SOME NEW RESULTS RELATED TO BESSEL AND GRÜSS INEQUALITIES FOR ORTHOGONAL FAMILIES IN INNER PRODUCT SPACES

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Abstract. Some new counterparts of Bessel's inequality for orthornormal families in real or complex inner product spaces are pointed out. Applications for some Grüss type inequalities are also emphasized.

1. Introduction

In [1], the author has proved the following result which provides both a Grüss type inequality for orthogonal families of vectors in real or complex inner products as well as, for $x = y$, a counterpart of Bessel's inequality.

Theorem 1. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, i.e., $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and $\|e_i\| = 1$, $i, j \in I$, F a finite part of I, $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$ $(i \in F)$, and $x, y \in H$. If either*

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} \Phi_i e_i - x, x - \sum_{i=1}^{n} \phi_i e_i \right\rangle \geq 0, \tag{1.1}$$
$$\operatorname{Re}\left\langle \sum_{i=1}^{n} \Gamma_i e_i - y, y - \sum_{i=1}^{n} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \tag{1.2}$$
$$\left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

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hold, then we have the inequality

$$
\begin{aligned}
(1.3)\qquad 0 &\leq \left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\
&\leq \frac{1}{4}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \\
&\qquad - \left[\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle\right]^{\frac{1}{2}} \\
&\qquad \times \left[\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

In the follow up paper [2], and by the use of a different technique, the author has pointed out the following result as well:

Theorem 2. *Let* $\{e_i\}_{i\in I}$, F, $\phi_i, \gamma_i, \Phi_i, \Gamma_i$ *and* x, y *be as in Theorem 1. If either (1.1) or (1.2) holds, then we have the inequality*

$$
\begin{aligned}
(1.4)\qquad 0 &\leq \left| \langle x, y\rangle - \sum_{i=1}^{n} \langle x, e_i\rangle \langle e_i, y\rangle \right| \\
&\leq \frac{1}{4}\left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \\
&\qquad - \sum_{i\in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i\rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i\rangle \right| \\
&\leq \frac{1}{4}\left(\sum_{i=1}^{n} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

It has also been shown that the bounds provided by the second inequality in (1.3) and the second inequality in (1.4) cannot be compared in general.

2. A New Counterpart of Bessel's Inequality

The following counterpart of Bessel's inequality holds.

Theorem 3. *Let* $\{e_i\}_{i\in I}$ *be a family of orthornormal vectors in* H, F *a finite part of* I, *and* ϕ_i, Φ_i $(i \in F)$, *real or complex numbers such that* $\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) > 0$. *If* $x \in H$ *is such that either*

(i) $\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i\right\rangle \geq 0$;
or, equivalently,

(ii) $\left\| x - \sum_{i\in F}\frac{\phi_i+\Phi_i}{2}e_i\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}$;

holds, then one has the inequality

$$\|x\|^2 \leq \frac{1}{4}\cdot\frac{\sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|\langle x, e_i\rangle|^2. \tag{2.1}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Firstly, we observe that for $y, a, A \in H$, the following are equivalent

$$\operatorname{Re}\langle A - y, y - a\rangle \geq 0 \tag{2.2}$$

and

$$\left\| y - \frac{a+A}{2}\right\| \leq \frac{1}{2}\|A - a\|. \tag{2.3}$$

Now, for $a = \sum_{i\in F}\phi_i e_i$, $A = \sum_{i\in F}\Phi_i e_i$, we have

$$\begin{aligned}\|A-a\| &= \left\|\sum_{i\in F}(\Phi_i-\phi_i)e_i\right\| = \left[\left\|\sum_{i\in F}(\Phi_i-\phi_i)e_i\right\|^2\right]^{\frac{1}{2}} \\ &= \left(\sum_{i\in F}|\Phi_i-\phi_i|^2\|e_i\|^2\right)^{\frac{1}{2}} = \left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},\end{aligned}$$

giving, for $y = x$, the desired equivalence.

