SOME COMPANIONS OF GRÜSS INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some companions of Grüss inequality in inner product spaces and applications for integrals are given.

1. Introduction

The following inequality of Grüss type in real or complex linear spaces is known (see [1]).

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $e \in H$, $\|e\| = 1$. If $\phi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the condition*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad and \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{1.1}$$

or, equivalently (see [2]),

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi| \quad and \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.2}$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| . \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The following particular instances for integrals and means are useful in applications.

Corollary 1. *Let $f, g : [a, b] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ be Lebesgue measurable and such that there exists the constants $\phi, \gamma, \Phi, \Gamma \in \mathbb{K}$ with the property*

$$\operatorname{Re} \left[(\Phi - f(x)) \left(\overline{f(x)} - \overline{\phi} \right) \right] \geq 0, \quad \operatorname{Re} \left[(\Gamma - g(x)) \left(\overline{g(x)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.4}$$

for a.e. $x \in [a, b]$, or, equivalently

$$\left| f(x) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \quad and \quad \left| g(x) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.5}$$

for a.e. $x \in [a, b]$.

Date: March 12, 2003.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 46C05.

Key words and phrases. Grüss inequality, Inner products, Integral inequality, Inequalities for sums.

Then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\overline{g(x)}dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b\overline{g(x)}dx\right| \le \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is incorporated in

Corollary 2. *Let $\mathbf{x},\mathbf{y}\in\mathbb{K}^n$ and $\phi,\gamma,\Phi,\Gamma\in\mathbb{K}$ be such that*

$$\operatorname{Re}\left[(\Phi-x_i)\left(\overline{x_i}-\overline{\phi}\right)\right]\ge 0 \quad \text{and} \quad \operatorname{Re}\left[(\Gamma-y_i)\left(\overline{y_i}-\overline{\gamma}\right)\right]\ge 0, \tag{1.7}$$

for each $i\in\{1,\dots,n\}$, or, equivalently,

$$\left|x_i-\frac{\phi+\Phi}{2}\right|\le\frac{1}{2}|\Phi-\phi| \quad \text{and} \quad \left|y_i-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma|, \tag{1.8}$$

for each $i\in\{1,\dots,n\}$.

Then we have the inequality

$$\left|\frac{1}{n}\sum_{i=1}^n x_i\overline{y_i}-\frac{1}{n}\sum_{i=1}^n x_i\cdot\frac{1}{n}\sum_{i=1}^n\overline{y_i}\right|\le\frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.9}$$

The constant $\frac{1}{4}$ is best possible in (1.9).

For some recent results related to Grüss type inequalities in inner product spaces, see [2]. More applications of Theorem 1 for integral and discrete inequalities may be found in [3].

It is the main aim of this paper to provide other inequalities of Grüss type in the general seting of inner product spaces over the real or complex number field $\mathbb{K}$. Applications for Lebsegue integrals are pointed out as well.

2. A Grüss Type Inequality

The following Grüss type inequality in inner product spaces holds.

Theorem 2. *Let $x,y,e\in H$ with $\|e\|=1$, and the scalars $a,A,b,B\in\mathbb{K}$ ($\mathbb{K}=\mathbb{C},\mathbb{R}$) such that $\operatorname{Re}(\bar{a}A)>0$ and $\operatorname{Re}\left(\bar{b}B\right)>0$. If*

$$\operatorname{Re}\langle Ae-x,x-ae\rangle\ge 0 \quad \text{and} \quad \operatorname{Re}\langle Be-y,y-be\rangle\ge 0 \tag{2.1}$$

or, equivalently (see [2]),

$$\left\|x-\frac{a+A}{2}e\right\|\le\frac{1}{2}|A-a| \quad \text{and} \quad \left\|y-\frac{b+B}{2}e\right\|\le\frac{1}{2}|B-b|, \tag{2.2}$$

then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\le\frac{1}{4}M(a,A)\,M(b,B)\,|\langle x,e\rangle\langle e,y\rangle|, \tag{2.3}$$

where $M(\cdot,\cdot)$ is defined by

$$M(a,A):=\left[\frac{(|A|-|a|)^2+4\left[|A\bar{a}|-\operatorname{Re}(A\bar{a})\right]}{\operatorname{Re}(\bar{a}A)}\right]^{\frac{1}{2}}. \tag{2.4}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Apply Schwartz's inequality in $(H;\langle\cdot,\cdot\rangle)$ for the vectors $x-\langle x,e\rangle e$ and $y-\langle y,e\rangle e$, to get (see also [1])

