AN OSTROWSKI LIKE INEQUALITY FOR CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we point out an Ostrowski type inequality for convex functions which complement in a sense the recent results for functions of bounded variation and absolutely continuous functions. Applications in connection with the Hermite-Hadamard inequality are also considered.

1. Introduction

In 1938, A. Ostrowski [9] proved the following integral inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left\| f' \right\|_\infty \tag{1.1}$$

provided f is differentiable and $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In the last 5 years, many authors have concentrated their efforts in generalising (1.1) and have applied the obtained results in different fields, including Numerical Integration, Probability Theory and Statistics, Information Theory, etc. For a comprehensive approach in the field, see the recent book [5] where many other references may be found.

One direction of generalising (1.1) was pointed out by the author in [2] – [4]. Let us recall here a couple of the main results obtained in the above papers.

Theorem 1. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0,\dots,k+1)$ be $k+2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,\dots,k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality:*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\ &\leq \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0,\dots,k-1 \right\} \right] \bigvee_a^b (f), \end{aligned} \tag{1.2}$$

where $\nu(h) := \max\{h_i | i = 0,\dots,k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0,\dots,k-1)$ and $\bigvee_a^b (f)$ is the total variation of f on $[a,b]$.

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The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

If one would assume more for the function f, for example, absolute continuity, then the following result holds.

Theorem 2. *Under the assumptions of Theorem 1 for* I_k *and* α_i $(i = 0, \ldots, k+1)$ *and if* $f : [a, b] \to \mathbb{R}$ *is absolutely continuous on* $[a, b]$, *then*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{1.3}$$

$$\leq \begin{cases} \left[\frac{1}{4} \sum\limits_{i=0}^{k-1} h_i^2 + \sum\limits_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\sum\limits_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p [a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \nu(h) + \max \left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0, \ldots, k-1 \right\} \right] \|f'\|_1, \end{cases}$$

where $\|\cdot\|_p$ $(p \in [1, \infty])$ *are the Lebesgue norms, i.e.,*

$$\|h\|_\infty : = ess \sup_{t \in [a,b]} |h(t)|,$$

$$\|h\|_p : = \left(\int_a^b |h(t)|^p\,dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ *and* $\frac{1}{2}$ *are best in the sense mentioned above.*

In this paper, the case of convex functions $f : [a, b] \to \mathbb{R}$ is examined. Some particular cases in connection with the well known Hermite-Hadamard inequality for convex functions are also considered.

2. The Results

The following result holds.

Theorem 3. *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *be a division of the interval* $[a, b]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a, b] \to \mathbb{R}$ *is a convex function on* $[a, b]$, *then we have the inequality:*

$$\begin{aligned} & \frac{1}{2} \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1})^2 f'_+(\alpha_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_-(\alpha_{i+1}) \right] \\ \leq\ & \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2} \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1})^2 f'_-(x_{i+1}) - (\alpha_{i+1} - x_i)^2 f'_+(x_i) \right]. \end{aligned} \tag{2.1}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Proof. Using the integration by parts formula, we may prove the equality (see for example [3]):

$$\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i) f(x_{i+1})-\int_a^b f(t)\,dt=\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(t-\alpha_{i+1}) f'(t)\,dt \tag{2.2}$$

for any locally absolutely continuous function $f:(a,b)\to\mathbb{R}$.