Now, observe that

$$\begin{aligned}&\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i\right\rangle \\ &\qquad = \sum_{i\in F}\operatorname{Re}\left[\Phi_i\overline{\langle x, e_i\rangle} + \overline{\phi_i}\langle x, e_i\rangle\right] - \|x\|^2 - \sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right),\end{aligned}$$

giving, from (i), that

$$\|x\|^2 + \sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right) \leq \sum_{i\in F}\operatorname{Re}\left[\Phi_i\overline{\langle x, e_i\rangle} + \overline{\phi_i}\langle x, e_i\rangle\right]. \tag{2.4}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha}q^2 \geq 2pq, \quad \alpha > 0,\ p, q \geq 0;$$

we deduce

$$2\|x\| \leq \frac{\|x\|^2}{\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}}} + \left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}}. \tag{2.5}$$

Dividing (2.4) by $\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}} > 0$ and using (2.5), we obtain

$$\|x\| \leq \frac{1}{2}\frac{\sum_{i\in F}\operatorname{Re}\left[\Phi_i\overline{\langle x,e_i\rangle}+\overline{\phi_i}\langle x,e_i\rangle\right]}{\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}}}, \tag{2.6}$$

which is also an interesting inequality in itself.

Using the Cauchy-Buniakowsky-Schwartz inequality for real numbers, we get

$$\begin{aligned}\sum_{i\in F}\operatorname{Re}\left[\Phi_i\overline{\langle x,e_i\rangle}+\overline{\phi_i}\langle x,e_i\rangle\right] &\leq \sum_{i\in F}\left|\Phi_i\overline{\langle x,e_i\rangle}+\overline{\phi_i}\langle x,e_i\rangle\right| \\ &\leq \sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)|\langle x,e_i\rangle| \\ &\leq \left[\sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)^2\right]^{\frac{1}{2}}\left[\sum_{i\in F}|\langle x,e_i\rangle|^2\right]^{\frac{1}{2}}.\end{aligned} \tag{2.7}$$

Making use of (2.6) and (2.7), we deduce the desired result (2.1).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (2.1) holds with a constant $c > 0$, i.e.,

$$\|x\|^2 \leq c\cdot\frac{\sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|\langle x,e_i\rangle|^2, \tag{2.8}$$

provided x, $\phi_i, \Phi_i, i \in F$ satisfies (i).

Choose $F = \{1\}$, $e_1 = e$, $\|e\| = 1$, $\phi_i = m$, $\Phi_i = M$ with $m, M > 0$, then, by (2.8) we get

$$\|x\|^2 \leq c\frac{(M+m)^2}{mM}|\langle x,e\rangle|^2 \tag{2.9}$$

provided

$$\operatorname{Re}\langle Me-x, x-me\rangle \geq 0. \tag{2.10}$$

If $x = me$, then obviously (2.10) holds, and by (2.9) we get

$$m^2 \leq c\frac{(M+m)^2}{mM}m^2$$

giving $mM \leq c(M+m)^2$ for $m, M > 0$. Now, if in this inequality we choose $m = 1-\varepsilon$, $M = 1+\varepsilon$ $(\varepsilon \in (0,1))$, then we get $1-\varepsilon^2 \leq 4c$ for $\varepsilon \in (0,1)$, from where we deduce $c \geq \frac{1}{4}$. ∎

Remark 1. *By the use of (2.6), the second inequality in (2.7) and the Hölder inequality, we may state the following counterparts of Bessel's inequality as well:*

$$\|x\|^2 \leq \frac{1}{2} \cdot \frac{1}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} \times \begin{cases} \max\limits_{i\in F} \{|\Phi_i| + |\phi_i|\} \sum\limits_{i\in F} |\langle x, e_i\rangle| \\ \left[\sum\limits_{i\in F} (|\Phi_i| + |\phi_i|)^p\right]^{\frac{1}{p}} \left(\sum\limits_{i\in F} |\langle x, e_i\rangle|^q\right)^{\frac{1}{q}}, \\ \qquad \text{for } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \max_{i\in F} |\langle x, e_i\rangle| \sum\limits_{i\in F} [|\Phi_i| + |\phi_i|]. \end{cases} \tag{2.11}$$

The following corollary holds.