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|^2\leq\left(\|x\|^2-|\langle x,e\rangle|^2\right)\left(\|y\|^2-|\langle y,e\rangle|^2\right). \tag{2.5}$$

Now, assume that $u,v\in H$, and $c,C\in\mathbb{K}$ with $\operatorname{Re}(\bar{c}C)>0$ and $\operatorname{Re}\langle Cv-u,u-cv\rangle\geq 0$. This last inequality is equivalent to

$$\|u\|^2+\operatorname{Re}(\bar{c}C)\|v\|^2\leq\operatorname{Re}\left[C\overline{\langle u,v\rangle}+\bar{c}\langle u,v\rangle\right]. \tag{2.6}$$

Dividing this inequality by $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}}>0$, we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u\|^2+[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v\|^2\leq\frac{\operatorname{Re}\left[C\overline{\langle u,v\rangle}+\bar{c}\langle u,v\rangle\right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}. \tag{2.7}$$

On the other hand, by the elementary inequality

$$\alpha p^2+\frac{1}{\alpha}q^2\geq 2pq,\quad \alpha>0,\ p,q\geq 0,$$

we deduce

$$2\|u\|\|v\|\leq\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u\|^2+[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v\|^2. \tag{2.8}$$

Making use of (2.7) and (2.8) and the fact that for any $z\in\mathbb{C}$, $\operatorname{Re}(z)\leq|z|$, we get

$$\|u\|\|v\|\leq\frac{\operatorname{Re}\left[C\overline{\langle u,v\rangle}+\bar{c}\langle u,v\rangle\right]}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\leq\frac{|c|+|C|}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}|\langle u,v\rangle|.$$

Consequently

$$\begin{aligned}\|u\|^2\|v\|^2-|\langle u,v\rangle|^2&\leq\left[\frac{(|c|+|C|)^2}{4[\operatorname{Re}(\bar{c}C)]}-1\right]|\langle u,v\rangle|^2\\&=\frac{1}{4}\frac{(|c|-|C|)^2+4[|\bar{c}C|-\operatorname{Re}(\bar{c}C)]}{\operatorname{Re}(\bar{c}C)}|\langle u,v\rangle|^2\\&=\frac{1}{4}M^2(c,C)|\langle u,v\rangle|^2.\end{aligned} \tag{2.9}$$

Now, if we write (2.9) for the choices $u=x$, $v=e$ and $u=y$, $v=e$ respectively and use (2.5), we deduce the desired result (2.2). The sharpness of the constant will be proved in the case where H is a real inner product space. ∎

The following corollary which provides a simpler Grüss type inequality for real constants (and in particular, for real inner product spaces) holds.

Corollary 3. *With the assumptions of Theorem 2 and if $a,b,A,B\in\mathbb{R}$ are such that $A>a>0$, $B>b>0$ and*

$$\left\|x-\frac{a+A}{2}e\right\|\leq\frac{1}{2}(A-a)\quad\text{and}\quad\left\|y-\frac{b+B}{2}e\right\|\leq\frac{1}{2}(B-b), \tag{2.10}$$

then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}\cdot\frac{(A-a)(B-b)}{\sqrt{abAB}}|\langle x,e\rangle\langle e,y\rangle|. \tag{2.11}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The prove the sharpness of the constant $\frac{1}{4}$ assume that the inequality (2.11) holds in real inner product spaces with $x = y$ and for a constant $k > 0$, i.e.,

$$\|x\|^2 - |\langle x, e\rangle|^2 \le k \cdot \frac{(A-a)^2}{aA} |\langle x, e\rangle|^2 \quad (A > a > 0), \tag{2.12}$$

provided $\left\|x - \frac{a+A}{2}e\right\| \le \frac{1}{2}(A-a)$, or equivalently, $\langle Ae - x, x - ae\rangle \ge 0$.

