ON SOME GRONWALL TYPE INEQUALITIES INVOLVING ITERATED INTEGRALS

Yeol Je Cho
Department of Mathematics Education
The Research Institute of Natural Sciences
College of Education, Gyeongsang National University
Chinju 660-701, Republic of Korea
E-mail: yjcho@nongde.gsnu.ac.kr

Sever S. Dragomir
School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428, MCMC, Melbourne
Victoria 8001, Australia
E-mail: sever.dragomir@vu.edu.au

Young-Ho Kim
Department of Applied Mathematics
Changwon National University
Changwon 641-773, Republic of Korea
E-mail: yhkim@sarim.changwon.ac.kr

Abstract: In this paper, some new Gronwall type inequalities involving iterated integrals are given.

1. Introduction

Let $u : [\alpha, \alpha + h] \to R$ be a continuous real-valued function satisfying the

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inequality

$$0 \le u(t) \le \int_{\alpha}^{t} [a + bu(s)]\, ds, \quad t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \le ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall [9] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several results on this subject, the inequality of Bellman [3] is very well known:

Let $x(t)$ and $k(t)$ be real valued nonnegative continuous functions for $t \ge \alpha$. If a is a constant, $a \ge 0$, and

$$x(t) \le a + \int_{\alpha}^{t} k(s)x(s)\, ds, \quad t \ge \alpha,$$

then

$$x(t) \le a \exp\left(\int_{\alpha}^{t} k(s)\, ds \right), \quad t \ge \alpha.$$

It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [9] and Guiliano [10]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [14], Bihari [4], and Langenhop [11]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Bykov proved the following interesting integral inequality, which appear in [1, p. 98]:

Let $u(t)$, $b(t)$, $k(t,s)$ and $h(t,s,\sigma)$ be nonnegative continuous functions for $\alpha \le \tau \le s \le t \le \beta$ and suppose that

$$\begin{aligned} u(t) \le a &+ \int_\alpha^t b(s)u(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u(\tau)\,d\tau ds \\ &+ \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)u(\sigma)\,d\sigma\,d\tau ds \end{aligned} \tag{1.1}$$

for any $t \in [\alpha,\beta]$, where $a \ge 0$ is a constant. Then

$$\begin{aligned} u(t) \le a \exp\biggl(&\int_\alpha^t b(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)\,d\tau ds \\ &+ \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)\,d\sigma\,d\tau\,ds\biggr), \quad t \in [\alpha,\beta]. \end{aligned}$$

In this paper, we consider simple inequalities involving iterated integrals in the inequality (1.1) for the case when the function u in the right-hand side of the inequality (1.1) is replaced by the function u^p for some p, and the constant a is replaced by a nonnegative, nondecreasing function $a(t)$. We also provide some related integral inequalities involving iterated integrals.

2. The case $p > 1$

In this section, we state and prove some new nonlinear integral inequalities involving iterated integrals. Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued. Before considering our first integral inequality involving iterated integrals, we need the following lemma, which appears in [1, p. 2].

Lemma 2.1. *Let $b(t)$ and $f(t)$ be continuous function for $t \ge \alpha$, let $v(t)$ be a differentiable function for $t \ge \alpha$ and suppose that*

$$v'(t) \le b(t)v(t) + f(t), \quad t \ge \alpha,$$

and $v(\alpha) \le v_0$. Then we have

$$v(t) \le v_0 \exp\biggl(\int_\alpha^t b(s)\,ds\biggr) + \int_\alpha^t f(s) \exp\biggl(\int_s^t b(\tau)\,d\tau\biggr)\,ds, \quad t \ge \alpha.$$

Theorem 2.2. *Let $u(t)$, $b(t)$, $k(t,s)$ and $h(t,s,\tau)$ be nonnegative continuous functions for $\alpha \leq \tau \leq s \leq t \leq \beta$ and let $p > 1$ be a constant. Suppose $a(t) \geq 0$ is nondecreasing in $J = [\alpha, \beta]$ and*

$$\begin{aligned} u(t) \leq a(t) &+ \int_\alpha^t b(s)u^p(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u^p(\tau)\,d\tau ds \\ &+ \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau ds, \quad t \in [\alpha, \beta]. \end{aligned} \tag{2.1}$$