Corollary 1. *With the assumption of Theorem 3 and if either (i) or (ii) holds, then*

$$0 \leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \leq \frac{1}{4} M^2 (\boldsymbol{\Phi}, \boldsymbol{\phi}, F) \sum_{i\in F} |\langle x, e_i\rangle|^2, \tag{2.12}$$

where

$$M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F) := \left[\frac{\sum_{i\in F} \left\{(|\Phi_i| + |\phi_i|)^2 + 4\left[\left|\Phi_i \overline{\phi_i}\right| - \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]\right\}}{\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)}\right]^{\frac{1}{2}}. \tag{2.13}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The inequality (2.12) follows by (2.1) on subtracting the same quantity $\sum_{i\in F} |\langle x, e_i\rangle|^2$ from both sides.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.12) holds with $c > 0$, i.e.,

$$0 \leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \leq c M^2 (\boldsymbol{\Phi}, \boldsymbol{\phi}, F) \sum_{i\in F} |\langle x, e_i\rangle|^2 \tag{2.14}$$

provided the condition (i) holds.

Choose $F = \{1\}$, $e_1 = e$, $\|e\| = 1$, $\phi_i = \phi$, $\Phi_i = \Phi$, $\phi, \Phi > 0$ in (2.14) to get

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq c \frac{(\Phi - \phi)^2}{\phi\Phi} |\langle x, e\rangle|^2, \tag{2.15}$$

provided

$$\langle \Phi e - x, x - \phi e\rangle \geq 0. \tag{2.16}$$

If $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ then we have

$$\|x\|^2 - |\langle x, e\rangle|^2 = x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{1}{2}(x_1 - x_2)^2,$$

$$|\langle x, e\rangle|^2 = \frac{(x_1 + x_2)^2}{2}$$

and by (2.15) we get

$$\frac{(x_1-x_2)^2}{2} \le c\frac{(\Phi-\phi)^2}{\phi\Phi}\cdot\frac{(x_1+x_2)^2}{2}. \tag{2.17}$$

Now, if we let $x_1 = \frac{\phi}{\sqrt{2}}$, $x_2 = \frac{\Phi}{\sqrt{2}}$ $(\phi, \Phi > 0)$ then obviously

$$\langle \Phi e - x, x - \phi e\rangle = \sum_{i=1}^{2}\left(\frac{\Phi}{\sqrt{2}} - x_i\right)\left(x_i - \frac{\phi}{\sqrt{2}}\right) = 0,$$

which shows that (2.16) is fulfilled, and thus by (2.17) we obtain

$$\frac{(\Phi-\phi)^2}{4} \le c\frac{(\Phi-\phi)^2}{\phi\Phi}\cdot\frac{(\Phi+\phi)^2}{4}$$

for any $\Phi > \phi > 0$. This implies

$$c(\Phi+\phi)^2 \ge \phi\Phi \tag{2.18}$$

for any $\Phi > \phi > 0$.

