right|\right|_{[a,b],1}; \\ \frac{1}{(q+1)^{1/q}} \left[\frac{2(b-a)^{q+1}}{4^{q+1}} + \left(tb+(1-t)a-\frac{3a+b}{4}\right)^{q+1} + \left(\frac{a+3b}{4} - tb - (1-t)a\right)^{q+1} \right]^{1/q} \\ \times (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[a,b],p}, p>1, \frac{1}{p}+\frac{1}{q}=1, f' \in L_p([a,b];X); \\ \left[\frac{1}{8}(b-a)^2 + \left(tb+(1-t)a-\frac{a+b}{2}\right)^2\right] \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty}, f' \in L_\infty([a,b];X). \end{cases}$$

The best inequality one can get from (3.4) is for $t = \frac{1}{2}$, obtaining

$$\left\| (B) \int_a^b f(t)\,dt - (b-a) \cdot \frac{f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)}{2} \right\| \tag{3.5}$$
$$\leq \begin{cases} \frac{1}{4}(b-a) \left|\left|\left| f' \right|\right|\right|_{[a,b],1}; \\ \frac{1}{4(q+1)^{1/q}} (b-a)^{1+1/q} \left|\left|\left| f' \right|\right|\right|_{[a,b],p}, p>1, \frac{1}{p}+\frac{1}{q}=1, f' \in L_p([a,b];X); \\ \frac{1}{8}(b-a)^2 \left|\left|\left| f' \right|\right|\right|_{[a,b],\infty}, f' \in L_\infty([a,b];X). \end{cases}$$

Remark 2. *One may realize that, instead of using the trapezoidal rule in approximating the Bochner integral* $(B)\int_a^b f(t)\,dt$*, that one should use the rule*

$$QT(f;a,b) := (b-a)\,.\frac{f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)}{2}, \tag{3.6}$$

that provides twice better theoretical accuracy.

4. The Case of Three Points

The case of three points is important for applications since it contains amongst other Simpson's quadrature rule.

The following proposition holds.

Proposition 2. *Let* $(X,\|\cdot\|)$ *be a Banach space with the Radon-Nikodym property and* $f:[a,b]\to X$ *be an absolutely continuous function on* $[a,b]$*. If* $a\le x_1\le x_2\le x_3\le b$ $(b>a)$ *and* $\alpha,\beta,\gamma\in[0,1]$ *with* $\alpha+\beta+\gamma=1$ *satisfies the condition*

$$(0\le)\,\frac{x_1-a}{b-a}\le\alpha\le\frac{x_2-a}{b-a}\le\alpha+\beta\le\frac{x_3-a}{b-a}\,(\le 1), \tag{4.1}$$

then we have the inequalities

$$\left\|(B)\int_a^b f(t)\,dt-(b-a)\left[\alpha f(x_1)+\beta f(x_2)+(1-\alpha-\beta)f(x_3)\right]\right\| \tag{4.2}$$

$$\le\begin{cases}(x_1-a)\,|||f'|||_{[a,x_1],1}+\left[\frac{1}{2}(x_2-x_1)+\left|\alpha b+(1-\alpha)a-\frac{x_1+x_2}{2}\right|\right]|||f'|||_{[x_1,x_2],1}\\ +\left[\frac{1}{2}(x_3-x_2)+\left|(\alpha+\beta)b+(1-\alpha-\beta)a-\frac{x_2+x_3}{2}\right|\right]|||f'|||_{[x_2,x_3],1}\\ +(b-x_2)\,|||f'|||_{[x_2,b],1}\,;\\ \\ \frac{1}{(q+1)^{1/q}}\Big\{(x_1-a)^{1+1/q}\,|||f'|||_{[a,x_1],p}\\ +\left[(\alpha b+(1-\alpha)a-x_1)^{q+1}+(x_2-\alpha b-(1-\alpha)a)^{q+1}\right]^{1/q}|||f'|||_{[x_1,x_2],p}\\ +\left[((\alpha+\beta)b+(1-\alpha-\beta)a-x_2)^{q+1}+(x_3-(\alpha+\beta)b-(1-\alpha-\beta)a)^{q+1}\right]^{1/q}|||f'|||_{[x_2,x_3],p}\\ +(b-x_3)^{1+1/q}\,|||f'|||_{[x_3,b],p}\Big\},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p([a,b];X)\,;\\ \\ \frac{(x_1-a)^2}{2}\,|||f'|||_{[a,x_1],\infty}+\left[\frac{1}{4}(x_2-x_1)^2+\left[\alpha b+(1-\alpha)a-\frac{x_1+x_2}{2}\right]^2\right]|||f'|||_{[x_1,x_2],\infty}\\ \left[\frac{1}{4}(x_3-x_2)^2+\left[(\alpha+\beta)b+(1-\alpha-\beta)a-\frac{x_2+x_3}{2}\right]^2\right]|||f'|||_{[x_2,x_3],\infty}\\ +\frac{(b-x_3)^2}{2}\,|||f'|||_{[x_3,b],\infty}\,,\ f'\in L_\infty([a,b];X)\,;\end{cases}$$