Since f is convex, then it is locally Lipschitzian on (a,b) and thus the above equality holds. Also, we have

$$f'_+(x_i)\le f'(t)\le f'_-(\alpha_{i+1}) \quad \text{for a.e.} \quad t\in[x_i,\alpha_{i+1}] \tag{2.3}$$

and

$$f'_+(\alpha_{i+1})\le f'(t)\le f'_-(x_{i+1}) \quad \text{for a.e.} \quad t\in[\alpha_{i+1},x_{i+1}]. \tag{2.4}$$

Using (2.3) and (2.4), we may write that

$$\begin{aligned} f'_-(\alpha_{i+1})\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt &\le \int_{x_i}^{\alpha_{i+1}} f'(t)(t-\alpha_{i+1})\,dt \\ &\le f'_+(x_i)\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} f'_+(\alpha_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt &\le \int_{\alpha_{i+1}}^{x_{i+1}} f'(t)(t-\alpha_{i+1})\,dt \\ &\le f'_-(x_{i+1})\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt. \end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6) and taking into account that

$$\int_{x_i}^{\alpha_{i+1}}(t-\alpha_{i+1})\,dt=-\frac{1}{2}(\alpha_{i+1}-x_i)^2$$

and

$$\int_{\alpha_{i+1}}^{x_{i+1}}(t-\alpha_{i+1})\,dt=\frac{1}{2}(x_{i+1}-\alpha_{i+1})^2,$$

we get

$$\begin{aligned} &\frac{1}{2}\left[(x_{i+1}-\alpha_{i+1})^2 f'_+(\alpha_{i+1})-(\alpha_{i+1}-x_i)^2 f'_-(\alpha_{i+1})\right] \\ \le\ &\int_{x_i}^{x_{i+1}}(t-\alpha_{i+1}) f'(t)\,dt \\ \le\ &\frac{1}{2}\left[(x_{i+1}-\alpha_{i+1})^2 f'_-(x_{i+1})-(\alpha_{i+1}-x_i)^2 f'_+(x_i)\right] \end{aligned} \tag{2.7}$$

for any $i=0,\dots,k-1$.

If we sum (2.7) over i from 0 to $k-1$ and use the identity (2.2), we deduce the desired result (2.1).

The sharpness will be proved in what follows for a particular case. ∎

It is natural to consider the following particular case.

Corollary 1. *Let L_k and f be as in the above theorem. Then we have the inequality*

$$
\begin{aligned}
(2.8)\quad 0 &\leq \frac{1}{8}\sum_{i=0}^{k-1}\left[f'_{+}\left(\frac{x_i+x_{i+1}}{2}\right)-f'_{-}\left(\frac{x_i+x_{i+1}}{2}\right)\right](x_{i+1}-x_i)^2 \\
&\leq \frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{k-1})f(b)\right] \\
&\quad -\int_a^b f(t)\,dt \\
&\leq \frac{1}{8}\sum_{i=0}^{k-1}\left[f'_{-}(x_{i+1})-f'_{+}(x_i)\right](x_{i+1}-x_i)^2 .
\end{aligned}
$$

The constant $\frac{1}{8}$ in both inequalities is sharp.

The proof follows by the above theorem choosing $\alpha_i=\frac{x_{i-1}+x_i}{2}$, $i=1,\dots,k$ and taking into account that (see also [2])

$$
\begin{aligned}
(2.9)\quad &\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) \\
&= \frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{k-1})f(b)\right].
\end{aligned}
$$

The following corollary for equidistant partitioning also holds.

Corollary 2. *Let*

$$I_k : x_i := a+(b-a)\cdot\frac{i}{k}\qquad(i=0,\dots,k)$$

be an equidistant partitioning of $[a,b]$. If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then we have the inequalities

$$
\begin{aligned}
(2.10)\quad 0 &\leq \frac{(b-a)^2}{8n^2}\sum_{i=0}^{k-1}\left\{f'_{+}\left[a+\left(i+\frac{1}{2}\right)\frac{b-a}{n}\right]\right. \\
&\qquad\qquad \left.-f'_{-}\left[a+\left(i+\frac{1}{2}\right)\frac{b-a}{n}\right]\right\} \\
&\leq \frac{1}{k}\cdot\frac{f(a)+f(b)}{2}(b-a) \\
&\qquad\qquad +\frac{b-a}{k}\sum_{i=1}^{k-1}f\left[\frac{(k-i)a+ib}{k}\right]-\int_a^b f(t)\,dt \\
&\leq \frac{(b-a)^2}{8n^2}\sum_{i=0}^{k-1}\left\{f'_{-}\left[a+(i+1)\cdot\frac{b-a}{n}\right]-f'_{+}\left[a+i\cdot\frac{b-a}{n}\right]\right\}.
\end{aligned}
$$

The following particular cases which hold when we assume differentiability conditions may be stated.