We choose $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then we have

$$\|x\|^2 - |\langle x, e\rangle|^2 = x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{(x_1 - x_2)^2}{2},$$
$$|\langle x, e\rangle|^2 = \frac{(x_1 + x_2)^2}{2},$$

and by (2.12) we get

$$\frac{(x_1 - x_2)^2}{2} \le k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(x_1 + x_2)^2}{2}. \tag{2.13}$$

Now, if we let $x_1 = \frac{a}{\sqrt{2}}$, $x_2 = \frac{A}{\sqrt{2}}$ $(A > a > 0)$, then obviously

$$\langle Ae - x, x - ae\rangle = \sum_{i=1}^{2} \left(\frac{A}{\sqrt{2}} - x_i\right)\left(x_i - \frac{a}{\sqrt{2}}\right) = 0,$$

which shows that the condition (2.10) is fulfilled, and by (2.13) we get

$$\frac{(A-a)^2}{4} \le k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(a+A)^2}{4}$$

for any $A > a > 0$. This implies

$$(A + a)^2 k \ge aA \tag{2.14}$$

for any $A > a > 0$.

Finally, let $a = 1 - q$, $A = 1 + q$, $q \in (0, 1)$. Then from (2.14) we get $4k \ge 1 - q^2$ for any $q \in (0, 1)$ which produces $k \ge \frac{1}{4}$. ∎

Remark 1. *If $\langle x, e\rangle, \langle y, e\rangle$ are assumed not to be zero, then the inequality (2.3) is equivalent to*

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle \langle e, y\rangle} - 1\right| \le \frac{1}{4} M(a, A) M(b, B), \tag{2.15}$$

while the inequality (2.11) is equivalent to

$$\left|\frac{\langle x, y\rangle}{\langle x, e\rangle \langle e, y\rangle} - 1\right| \le \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}}. \tag{2.16}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

3. Some Related Results

The following result holds.

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$. If $\gamma,\Gamma\in\mathbb{K}$, $e,x,y\in H$ with $\|e\|=1$ and $\lambda\in(0,1)$ are such that*

$$\operatorname{Re}\langle\Gamma e-(\lambda x+(1-\lambda)y),(\lambda x+(1-\lambda)y)-\gamma e\rangle\geq 0, \tag{3.1}$$

or, equivalently,

$$\left\|\lambda x+(1-\lambda)y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{3.2}$$

then we have the inequality

$$\operatorname{Re}[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle]\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \tag{3.3}$$

The constant $\frac{1}{16}$ is the best possible constant in (3.3) in the sense that it cannot be replaced by a smaller one.

Proof. We know that for any $z,u\in H$ one has

$$\operatorname{Re}\langle z,u\rangle\leq\frac{1}{4}\|z+u\|^2.$$

Then for any $a,b\in H$ and $\lambda\in(0,1)$ one has

$$\operatorname{Re}\langle a,b\rangle\leq\frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b\|^2. \tag{3.4}$$

Since

$$\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle=\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle\qquad(\text{as }\|e\|=1),$$

using (3.4), we have

$$\begin{aligned}\operatorname{Re}&[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle]\\&=\operatorname{Re}[\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle]\\&\leq\frac{1}{4\lambda(1-\lambda)}\|\lambda(x-\langle x,e\rangle e)+(1-\lambda)(y-\langle y,e\rangle e)\|^2\\&=\frac{1}{4\lambda(1-\lambda)}\|\lambda x+(1-\lambda)y-\langle\lambda x+(1-\lambda)y,e\rangle e\|^2.\end{aligned} \tag{3.5}$$

Since, for $m,e\in H$ with $\|e\|=1$, one has the equality

$$\|m-\langle m,e\rangle e\|^2=\|m\|^2-|\langle m,e\rangle|^2, \tag{3.6}$$

then by (3.5) we deduce the inequality

$$\begin{aligned}\operatorname{Re}&[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle]\\&\leq\frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)y\|^2-|\langle\lambda x+(1-\lambda)y,e\rangle|^2\right].\end{aligned} \tag{3.7}$$