$$\leq \begin{cases} \max\left\{x_1 - a, \frac{1}{2}(x_2 - x_1) + \left|\alpha b + (1-\alpha)a - \frac{x_1+x_2}{2}\right|, \right. \\ \left. \frac{1}{2}(x_2 - x_1) + \left|(\alpha+\beta)b + (1-\alpha-\beta)a - \frac{x_2+x_3}{2}\right|, b - x_2\right\} |||f'|||_{[a,b],1}; \\ \\ \frac{1}{(q+1)^{1/q}}\left\{(x_1-a)^{q+1} + (\alpha b + (1-\alpha)a - x_1)^{q+1} + (x_2 - \alpha b - (1-\alpha)a)^{q+1} \right. \\ + ((\alpha+\beta)b + (1-\alpha-\beta)a - x_2)^{q+1} + (x_3 - (\alpha+\beta)b - (1-\alpha-\beta)a)^{q+1} \\ \left. + (b-x_3)^{q+1}\right\} |||f'|||_{[a,b],p}, \ p > 1, \frac{1}{p} + \frac{1}{q} = 1, f' \in L_p([a,b];X); \\ \\ \left[\frac{(x_1-a)^2}{2} + \frac{1}{4}(x_2-x_1)^2 + \left[\alpha b + (1-\alpha)a - \frac{x_1+x_2}{2}\right]^2 \right. \\ \left. + \frac{1}{4}(x_3-x_2)^2 + \left[(\alpha+\beta)b + (1-\alpha-\beta)a - \frac{x_2+x_3}{2}\right]^2 + \frac{(b-x_3)^2}{2}\right] |||f'|||_{[a,b],\infty}, \\ f' \in L_\infty([a,b];X). \end{cases}$$

The following particular inequalities are of interest.

1.Assume that $x_1 = a, x_2 = \frac{a+b}{2}, x_3 = b$ and $\alpha, \beta \in [0,1]$ so that $0 \leq \alpha \leq \frac{1}{2} \leq \alpha + \beta \leq 1$, then we have the inequalities

$$\left\|(B)\int_a^b f(t)\,dt - (b-a)\left[\alpha f(a) + \beta f\left(\frac{a+b}{2}\right) + (1-\alpha-\beta)f(b)\right]\right\| \tag{4.3}$$

$$\leq \begin{cases} \left[\frac{1}{4}(b-a) + \left|\alpha b + (1-\alpha)a - \frac{3a+b}{4}\right|\right] |||f'|||_{\left[a,\frac{a+b}{2}\right],1} \\ + \left[\frac{1}{4}(b-a) + \left|(\alpha+\beta)b + (1-\alpha-\beta)a - \frac{a+3b}{4}\right|\right] |||f'|||_{\left[\frac{a+b}{2},b\right],1}; \\ \\ \frac{1}{(q+1)^{1/q}}\left\{\left[\alpha^{q+1}(b-a)^{q+1} + \left(\frac{a+b}{2} - \alpha b - (1-\alpha)a\right)^{q+1}\right]^{1/q} |||f'|||_{\left[a,\frac{a+b}{2}\right],p} \right. \\ \left. + \left[\left((\alpha+\beta)b + (1-\alpha-\beta)a - \frac{a+b}{2}\right)^{q+1} + (1-\alpha-\beta)^{q+1}(b-a)^{q+1}\right]^{1/q}\right\} |||f'|||_{\left[\frac{a+b}{2},b\right],p} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, f' \in L_p([a,b];X); \\ \\ \left[\frac{1}{16}(b-a)^2 + \left[\alpha b + (1-\alpha)a - \frac{3a+b}{4}\right]^2\right] |||f'|||_{\left[a,\frac{a+b}{2}\right],\infty} \\ \left[\frac{1}{16}(b-a)^2 + \left[(\alpha+\beta)b + (1-\alpha-\beta)a - \frac{a+3b}{4}\right]^2\right] |||f'|||_{\left[\frac{a+b}{2},b\right],\infty} \\ f' \in L_\infty([a,b];X); \end{cases}$$

$$\leq \begin{cases} \max\left\{\left[\frac{1}{4}(b-a)+\left|\alpha b+(1-\alpha)a-\frac{3a+b}{4}\right|\right], \right. \\ \left.\left[\frac{1}{4}(b-a)+\left|(\alpha+\beta)b+(1-\alpha-\beta)a-\frac{a+3b}{4}\right|\right]\right\}\||f'|\|_{[a,b],1} \\ \\ \frac{1}{(q+1)^{1/q}}\left\{\alpha^{q+1}(b-a)^{q+1}+\left(\frac{a+b}{2}-\alpha b-(1-\alpha)a\right)^{q+1}\right. \\ \left.+\left((\alpha+\beta)b+(1-\alpha-\beta)a-\frac{a+b}{2}\right)^{q+1}+(1-\alpha-\beta)^{q+1}(b-a)^{q+1}\right\}^{1/q}\||f'|\|_{[a,b],p} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, f'\in L_p([a,b];X); \\ \\ \left\{\frac{1}{8}(b-a)^2+\left[\alpha b+(1-\alpha)a-\frac{3a+b}{4}\right]^2\right. \\ \left.+\left[(\alpha+\beta)b+(1-\alpha-\beta)a-\frac{a+3b}{4}\right]^2\right\}\||f'|\|_{[a,b],\infty} \\ f'\in L_\infty([a,b];X). \end{cases}$$