Finally, let $\phi = 1-\varepsilon$, $\Phi = 1+\varepsilon$, $\varepsilon \in (0,1)$. Then from (2.18) we get $4c \ge 1-\varepsilon^2$ for any $\varepsilon \in (0,1)$ which produces $c \ge \frac{1}{4}$. ∎

Remark 2. *If $\{e_i\}_{i\in I}$ is an orthornormal family in the real inner product $(H; \langle\cdot,\cdot\rangle)$ and $M_i, m_i \in \mathbb{R}$, $i \in F$ (F is a finite part of I) and $x \in H$ are such that $M_i, m_i \ge 0$ for $i \in F$ with $\sum_{i\in F} M_i m_i \ge 0$ and*

$$\left\langle \sum_{i\in F} M_i e_i - x, x - \sum_{i\in F} m_i e_i \right\rangle \ge 0,$$

then we have the inequality

$$0 \le \|x\|^2 - \sum_{i\in F}[\langle x, e_i\rangle]^2 \le \frac{1}{4}\cdot\frac{\sum_{i\in F}(M_i - m_i)^2}{\sum_{i\in F} M_i m_i}\cdot\sum_{i\in F}[\langle x, e_i\rangle]^2. \tag{2.19}$$

The constant $\frac{1}{4}$ is best possible.

The following counterpart of the Schwarz's inequality in inner product spaces holds.

Corollary 2. *Let $x, y \in H$ and $\delta, \Delta \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ with the property that $\operatorname{Re}(\Delta\overline{\delta}) > 0$. If either*

$$\operatorname{Re}\langle \Delta y - x, x - \delta y\rangle \ge 0 \tag{2.20}$$

or, equivalently,

$$\left\| x - \frac{\delta+\Delta}{2}\cdot y \right\| \le \frac{1}{2}|\Delta - \delta|\,\|y\| \tag{2.21}$$

holds, then we have the inequalities

$$\begin{aligned} \|x\|\,\|y\| &\le \frac{1}{2}\cdot\frac{\operatorname{Re}\left[\Delta\overline{\langle x, y\rangle} + \overline{\delta}\langle x, y\rangle\right]}{\sqrt{\Delta\overline{\delta}}} \\ &\le \frac{1}{2}\cdot\frac{|\Delta| + |\delta|}{\sqrt{\Delta\overline{\delta}}}\,|\langle x, y\rangle|, \end{aligned} \tag{2.22}$$

$$0 \leq \|x\| \|y\| - |\langle x, y\rangle| \leq \frac{1}{2} \cdot \frac{\left(\sqrt{|\Delta|} - \sqrt{|\delta|}\right)^2 + 2\left(\sqrt{\Delta\overline{\delta}} - \sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}\right)}{\sqrt{\Delta\overline{\delta}}} |\langle x, y\rangle|, \tag{2.23}$$

$$\|x\|^2 \|y\|^2 \leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)} |\langle x, y\rangle|^2, \tag{2.24}$$

and

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2 + 4\left(\left|\Delta\overline{\delta}\right| - \operatorname{Re}\left(\Delta\overline{\delta}\right)\right)}{\operatorname{Re}\left(\Delta\overline{\delta}\right)} |\langle x, y\rangle|^2. \tag{2.25}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Proof. The inequality (2.22) follows from (2.6) on choosing $F = \{1\}$, $e_1 = e = \frac{y}{\|y\|}$, $\Phi_1 = \Phi = \Delta \|y\|$, $\phi_1 = \phi = \delta \|y\|$ $(y \neq 0)$. The inequality (2.23) is equivalent with (2.22). The inequality (2.24) follows from (2.1) for $F = \{1\}$ and the same choices as above. Finally, (2.25) is obviously equivalent with (2.24). ∎

3. Some Grüss Type Inequalities

The following result holds.

Theorem 4. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ and $x, y \in H$. If either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0, \quad \operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \geq 0, \tag{3.1}$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}, \quad \left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}}, \tag{3.2}$$

hold, then we have the inequality

$$0 \leq \left| \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \right| \leq \frac{1}{4} M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F) M(\boldsymbol{\Gamma}, \boldsymbol{\gamma}, F) \left(\sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}}, \tag{3.3}$$

where $M(\boldsymbol{\Phi}, \boldsymbol{\phi}, F)$ is defined in (2.13).

The constant $\frac{1}{4}$ is best possible.