Now, if we apply Grüss' inequality

$$0\leq\|a\|^2-|\langle a,e\rangle|^2\leq\frac{1}{4}|\Gamma-\gamma|^2$$

provided

$$\operatorname{Re}\langle\Gamma e-a,a-\gamma e\rangle\geq 0,$$

for $a=\lambda x+(1-\lambda)y$, we have

$$\left\|\lambda x+(1-\lambda)y\right\|^2-\left|\left\langle \lambda x+(1-\lambda)y,e\right\rangle\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2. \tag{3.8}$$

Utilising (3.7) and (3.8) we deduce the desired inequality (3.3). To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C>0$, provided (3.1) is valid, i.e.,

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\leq C\cdot\frac{1}{\lambda(1-\lambda)}\left|\Gamma-\gamma\right|^2. \tag{3.9}$$

If in (3.9) we choose $x=y$, provided (3.1) holds with $x=y$ and $\lambda\in(0,1)$, then

$$\|x\|^2-\left|\langle x,e\rangle\right|^2\leq C\cdot\frac{1}{\lambda(1-\lambda)}\left|\Gamma-\gamma\right|^2, \tag{3.10}$$

provided

$$\operatorname{Re}\langle \Gamma e-x,x-\gamma e\rangle\geq 0.$$

Since we know, in Grüss' inequality, the constant $\frac{1}{4}$ is best possible, then by (3.10), one has

$$\frac{1}{4}\leq\frac{C}{\lambda(1-\lambda)}\quad\text{for}\quad\lambda\in(0,1),$$

giving, for $\lambda=\frac{1}{2}$, $C\geq\frac{1}{16}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above result.

Corollary 4. *Assume that γ,Γ,e,x,y and λ are as in Theorem 3. If*

$$\operatorname{Re}\left\langle \Gamma e-(\lambda x\pm(1-\lambda)y),(\lambda x\pm(1-\lambda)y)-\gamma e\right\rangle\geq 0, \tag{3.11}$$

or, equivalently,

$$\left\|\lambda x\pm(1-\lambda)y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|^2, \tag{3.12}$$

then we have the inequality

$$\left|\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\right|\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\left|\Gamma-\gamma\right|^2. \tag{3.13}$$

The constant $\frac{1}{16}$ is best possible in (3.13).

Proof. Using Theorem 3 for $(-y)$ instead of y, we have that

$$\operatorname{Re}\left\langle \Gamma e-(\lambda x-(1-\lambda)y),(\lambda x-(1-\lambda)y)-\gamma e\right\rangle\geq 0,$$

which implies that

$$\operatorname{Re}\left[-\langle x,y\rangle+\langle x,e\rangle\langle e,y\rangle\right]\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\left|\Gamma-\gamma\right|^2$$

giving

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\geq-\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\left|\Gamma-\gamma\right|^2. \tag{3.14}$$

Consequently, by (3.3) and (3.14) we deduce the desired inequality (3.13). ∎

Remark 2. *If $M, m \in \mathbb{R}$ with $M > m$ and, for $\lambda \in (0,1)$,*

$$\left\| \lambda x + (1-\lambda) y - \frac{M+m}{2} e \right\| \leq \frac{1}{2} (M-m) \tag{3.15}$$

then

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{16} \cdot \frac{1}{\lambda (1-\lambda)} (M-m)^2 .$$

If (3.15) holds with "$\pm$" instead of "$+$", then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{16} \cdot \frac{1}{\lambda (1-\lambda)} (M-m)^2 . \tag{3.16}$$

Remark 3. *If $\lambda = \frac{1}{2}$ in (3.1) or (3.2), then we obtain the result from* [2], *i.e.,*

$$\operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0 \tag{3.17}$$

or, equivalently

$$\left\| \frac{x+y}{2} - \frac{\gamma+\Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.18}$$

implies

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2 . \tag{3.19}$$

The constant $\frac{1}{4}$ is best possible in (3.19).