It is easy to see that, the best inequality one can derive from (4.3) is the one for $\alpha=\frac{1}{4}$ and $\beta=\frac{3}{4}$, getting

$$\left\|(B)\int_a^b f(t)\,dt-(b-a)\left[\frac{f(a)+f(b)}{4}+\frac{1}{2}f\left(\frac{a+b}{2}\right)\right]\right\| \tag{4.4}$$

$$\leq \begin{cases} \frac{1}{4}(b-a)\||f'|\|_{[a,b],1}; \\ \\ \frac{1}{2^{2+1/q}(q+1)^{1/q}}(b-a)^{1+1/q}\left\{\||f'|\|_{\left[a,\frac{a+b}{2}\right],p}+\||f'|\|_{\left[\frac{a+b}{2},b\right],p}\right\} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, f'\in L_p([a,b];X); \\ \\ \frac{1}{16}(b-a)^2\left[\||f'|\|_{\left[a,\frac{a+b}{2}\right],\infty}+\||f'|\|_{\left[\frac{a+b}{2},b\right],\infty}\right], \ f'\in L_\infty([a,b];X); \end{cases}$$

$$\leq \begin{cases} \frac{1}{4}(b-a)\||f'|\|_{[a,b],1}; \\ \\ \frac{1}{2^{2+1/q}(q+1)^{1/q}}(b-a)^{1+1/q}\||f'|\|_{[a,b],p}, \ p>1, \frac{1}{p}+\frac{1}{q}=1, f'\in L_p([a,b];X); \\ \\ \frac{1}{8}(b-a)^2\||f'|\|_{[a,b],\infty} \ \ f'\in L_\infty([a,b];X). \end{cases}$$

The inequality (4.3) incorporates *Simpson's rule* as well. Indeed, if we choose $\alpha=\frac{1}{6},\beta=\frac{4}{6}$, then we get from (4.3) the following result

$$\left\|(B)\int_a^b f(t)\,dt-\frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right\| \tag{4.5}$$

$$\leq \begin{cases} \frac{1}{3}(b-a)\,|||f'|||_{[a,b],1}\,; \\ \\ \frac{1}{(q+1)^{1/q}}\frac{\left(2^{q+1}+1\right)^{1/q}}{6^{1+1/q}}(b-a)^{1+1/q}\left\{|||f'|||_{\left[a,\frac{a+b}{2}\right],p}+|||f'|||_{\left[\frac{a+b}{2},b\right],p}\right\} \\ p>1,\,\frac{1}{p}+\frac{1}{q}=1,\,f'\in L_p([a,b];X)\,; \\ \\ \frac{5}{72}(b-a)^2\left[|||f'|||_{\left[a,\frac{a+b}{2}\right],\infty}+|||f'|||_{\left[\frac{a+b}{2},b\right],\infty}\right],\ f'\in L_\infty([a,b];X)\,; \end{cases}$$

$$\leq \begin{cases} \frac{1}{3}(b-a)\,|||f'|||_{[a,b],1}\,; \\ \\ \frac{\left(2^{q+1}+1\right)^{1/q}}{2\cdot 3^{1+1/q}(q+1)^{1/q}}(b-a)^{1+1/q}\,|||f'|||_{[a,b],p}\,,\ p>1,\,\frac{1}{p}+\frac{1}{q}=1,\,f'\in L_p([a,b];X)\,; \\ \\ \frac{5}{36}(b-a)^2\,|||f'|||_{[a,b],\infty}\ \ f'\in L_\infty([a,b];X)\,. \end{cases}$$

Remark 3. *It is obvious that, if the values in* $a, \frac{a+b}{2}$ *and* b *of the function* $f : [a,b] \to X$ *are available, then one should choose the rule*

$$QS(f;a,b) := (b-a)\left[\frac{f(a)+f(b)}{4}+\frac{1}{2}f\left(\frac{a+b}{2}\right)\right]$$

that provides a better approximation for the Bochner integral $(B)\int_a^b f(t)\,dt$ *than the classical Simpson's rule.*

2. If one chooses $x_1 = \frac{3a+b}{4}, x_2 = \frac{a+b}{2}, x_3 = \frac{a+3b}{4}$, then by the use of inequality (4.2), that one can derive estimates for the norm of difference

$$(B)\int_a^b f(t)\,dt-(b-a)\left[\alpha f\left(\frac{3a+b}{4}\right)+\beta f\left(\frac{a+b}{2}\right)+(1-\alpha-\beta)\,f\left(\frac{a+3b}{4}\right)\right]$$

in terms of the Lebesgue norms of the derivative f'.

We omit the details.

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