Proof. Using Schwartz's inequality in the inner product space $(H, \langle\cdot,\cdot\rangle)$ one has

$$\left|\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right|^2 \leq \left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2 \left\|y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \tag{3.4}$$

and since a simple calculation shows that

$$\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle = \langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle$$

and

$$\left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2 \leq \|x\|^2 - \sum_{i\in F}|\langle x,e_i\rangle|^2$$

for any $x, y \in H$, then by (3.4) and by the counterpart of Bessel's inequality in Corollary 1, we have

$$\begin{aligned} &\left|\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \\ &\leq \left(\|x\|^2 - \sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2 - \sum_{i\in F}|\langle y,e_i\rangle|^2\right) \\ &\leq \frac{1}{4}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}|\langle x,e_i\rangle|^2 \cdot \frac{1}{4}M^2(\mathbf{\Gamma},\boldsymbol{\gamma},F)\sum_{i\in F}|\langle y,e_i\rangle|^2 . \end{aligned} \tag{3.5}$$

Taking the square root in (3.5), we deduce (3.3).

The fact that $\frac{1}{4}$ is the best possible constant follows by Corollary 1 and we omit the details. ∎

The following corollary for real inner product spaces holds.

Corollary 3. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $M_i, m_i,\ N_i, n_i \geq 0,\ i \in F$ and $x, y \in H$ such that $\sum_{i\in F} M_i m_i > 0$, $\sum_{i\in F} N_i n_i > 0$ and*

$$\left\langle \sum_{i\in F} M_i e_i - x, x - \sum_{i\in F} m_i e_i\right\rangle \geq 0, \qquad \left\langle \sum_{i\in F} N_i e_i - y, y - \sum_{i\in F} n_i e_i\right\rangle \geq 0. \tag{3.6}$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \left|\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle y,e_i\rangle\right|^2 \\ &\leq \frac{1}{16}\cdot\frac{\sum_{i\in F}(M_i-m_i)^2\sum_{i\in F}(N_i-n_i)^2\sum_{i\in F}|\langle x,e_i\rangle|^2\sum_{i\in F}|\langle y,e_i\rangle|^2}{\sum_{i\in F}M_im_i\sum_{i\in F}N_in_i}. \end{aligned} \tag{3.7}$$

The constant $\frac{1}{16}$ is best possible.

In the case where the family $\{e_i\}_{i\in I}$ reduces to a single vector, we may deduce from Theorem 4 the following particular case first obtained in [3].

Corollary 4. *Let* $e \in H$, $\|e\| = 1$, $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}\left(\Phi\overline{\phi}\right), \operatorname{Re}\left(\Gamma\overline{\gamma}\right) > 0$ *and* $x, y \in H$ *such that either*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0, \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0, \tag{3.8}$$

or, equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|, \qquad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.9}$$

holds, then

$$0 \leq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} M(\Phi, \phi) M(\Gamma, \gamma) |\langle x, e\rangle \langle e, y\rangle|, \tag{3.10}$$

where

$$M(\Phi, \phi) := \left[\frac{(|\Phi| - |\phi|)^2 + 4\left[|\phi\Phi| - \operatorname{Re}\left(\Phi\overline{\phi}\right)\right]}{\operatorname{Re}\left(\Phi\overline{\phi}\right)} \right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 3. *If* H *is real,* $e \in H$, $\|e\| = 1$ *and* $a, b, A, B \in \mathbb{R}$ *are such that* $A > a > 0$, $B > b > 0$ *and*

$$\left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2}(A - a), \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2}(B - b), \tag{3.11}$$

then

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}} |\langle x, e\rangle \langle e, y\rangle|. \tag{3.12}$$

The constant $\frac{1}{4}$ *is best possible.*

If $\langle x, e\rangle, \langle y, e\rangle \neq 0$, then the following equivalent form of (3.12) also holds

$$\left| \frac{\langle x, y\rangle}{\langle x, e\rangle \langle e, y\rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}}. \tag{3.13}$$

4. Some Companion Inequalities

The following companion of the Grüss inequality also holds.