For $\lambda = \frac{1}{2}$, Corollary 4 and Remark 2 will produce the corresponding results obtained in [2]. We omit the details.

4. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and $2-$integrable on Ω, i.e.,

$$\int_\Omega |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds

Proposition 1. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, are so that $\operatorname{Re}(\Phi \overline{\varphi}) > 0, \operatorname{Re}(\Gamma \overline{\gamma}) > 0, \int_\Omega |h(s)|^2 d\mu(s) = 1$ and*

$$\int_\Omega \operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi} \overline{h(s)} \right) \right] d\mu(s) \geq 0 \tag{4.1}$$

$$\int_\Omega \operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma} \overline{h(s)} \right) \right] d\mu(s) \geq 0$$

or, equivalently

$$\left(\int_\Omega \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi| , \tag{4.2}$$

$$\left(\int_\Omega \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma| ,$$

then we have the following Grüss type integral inequality

$$\left|\int_{\Omega} f(s)\overline{g(s)}d\mu(s) - \int_{\Omega} f(s)\overline{h(s)}d\mu(s)\int_{\Omega} h(s)\overline{g(s)}d\mu(s)\right| \leq \frac{1}{4}M(\varphi,\Phi)M(\gamma,\Gamma)\left|\int_{\Omega} f(s)\overline{h(s)}d\mu(s)\int_{\Omega} h(s)\overline{g(s)}d\mu(s)\right| \tag{4.3}$$

where

$$M(\varphi,\Phi) := \left[\frac{(|\Phi| - |\varphi|)^2 + 4[|\Phi\overline{\varphi}| - \operatorname{Re}(\Phi\overline{\varphi})]}{\operatorname{Re}(\Phi\overline{\varphi})}\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 3 on choosing $H = L^2(\Omega,\mathbb{K})$ with the inner product

$$\langle f,g\rangle := \int_{\Omega} f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 4. *It is obvious that a sufficient condition for* (4.1) *to hold is*

$$\operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \overline{\varphi}\overline{h(s)}\right)\right] \geq 0,$$

and

$$\operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \overline{\gamma}\overline{h(s)}\right)\right] \geq 0,$$

for $\mu-$a.e. $s \in \Omega$, or equivalently,

$$\left|f(s) - \frac{\Phi+\varphi}{2}h(s)\right| \leq \frac{1}{2}|\Phi-\varphi||h(s)| \quad \text{and}$$
$$\left|g(s) - \frac{\Gamma+\gamma}{2}h(s)\right| \leq \frac{1}{2}|\Gamma-\gamma||h(s)|,$$

for $\mu-$a.e. $s \in \Omega$.

The following result may be stated as well.

Corollary 5. *If $z,Z,t,T \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f,g \in L^2(\Omega,\mathbb{K})$ are such that:*

$$\operatorname{Re}\left[(Z - f(s))\left(\overline{f(s)} - \bar{z}\right)\right] \geq 0, \tag{4.4}$$
$$\operatorname{Re}\left[(T - g(s))\left(\overline{g(s)} - \bar{t}\right)\right] \geq 0 \quad \text{for a.e. } s \in \Omega$$

or, equivalently

$$\left|f(s) - \frac{z+Z}{2}\right| \leq \frac{1}{2}|Z-z|, \tag{4.5}$$
$$\left|g(s) - \frac{t+T}{2}\right| \leq \frac{1}{2}|T-t| \quad \text{for a.e. } s \in \Omega;$$

then we have the inequality

$$\left|\frac{1}{\mu(\Omega)}\int_{\Omega} f(s)\overline{g(s)}d\mu(s) - \frac{1}{\mu(\Omega)}\int_{\Omega} f(s)d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_{\Omega}\overline{g(s)}d\mu(s)\right| \leq \frac{1}{4}M(z,Z)M(t,T)\left|\frac{1}{\mu(\Omega)}\int_{\Omega} f(s)d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_{\Omega}\overline{g(s)}d\mu(s)\right|. \tag{4.6}$$