Then we have

$$u(t) \leq a(t)\left[1 - (p-1)\int_\alpha^t B(s)a^{p-1}(s)\,ds\right]^{\frac{1}{1-p}}, \quad t \in [\alpha, \beta_p), \tag{2.2}$$

where

$$\beta_p = \sup\left\{t \in J : (p-1)\int_\alpha^t B(s)a^{p-1}(s)\,ds < 1\right\}$$

and

$$B(t) = b(t) + \int_\alpha^t k(t,s)\,ds + \int_\alpha^t \int_\alpha^s h(t,s,\tau)\,d\tau\,ds.$$

Proof. We denote the right-hand side of (2.1) by $a(t) + v(t)$. Then, for $\alpha \leq t \leq T < \beta_p$, (2.1) implies $v(\alpha) = 0$, the function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$,

$$u(t) \leq a(t) + v(t) \tag{2.3}$$

and

$$\begin{aligned} v'(t) &= b(t)u^p(t) + \int_\alpha^t k(t,\tau)u^p(\tau)\,d\tau + \int_\alpha^t \int_\alpha^\tau h(t,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau \\ &\leq B(t)[a(t) + v(t)]^p \\ &\leq B(t)[a(t) + v(t)]^{p-1}[a(T) + v(t)], \end{aligned}$$

that is,

$$v'(t) \leq R(t)[a(T) + v(t)], \tag{2.4}$$

where $R(t) = B(t)[a(t) + v(t)]^{p-1}$. Lemma 2.1 and (2.4) imply

$$v(t) \le a(T) \int_\alpha^t R(s) \exp\biggl(\int_s^t R(\tau)\, d\tau \biggr)\, ds$$

and so

$$v(t) + a(T) \le a(T) \exp\biggl(\int_\alpha^t R(s)\, ds \biggr), \quad \alpha \le t \le T.$$

Hence, for $t = T$,

$$v(t) + a(t) \le a(t) \exp\biggl(\int_\alpha^t R(s)\, ds \biggr). \tag{2.5}$$

From (2.5), we successively obtain

$$\begin{aligned}
&[v(t) + a(t)]^{p-1} \le a^{p-1}(t) \exp\biggl(\int_\alpha^t (p-1) R(s)\, ds \biggr), \\
&R(t) \le B(t) a^{p-1}(t) \exp\biggl(\int_\alpha^t (p-1) R(s)\, ds \biggr), \\
&Z(t) \le (p-1) B(t) a^{p-1}(t) \exp\biggl(\int_\alpha^t Z(s)\, ds \biggr),
\end{aligned}$$

where $Z(t) = (p-1)R(t)$. Consequently, we have

$$Z(t) \exp\biggl(-\int_\alpha^t Z(s)\, ds \biggr) \le (p-1) B(t) a^{p-1}(t)$$

or

$$\frac{d}{dt}\biggl[-\exp\biggl(-\int_\alpha^t Z(s)\, ds \biggr) \biggr] \le (p-1) B(t) a^{p-1}(t).$$

Integrating this from α to t yields

$$1 - \exp\biggl(-\int_\alpha^t Z(s)\, ds \biggr) \le \int_\alpha^t (p-1) B(s) a^{p-1}(s)\, ds,$$

from which we conclude that

$$\exp\biggl(\int_\alpha^t R(s)\, ds \biggr) \le \biggl[1 - (p-1) \int_\alpha^t B(s) a^{p-1}(s)\, ds \biggr]^{\frac{1}{1-p}}.$$

This, together with (2.3) and (2.5), implies (2.2). This completes the proof.

In the same manner, we can prove the following theorem:

Theorem 2.3. *Let $u(t)$, $b(t)$, $k(t,s)$ and $\sigma(t)$ be nonnegative continuous functions for $\alpha \le s \le t \le \beta$ and let $p > 1$ be a constant. Suppose that $\sigma(t)$ is nondecreasing in $J = [\alpha, \beta]$ and*

$$u(t) \le \sigma(t)\left\{a_1 + \int_\alpha^t b(s)u^p(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u^p(\tau)\,d\tau ds\right\}$$

for any $t \in [\alpha, \beta]$, where $a_1 \ge 0$ is a constant. Then we have

$$u(t) \le a_1\sigma(t)\exp(\sigma(t))\left[1 - (p-1)a_1^{p-1}\int_\alpha^t B_1(s)\sigma^{p-1}(s)\exp(\sigma(s))\,ds\right]^{\frac{1}{1-p}}$$

for any $t \in [\alpha, \beta_p)$, where $B_1(t) = b(t) + \int_\alpha^t k(t,\tau)\,d\tau$ and

$$\beta_p = \sup\{t \in J : (p-1)a_1^{p-1}\int_\alpha^t B_1(s)\sigma^{p-1}(s)\exp(\sigma(s))\,ds < 1\}.$$