Theorem 5. *Let* $\{e_i\}_{i \in I}$ *be a family of orthornormal vectors in* H, F *a finite part of* I, $\phi_i, \Phi_i \in \mathbb{K}$, $(i \in F)$, $x, y \in H$ *and* $\lambda \in (0, 1)$, *such that either*

$$\operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - (\lambda x + (1 - \lambda) y), \lambda x + (1 - \lambda) y - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0 \tag{4.1}$$

or, equivalently,

$$\left\| \lambda x + (1 - \lambda) y - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \tag{4.2}$$

holds. Then we have the inequality

$$\begin{aligned} \operatorname{Re}\left[\langle x, y\rangle - \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle \right] \qquad \\ \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} M^2(\boldsymbol{\Phi}, \boldsymbol{\phi}, F) \sum_{i \in F} |\langle \lambda x + (1 - \lambda) y, e_i\rangle|^2. \end{aligned} \tag{4.3}$$

The constant $\frac{1}{16}$ *is the best possible constant in (4.3) in the sense that it cannot be replaced by a smaller constant.*

Proof. Using the known inequality

$$\operatorname{Re}\langle z,u\rangle \leq \frac{1}{4}\|z+u\|^2$$

we may state that for any $a,b\in H$ and $\lambda\in(0,1)$

$$\operatorname{Re}\langle a,b\rangle \leq \frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b\|^2. \tag{4.4}$$

Since

$$\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle = \left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle,$$

for any $x,y\in H$, then, by (4.4), we get

$$\begin{aligned}
&\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \\
&= \operatorname{Re}\left[\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right] \\
&\leq \frac{1}{4\lambda(1-\lambda)}\left\|\lambda\left(x-\sum_{i\in F}\langle x,e_i\rangle e_i\right)+(1-\lambda)\left(y-\sum_{i\in F}\langle y,e_i\rangle e_i\right)\right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)}\left\|\lambda x+(1-\lambda)y-\sum_{i\in F}\langle \lambda x+(1-\lambda)y,e_i\rangle e_i\right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)y\|^2-\sum_{i\in F}|\langle \lambda x+(1-\lambda)y,e_i\rangle|^2\right].
\end{aligned} \tag{4.5}$$

If we apply the counterpart of Bessel's inequality from Corollary 1 for $\lambda x+(1-\lambda)y$, we may state that

$$\begin{aligned}
&\|\lambda x+(1-\lambda)y\|^2-\sum_{i\in F}|\langle \lambda x+(1-\lambda)y,e_i\rangle|^2 \\
&\qquad \leq \frac{1}{4}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}|\langle \lambda x+(1-\lambda)y,e_i\rangle|^2.
\end{aligned} \tag{4.6}$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $\frac{1}{16}$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x=y$, then it becomes (i) of Theorem 3, implying for $\lambda=\frac{1}{2}$ (2.12), for which, we have shown that $\frac{1}{4}$ was the best constant. ∎

Remark 4. *If in Theorem 5, we choose* $\lambda=\frac{1}{2}$*, then we get*

$$\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \leq \frac{1}{4}M^2(\mathbf{\Phi},\boldsymbol{\phi},F)\sum_{i\in F}\left|\left\langle \frac{x+y}{2},e_i\right\rangle\right|^2, \tag{4.7}$$

provided

$$\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i\in F}\phi_i e_i \right\rangle \geq 0$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i \right\| \leq \frac{1}{2}\left(\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2\right)^{\frac{1}{2}}. \tag{4.8}$$

5. Integral Inequalities

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Let $\rho\geq 0$ be a $\mu-$measurable function on Ω. Denote by $L^2_\rho(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s) < \infty. \tag{5.1}$$

Consider the family $\{f_i\}_{i\in I}$ of functions in $L^2_\rho(\Omega,\mathbb{K})$ with the properties that

$$\int_\Omega \rho(s) f_i(s)\overline{f_j}(s)\,d\mu(s) = \delta_{ij}, \quad i,j\in I, \tag{5.2}$$

where δ_{ij} is 0 if $i\neq j$ and $\delta_{ij}=1$ if $i=j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2_\rho(\Omega,\mathbb{K})$.