Remark 5. *The case of real functions incorporates the following interesting inequality*

$$\left|\frac{\mu(\Omega)\int_\Omega f(s)\,g(s)\,d\mu(s)}{\int_\Omega f(s)\,d\mu(s)\int_\Omega g(s)\,d\mu(s)}-1\right|\leq\frac{1}{4}\cdot\frac{(Z-z)(T-t)}{\sqrt{ztZT}} \tag{4.7}$$

provided $\mu(\Omega)<\infty$,

$$z\leq f(s)\leq Z, t\leq g(s)\leq T$$

for $\mu-a.e.$ $s\in\Omega$, *where* z,t,Z,T *are real numbers and the integrals at the denominator are not zero. Here the constant* $\frac{1}{4}$ *is best possible in the sense mentioned above.*

Using Theorem 3 we may state the following result as well.

Proposition 2. *If* $f,g,h\in L^2(\Omega,\mathbb{K})$ *and* $\gamma,\Gamma\in\mathbb{K}$ *are such that* $\int_\Omega|h(s)|^2\,d\mu(s)=1$ *and*

$$\int_\Omega\{\operatorname{Re}[\Gamma h(s)-(\lambda f(s)+(1-\lambda)g(s))] \times\left[\lambda\overline{f(s)}+(1-\lambda)\overline{g(s)}-\bar{\gamma}\bar{h}(s)\right]\Big\}\,d\mu(s)\geq 0 \tag{4.8}$$

or, equivalently,

$$\left(\int_\Omega\left|\lambda f(s)+(1-\lambda)g(s)-\frac{\gamma+\Gamma}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}|\Gamma-\gamma|, \tag{4.9}$$

then we have the inequality

$$\begin{aligned} I:=&\int_\Omega\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\\ &-\operatorname{Re}\left[\int_\Omega f(s)\overline{h(s)}d\mu(s)\cdot\int_\Omega h(s)\overline{g(s)}d\mu(s)\right]\\ \leq&\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \end{aligned} \tag{4.10}$$

The constant $\frac{1}{16}$ *is best possible.*

If (4.8) and (4.9) hold with " $\pm$ *" instead of "* $+$ *" (see Corollary 4), then*

$$|I|\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \tag{4.11}$$

Remark 6. *It is obvious that a sufficient condition for (4.8) to hold is*

$$\operatorname{Re}\left\{[\Gamma h(s)-(\lambda f(s)+(1-\lambda)g(s))]\cdot\left[\lambda\overline{f(s)}+(1-\lambda)\overline{g(s)}-\bar{\gamma}\bar{h}(s)\right]\right\}\geq 0 \tag{4.12}$$

for a.e. $s\in\Omega$, *or equivalently*

$$\left|\lambda f(s)+(1-\lambda)g(s)-\frac{\gamma+\Gamma}{2}h(s)\right|\leq\frac{1}{2}|\Gamma-\gamma||h(s)| \tag{4.13}$$

for a.e. $s\in\Omega$.

Finally, the following corollary holds.

Corollary 6. *If* $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ *and* $f, g \in L^2(\Omega, \mathbb{K})$ *are such that*

$$\operatorname{Re}\left[(Z - (\lambda f(s) + (1-\lambda) g(s)))\left(\lambda\overline{f(s)} + (1-\lambda)\overline{g(s)} - \overline{z}\right)\right] \geq 0 \tag{4.14}$$

for a.e. $s \in \Omega$, *or, equivalently*

$$\left|\lambda f(s) + (1-\lambda) g(s) - \frac{z+Z}{2}\right| \leq \frac{1}{2}|Z - z|, \tag{4.15}$$

for a.e. $s \in \Omega$, *then we have the inequality*

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)} \int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \\ &\qquad - \operatorname{Re}\left[\frac{1}{\mu(\Omega)}\int_\Omega f(s)\, d\mu(s) \cdot \frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)} d\mu(s)\right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2 . \end{aligned}$$

If (4.14) and (4.15) hold with " $\pm$ *" instead of "* $+$ *", then*

$$|J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2 . \tag{4.16}$$

Remark 7. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

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