Corollary 3. *If $\alpha_i \in (a,b)$ for $i = 1, \ldots, k$ are points of differentiability for f, then we have the inequality*

$$\sum_{i=0}^{k-1} (x_{i+1} - x_i) \left(\frac{x_i + x_{i+1}}{2} - \alpha_{i+1} \right) f'(\alpha_{i+1}) \leq \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_{i+1}) - \int_a^b f(t)\, dt. \tag{2.11}$$

If we denote by $\nu(I_n) := \max\{x_{i+1} - x_i | i = 0, \ldots, k-1\}$, then the following corollary also holds.

Corollary 4. *If x_i $(i = 1, \ldots, k-1)$ are points of differentiability for f then*

$$\frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=0}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right] - \int_a^b f(t)\, dt \leq \frac{1}{8}[\nu(I_n)]^2 \left[f'_-(b) - f'_+(a)\right]. \tag{2.12}$$

3. Some Particular Inequalities

(1) If we choose $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = x \in (a,b)$, $\alpha_2 = b$, then from (2.1) we deduce (see also [6])

$$\begin{aligned} \frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] &\leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\, dt \\ &\leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_-(a)\right]. \end{aligned} \tag{3.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [6]).

If $x = \frac{a+b}{2}$, then by (3.1) one deduces (see also [6])

$$\begin{aligned} 0 &\leq \frac{1}{8}(b-a)^2 \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f(a) + f(b)}{2} \cdot (b-a) - \int_a^b f(t)\, dt \\ &\leq \frac{1}{8}(b-a)^2 \left[f'_-(b) - f'_+(a)\right] \end{aligned} \tag{3.2}$$

and the constant $\frac{1}{8}$ in both inequalities is sharp (see for example [6]).

If one would assume that $x \in (a,b)$ is a point of differentiability, then

$$(b-a)\left(\frac{a+b}{2} - x\right) f'(x) \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\, dt. \tag{3.3}$$

(2) If we choose $a=x_0<x<x_2=b$ and the numbers $\alpha_0=a$, $\alpha\in(a,x]$, $\beta\in[x,b)$ and $\alpha_3=b$, then by Theorem 3, we deduce

$$\begin{aligned}
&\frac{1}{2}\left[(x-\alpha)^2 f'_+(\alpha)-(\alpha-a)^2 f'_-(\alpha)+(b-\beta)^2 f'_+(\beta)-(\beta-x)^2 f'_-(\beta)\right] \\
\leq\;& (\alpha-a) f(a)+(\beta-\alpha) f(x)+(b-\beta) f(b)-\int_a^b f(t)\,dt \\
\leq\;& \frac{1}{2}\left[(x-\alpha)^2 f'_-(x)-(\alpha-a)^2 f'_+(a)+(b-\beta)^2 f'_-(b)-(\beta-x)^2 f'_+(x)\right].
\end{aligned} \tag{3.4}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

(a) Note that if we let $\alpha\to a+$ and $\beta\to b-$, then from (3.4), by taking into account firstly that $(x-\alpha)^2 f'_+(a)\leq(x-\alpha)^2 f'_+(\alpha)$ and $-(\beta-x)^2 f'_-(b)\leq-(\beta-x)^2 f'_-(\beta)$, we may deduce the inequality obtained in [7]:

$$\begin{aligned}
&\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\
\leq\;& \int_a^b f(t)\,dt-(b-a) f(x) \\
\leq\;& \frac{1}{2}\left[(\beta-x)^2 f'_-(b)+(x-a)^2 f'_+(a)\right].
\end{aligned} \tag{3.5}$$