The following proposition holds.

Proposition 1. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2_\rho(\Omega,\mathbb{K})$, F a finite subset of I, $\phi_i,\Phi_i\in\mathbb{K}$ $(i\in F)$ such that $\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)>0$ and $f\in L^2_\rho(\Omega,\mathbb{K})$, so that either*

$$\int_\Omega \rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\;\overline{f_i}(s)\right)\right]d\mu(s)\geq 0 \tag{5.3}$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s) - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4}\sum_{i\in F}\left|\Phi_i-\phi_i\right|^2. \tag{5.4}$$

Then we have the inequality

$$\left(\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2}\cdot\frac{1}{\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}}} \tag{5.5}$$

$$\times\begin{cases} \max_{i\in F}\left\{|\Phi_i|+|\phi_i|\right\}\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right| \\ \left[\sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)^p\right]^{\frac{1}{p}}\left(\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right|^q\right)^{\frac{1}{q}}, \\ \qquad\qquad\qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \max_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right|\sum_{i\in F}\left[|\Phi_i|+|\phi_i|\right]. \end{cases}$$

In particular, we have

$$\int_{\Omega}\rho(s)|f(s)|^2\,d\mu(s) \le \frac{1}{4}\cdot\frac{\sum_{i\in F}\left(|\Phi_i|+|\phi_i|\right)^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right|^2. \tag{5.6}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof is obvious by Theorem 3 and Remark 1. We omit the details.

The following proposition also holds.

Proposition 2. *Assume that f_i, f, ϕ_i, Φ_i and F satisfy the assumptions of Proposition 1. Then we have the following counterpart of Bessel's inequality:*

$$\begin{aligned} 0 &\le \int_{\Omega}\rho(s)f^2(s)\,d\mu(s)-\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right|^2 \\ &\le \frac{1}{4}M^2(\boldsymbol{\Phi},\boldsymbol{\phi},F)\cdot\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)\,d\mu(s)\right|^2, \end{aligned} \tag{5.7}$$

where, as above,

$$M(\boldsymbol{\Phi},\boldsymbol{\phi},F):=\left[\frac{\sum_{i\in F}\left\{\left(|\Phi_i|-|\phi_i|\right)^2+4\left[|\phi_i\Phi_i|-\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]\right\}}{\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\right]^{\frac{1}{2}}. \tag{5.8}$$

The constant $\frac{1}{4}$ is the best possible.

The following Grüss type inequality also holds.

Proposition 3. *Let $\{f_i\}_{i\in I}$ and F be as in Proposition 1. If $\phi_i,\Phi_i,\gamma_i,\Gamma_i\in\mathbb{K}$ $(i\in F)$ and $f,g\in L^2_{\rho}(\Omega,\mathbb{K})$ so that either*

$$\begin{aligned} &\int_{\Omega}\rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s)-f(s)\right)\left(\overline{f}(s)-\sum_{i\in F}\overline{\phi_i}\ \overline{f_i}(s)\right)\right]d\mu(s)\ge 0, \\ &\int_{\Omega}\rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Gamma_i f_i(s)-g(s)\right)\left(\overline{g}(s)-\sum_{i\in F}\overline{\gamma_i}\ \overline{f_i}(s)\right)\right]d\mu(s)\ge 0, \end{aligned} \tag{5.9}$$

or, equivalently,

$$\int_{\Omega}\rho(s)\left|f(s)-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot f_i(s)\right|^2 d\mu(s)\le\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2, \tag{5.10}$$