Let $\alpha < \beta$, and set $J_i = \{(t_1, t_2, \ldots, t_i) \in R^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\}$, $i = 1, \cdots, n$.

Theorem 2.4. *Let $u(t), a(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$ and let $p > 1$ be a constant. Suppose that $\frac{a(t)}{b(t)}$ is nondecreasing in J and*

$$\begin{aligned} u(t) \le a(t) + b(t)\Big[&\int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)u^p(t_n)\,dt_n\right)\cdots\right)dt_1\Big] \end{aligned} \tag{2.6}$$

for any $t \in J$, where $k_i(t, t_1, \ldots, t_i)$ are nonnegative continuous functions in J_{i+1} for $i = 1, 2, \cdots, n$. Suppose thta the partial derivatives $\frac{\partial k_i}{\partial t}(t, t_1, \cdots, t_i)$ exist and are nonnegative and continuous in J_{i+1} for $i = 1, 2, \cdots, n$. Then, for any $t \in J$,

$$u(t) \le a(t)\left[1 - (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1}(R[b^p](s) + Q[b^p](s))\,ds\right]^{\frac{1}{1-p}} \tag{2.7}$$

for any $t \in [\alpha, \tilde{\beta}_p)$*, where*

$$\tilde{\beta}_p = \sup\{t \in J : (p-1)a_1^{p-1} \int_\alpha^t (a(s)/b(s))^{p-1}(R[b^p](s) + Q[b^p](s)\, ds < 1\},$$

$$\begin{aligned} R[w](t) &= k_1(t,t)w(t) + \int_\alpha^t k_2(t,t,t_2)w(t_2)dt_2 \\ &\quad + \sum_{i=3}^n \int_\alpha^t \left(\int_\alpha^{t_2} \cdots \left(\int_\alpha^{t_{i-1}} k_i(t,t,t_2,\cdots,t_i)w(t_i)\, dt_i\right)\cdots\right) dt_2, \\ Q[w](t) &= \int_\alpha^t \frac{\partial k_1}{\partial t}(t,t_1)w(t_1)\, dt_1 \\ &\quad + \sum_{i=2}^n \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{i-1}} \frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)w(t_i)\, dt_i\right)\cdots\right) dt_1 \end{aligned}$$

for each continuous function $w(t)$ *in* J*.*

Proof. First, we note that $R[w]$ and $Q[w]$ are linear functionals and

$$R[w_1] \le R[w_2], \quad Q[w_1] \le Q[w_2]$$

if $w_1(t) \le w_2(t)$ for any $t \in J$ and

$$R[w_1 w_2] \le R[w_1]w_2, \quad Q[w_1 w_2] \le Q[w_1]w_2$$

if $w_1(t)$ is nonnegative in J and $w_2(t)$ is nondecreasing in J. We set

$$\begin{aligned} v(t) &= \int_\alpha^t k_1(t,t_1)u^p(t_1)\, dt_1 + \cdots \\ &\quad + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)u^p(t_n)\, dt_n\right)\cdots\right)dt_1. \end{aligned}$$

Then, for $\alpha \le t \le T < \beta_p$, (2.6) implies $v(\alpha) = 0$, the function $v(t)$ is nondecreasing,

$$u(t) \le a(t) + b(t)v(t) \tag{2.8}$$

and we have

$$v'(t) = R[u^p](t) + Q[u^p](t) \le (R[b^p](t) + Q[b^p](t))(\frac{a(t)}{b(t)} + v(t))^p,$$

that is,

$$v'(t) \le R(t)[a(T)/b(T) + v(t)], \tag{2.9}$$

where $R(t) = (R[b^p](t) + Q[b^p](t))[a(t)/b(t) + v(t)]^{p-1}$. Lemma 2.1 and (2.9) imply

$$v(t) + \frac{a(T)}{b(T)} \le \frac{a(T)}{b(T)} \exp\left(\int_\alpha^t R(s)\,ds\right), \quad \alpha \le t \le T.$$

Hence, for $t = T$,

$$v(t) + \frac{a(t)}{b(t)} \le \frac{a(t)}{b(t)} \exp\left(\int_\alpha^t R(s)\,ds\right). \tag{2.10}$$

From (2.10), we successively obtain

$$\left[v(t) + \frac{a(t)}{b(t)}\right]^{p-1} \le \left[\frac{a(t)}{b(t)}\right]^{p-1} \exp\left(\int_\alpha^t (p-1)R(s)\,ds\right),$$

$$R(t) \le (R[b^p](t) + Q[b^p](t))\left[\frac{a(t)}{b(t)}\right]^{p-1} \exp\left(\int_\alpha^t (p-1)R(s)\,ds\right),$$

$$Z(t) \le (p-1)(R[b^p](t) + Q[b^p](t))\left[\frac{a(t)}{b(t)}\right]^{p-1} \exp\left(\int_\alpha^t (p-1)R(s)\,ds\right),$$

where $Z(t) = (p-1)R(t)$. Consequently, we have

$$\frac{d}{dt}\left[-\exp\left(-\int_\alpha^t Z(s)\,ds\right)\right] \le (p-1)(R[b^p](t) + Q[b^p](t))\left[\frac{a(t)}{b(t)}\right]^{p-1}.$$

Integrating this from α to t yields

$$\begin{aligned} &1 - \exp\left(-\int_\alpha^t Z(s)\,ds\right) \\ &\le (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} (R[b^p](s) + Q[b^p](s))\,ds, \end{aligned}$$

from which we conclude that

$$\begin{aligned} &\exp\left(\int_\alpha^t R(s)\,ds\right) \\ &\le \left[1 - (p-1)\int_\alpha^t \left(\frac{a(s)}{b(s)}\right)^{p-1} (R[b^p](s) + Q[b^p](s))\,ds\right]^{\frac{1}{1-p}}. \end{aligned}$$

This, together with (2.8) and (2.10), implies (2.7). This completes the proof.

3. The case $p > 0$ $(p \neq 1)$

In this section, we use another method for studying nonlinear integral inequalities. Before considering the first result of the integral inequality, we need the following lemma, which appears in [1, p. 38].

Lemma 3.1. *Let $v(t)$ be a positive differential function satisfying the inequality*

$$v'(t) \leq b(t)v(t) + k(t)v^p(t), \quad t \in J = [\alpha, \beta],$$

where the functions b and k are continuous in J and $p \geq 0$ $(p \neq 1)$ is a constant. Then we have

$$v(t) \leq \exp\biggl(\int_\alpha^t b(s)\,ds\biggr)\biggl[v^q(\alpha) + q\int_\alpha^t k(s)\exp\biggl(-q\int_\alpha^s b(\tau)\,d\tau\biggr)\,ds\biggr]^{1/q}$$

for any $t \in [\alpha, \beta_1)$, where β_1 is chosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha, \beta_1)$.

An essential element in the investigation of the integral inequalities in the following theorems is the application of the result of Lemma 3.1.

Theorem 3.2. *Let $u(t)$, $b(t)$, $k(t,s)$, $h(t,s,\sigma)$ be nonnegative continuous functions for $\alpha \leq \sigma \leq s \leq t \leq \beta$ and suppose that*

$$\begin{aligned} u(t) \leq a &+ \int_\alpha^t b(s)u^p(s)\,ds + \int_\alpha^t\int_\alpha^s k(s,\tau)u^p(\tau)\,d\tau ds \\ &+ \int_\alpha^t\int_\alpha^s\int_\alpha^\tau h(s,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau ds \end{aligned} \tag{3.1}$$

for any $t \in [\alpha, \beta]$, where $a > 0$ and $p \geq 0$ $(p \neq 1)$ are a constants. Then we have

$$\begin{aligned} u(t) \leq \biggl[a^q + q\int_\alpha^t\biggl(b(s) &+ \int_\alpha^s k(s,\tau)\,d\tau \\ &+ \int_\alpha^s\int_\alpha^\tau h(s,\tau,\sigma)\,d\sigma\,d\tau\biggr)\,ds\biggr]^{1/q} \end{aligned} \tag{3.2}$$

for any $t \in [\alpha, \beta_1)$, where $q = 1 - p$ and β_1 is chosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha, \beta_1)$.