$$\int_{\Omega}\rho(s)\left|g(s)-\sum_{i\in F}\frac{\Gamma_i+\gamma_i}{2}\cdot f_i(s)\right|^2 d\mu(s)\le\frac{1}{4}\sum_{i\in F}|\Gamma_i-\gamma_i|^2,$$

then we have the inequality

$$\begin{aligned}&\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right.\\&\qquad\left.-\sum_{i\in F}\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)d\mu(s)\int_{\Omega}\rho(s)f_i(s)\overline{g(s)}d\mu(s)\right|\\&\le\frac{1}{4}M(\boldsymbol{\Phi},\boldsymbol{\phi},F)\,M(\boldsymbol{\Gamma},\boldsymbol{\gamma},F)\left(\sum_{i\in F}\left|\int_{\Omega}\rho(s)f(s)\overline{f_i}(s)d\mu(s)\right|^2\right)^{\frac{1}{2}}\\&\qquad\times\left(\sum_{i\in F}\left|\rho(s)f_i(s)\overline{g(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}},\end{aligned} \tag{5.11}$$

where $M(\boldsymbol{\Phi},\boldsymbol{\phi},F)$ is defined in (5.8).

The constant $\frac{1}{4}$ is the best possible.

The proof follows by Theorem 4 and we omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the counterpart of Bessel's inequality (5.7) or for the Grüss type inequality (5.11) to hold.

Corollary 5. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in the real Hilbert space $L^2_\rho(\Omega)$, F a finite part of I, $M_i,m_i\ge 0$ $(i\in F)$, with $\sum_{i\in F}M_im_i>0$ and $f\in L^2_\rho(\Omega)$ so that*

$$\sum_{i\in F}m_if_i(s)\le f(s)\le\sum_{i\in F}M_if_i(s)\quad\text{for}\quad\mu-a.e.\ s\in\Omega. \tag{5.12}$$

Then we have the inequality

$$\begin{aligned}0&\le\int_{\Omega}\rho(s)f^2(s)d\mu(s)-\sum_{i\in F}\left[\int_{\Omega}\rho(s)f(s)f_i(s)d\mu(s)\right]^2\\&\le\frac{1}{4}\cdot\frac{\sum_{i\in F}(M_i-m_i)^2}{\sum_{i\in F}M_im_i}\cdot\sum_{i\in F}\left[\int_{\Omega}\rho(s)f(s)f_i(s)d\mu(s)\right]^2.\end{aligned} \tag{5.13}$$

The constant $\frac{1}{4}$ is best possible.

Corollary 6. *Let $\{f_i\}_{i\in I}$ and F be as above. If $M_i,m_i,N_i,n_i\ge 0$ $(i\in F)$ with $\sum_{i\in F}M_im_i,\sum_{i\in F}N_in_i>0$ and $f,g\in L^2_\rho(\Omega)$ such that*

$$\sum_{i\in F}m_if_i(s)\le f(s)\le\sum_{i\in F}M_if_i(s) \tag{5.14}$$

and

$$\sum_{i\in F} n_i f_i(s) \leq g(s) \leq \sum_{i\in F} N_i f_i(s) \quad \text{for} \ \ \mu - a.e.\ s \in \Omega,$$

then we have the inequality

$$\begin{aligned}\Bigg| &\int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \\ &- \sum_{i\in F} \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \Bigg| \\ &\leq \frac{1}{4} \left(\frac{\sum_{i\in F} (M_i - m_i)^2}{\sum_{i\in F} M_i m_i} \right)^{\frac{1}{2}} \left(\frac{\sum_{i\in F} (N_i - n_i)^2}{\sum_{i\in F} N_i n_i} \right)^{\frac{1}{2}} \\ &\times \left[\sum_{i\in F} \left(\int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right)^2 \sum_{i\in F} \left(\int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right)^2 \right]^{\frac{1}{2}}.\end{aligned}$$

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