Proof. We denote the right-hand side of (3.1) by the function $v(t)$. Then the function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$, $u(t) \le v(t)$, $v(\alpha) = a$ and

$$\begin{aligned} v'(t) &= b(t)u^p(t) + \int_\alpha^t k(t,\tau)u^p(\tau)\,d\tau + \int_\alpha^t \int_\alpha^\tau h(t,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau \\ &\le b(t)v^p(t) + \int_\alpha^t k(t,\tau)v^p(\tau)\,d\tau + \int_\alpha^t \int_\alpha^\tau h(t,\tau,\sigma)v^p(\sigma)\,d\sigma\,d\tau \\ &\le \left(b(t) + \int_\alpha^t k(t,\tau)\,d\tau + \int_\alpha^t \int_\alpha^\tau h(t,\tau,\sigma)\,d\sigma\,d\tau\right)v^p(t). \end{aligned}$$

Therefore, applying Lemma 3.1, we arrive at (3.2). This completes the proof.

Theorem 3.3. *Let $u(t)$, $b(t)$, $k(t,s)$, $h(t,s,\sigma)$ be nonnegative continuous functions for $\alpha \le \sigma \le s \le t \le \beta$ and suppose that*

$$\begin{aligned} u(t) \le a(t) &+ \int_\alpha^t b(s)u^p(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u^p(\tau)\,d\tau ds \\ &+ \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau ds \end{aligned} \tag{3.3}$$

for any $t \in [\alpha, \beta]$, where $a(t)$ is a positive nondecreasing function and $p \ge 0$ $(p \ne 1)$ is a constant. Then we have

$$\begin{aligned} u(t) \le \Bigg[A^q(t) + q\int_\alpha^t \Bigg(b(s) &+ \int_\alpha^s k(s,\tau)\,d\tau \\ &+ \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)\,d\sigma\,d\tau\Bigg)\,ds\Bigg]^{\frac{1}{q}} \end{aligned} \tag{3.4}$$

for any $t \in [\alpha, \beta_1)$, where $q = 1 - p$, $A(t) = \sup_{s \in [\alpha,t]} a(s)$ and β_1 is chosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha, \beta_1)$.

Proof. The function $A(t)$ is nondecreasing in $t \in [\alpha, \beta]$. Thus (3.3) implies that, for all $\alpha \le t \le T \le \beta$,

$$\begin{aligned} u(t) \le A(T) &+ \int_\alpha^t b(s)u^p(s)\,ds + \int_\alpha^t \int_\alpha^s k(s,\tau)u^p(\tau)\,d\tau ds \\ &+ \int_\alpha^t \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)u^p(\sigma)\,d\sigma\,d\tau ds. \end{aligned} \tag{3.5}$$

We denote the right-hand side of (3.5) by the function $v(t)$. Then the function $v(t)$ is nondecreasing in $t \in [\alpha, \beta]$, $u(t) \le v(t)$, $v(\alpha) = A(T)$ and

$$v'(t) \le \left(b(t) + \int_\alpha^t k(t,\tau)\,d\tau + \int_\alpha^t \int_\alpha^\tau h(t,\tau,\sigma)\,d\sigma\,d\tau \right) v^p(t).$$

Consequently, Lemma 3.1 implies

$$u(t) \le \left[A^q(T) + q\int_\alpha^t \left(b(s) + \int_\alpha^s k(s,\tau)\,d\tau + \int_\alpha^s \int_\alpha^\tau h(s,\tau,\sigma)\,d\sigma\,d\tau \right) ds \right]^{\frac{1}{q}}$$

and, for $t = T$, we obtain (3.4). This completes the proof.

Let $\alpha < \beta$, and set

$$J_i = \{(t_1, t_2, \ldots, t_i) \in R^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\}$$

for $i = 1, \cdots, n$. By a similar reasoning to the proof of Theorem 3.2, we also can prove the following result:

Theorem 3.4. *Let $u(t)$, and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$ and suppose that*

$$u(t) \le b(t)\Big[a + \int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)u^p(t_n)\,dt_n \right) \cdots \right) dt_1 \Big]$$

for any $t \in J$, where $a > 0$ and $p \ge 0$ $(p \ne 1)$ is a constant, $k_i(t, t_1, \cdots, t_i)$ are nonnegative continuous functions in J_{i+1} for $i = 1, 2, \cdots, n$. Suppose that the partial derivatives $\frac{\partial k_i}{\partial t}(t, t_1, \cdots, t_i)$ exist and are nonnegative and continuous in J_{i+1} for $i = 1, 2, \cdots, n$. Then, for any $t \in J$,

$$u(t) \le b(t)\left[a^q + q\int_\alpha^t (R[b^p](s) + Q[b^p](s))\,ds \right]^{1/q} \tag{3.6}$$

for any $t \in [\alpha, \beta_1)$, where $q = 1 - p$, β_1 is chosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha, \beta_1)$,

$$\begin{aligned} R[w](t) &= k_1(t,t)w(t) + \int_\alpha^t k_2(t,t,t_2)w(t_2)dt_2 \\ &\quad + \sum_{i=3}^{n} \int_\alpha^t \left(\int_\alpha^{t_2} \cdots \left(\int_\alpha^{t_{i-1}} k_i(t,t,t_2,\cdots,t_i)w(t_i)\,dt_i \right) \cdots \right) dt_2, \\ Q[w](t) &= \int_\alpha^t \frac{\partial k_1}{\partial t}(t,t_1)w(t_1)\,dt_1 \\ &\quad + \sum_{i=2}^{n} \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{i-1}} \frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)w(t_i)\,dt_i \right) \cdots \right) dt_1 \end{aligned}$$

for each continuous function $w(t)$ in J.

Proof. We set

$$\begin{aligned} v(t) &= a + \int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots \\ &\quad + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)u^p(t_n)\,dt_n \right) \cdots \right) dt_1. \end{aligned}$$

Since $v(\alpha) = a$, $u(t) \le b(t)v(t)$ and $v(t)$ is nondecreasing and continuous in J, we have

$$\begin{aligned} v'(t) &= R[u^p](t) + Q[u^p](t) \le R[b^p u^p](t) + Q[b^p u^p](t) \\ &\le (R[b^p](t) + Q[b^p](t))v^p(t), \end{aligned}$$

from which, by the same method as in the proof of Theorem 3.2, we find the inequality (3.6). This completes the proof.

Corollary 3.5. *Let $u(t)$ be nonnegative continuous function for $\alpha \le t \le \beta$and suppose that*

$$u(t) \le a + \int_\alpha^t k_1(t,s)u^p(s)\,ds + \int_\alpha^t \left(\int_\alpha^s h(t,s,\sigma)u^p(\sigma)\,d\sigma \right) ds,$$

where $a > 0$ and $p \ge 0$ $(p \ne 1)$ is a constant, $k(t,s)$ and $h(t,s,\sigma)$ are nonnegative continuous functions for $\alpha \le \sigma \le s \le t \le \beta$. Suppose that *the partial derivatives* $\frac{\partial k}{\partial t}(t,s)$ *and* $\frac{\partial h}{\partial t}(t,s,\sigma)$ *exist and are nonnegative and continuous for* $\alpha \le \sigma \le s \le t \le \beta$. *Then, for any* $t \in J$,

$$u(t) \le \left[a^q + q\int_\alpha^t (R(s)+Q(s))\,ds\right]^{1/q}, \quad t \in [\alpha, \beta_1),$$

where $q = 1-p$, β_1 *is chosen so that the expression between* $[\cdots]$ *is positive in the subinterval* $[\alpha, \beta_1)$,

$$R(t) = k(t,t) + \int_\alpha^t h(t,t,\sigma)d\sigma$$

and

$$Q(t) = \int_\alpha^t \frac{\partial k}{\partial t}(t,\sigma)\,d\sigma + \int_\alpha^t \left(\int_\alpha^s \frac{\partial h}{\partial t}(t,s,\sigma)\,d\sigma\right) ds.$$

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