The constant $\frac{1}{2}$ is sharp in both inequalities (see for example [7]).
If in (3.5) we choose $x=\frac{a+b}{2}$, then (see also [7])

$$\begin{aligned}
0 \leq\;& \frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\
\leq\;& \int_a^b f(t)\,dt-(b-a) f\left(\frac{a+b}{2}\right) \\
\leq\;& \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right]
\end{aligned} \tag{3.6}$$

and the constant $\frac{1}{8}$ is sharp in both inequalities.
We may state now the following result for convex functions improving Hermite-Hadamard integral inequalities.

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}
0 \leq\;& \frac{1}{8}(b-a)\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\
\leq\;& \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right) \\
\leq\;& \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \\
\leq\;& \frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right].
\end{aligned} \tag{3.7}$$

The constant $\frac{1}{8}$ is sharp in both parts.

If one would assume that $x \in (a,b)$ is a differentiability point for f, then we have the inequality [7]

$$(b-a)\left(\frac{a+b}{2}-x\right)f'(x) \leq \int_a^b f(t)\,dt-(b-a)f(x). \tag{3.8}$$

(b) If we choose $\alpha=\frac{a+x}{2}$ and $\beta=\frac{x+b}{2}$, then by (3.4) we have the three point inequality:

$$\begin{aligned} 0 &\leq \frac{1}{8}\left\{(x-a)^2\left[f'_+\left(\frac{a+x}{2}\right)-f'_-\left(\frac{a+x}{2}\right)\right]\right. \\ &\qquad \left.+(b-x)^2\left[f'_+\left(\frac{x+b}{2}\right)-f'_-\left(\frac{x+b}{2}\right)\right]\right\} \\ &\leq \frac{1}{2}\left[(x-a)f(a)+f(x)(b-a)+(b-x)f(b)\right]-\int_a^b f(t)\,dt \\ &\leq \frac{1}{8}\left\{(x-a)^2\left[f'_+(x)-f'_-(a)\right]+(b-x)^2\left[f'_-(b)-f'_+(x)\right]\right\} \end{aligned} \tag{3.9}$$

for any $x\in(a,b)$. The constant $\frac{1}{8}$ is sharp in both parts.
If in (3.9) we choose $x=\frac{a+b}{2}$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{32}(b-a)^2\left[f'_+\left(\frac{3a+b}{4}\right)-f'_-\left(\frac{3a+b}{4}\right)\right. \\ &\qquad \left.+f'_+\left(\frac{a+3b}{4}\right)-f'_-\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right](b-a)-\int_a^b f(t)\,dt \\ &\leq \frac{1}{32}(b-a)^2\left[f'_-(b)-f'_+\left(\frac{a+b}{2}\right)+f'_-\left(\frac{a+b}{2}\right)-f'_+(a)\right] \end{aligned} \tag{3.10}$$

If one would assume that f is differentiable in $\frac{a+b}{2}$, then we get the following reverse of Bullen's inequality

$$\begin{aligned} 0 &\leq \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right](b-a)-\int_a^b f(t)\,dt \\ &\leq \frac{1}{32}(b-a)^2\left[f'_-(b)-f'_+(a)\right]. \end{aligned} \tag{3.11}$$

The constant $\frac{1}{32}$ is sharp.

(c) Now, if we choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ and $x \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$ in (3.4), then we have the inequalities

$$\begin{aligned} & \frac{1}{2}\left[\left(x-\frac{5a+b}{6}\right)^2 f'_+\left(\frac{5a+b}{6}\right)-\frac{(b-a)^2}{36}f'_-\left(\frac{5a+b}{6}\right)\right. \\ & \left.+\frac{(b-a)^2}{36}f'_+\left(\frac{a+5b}{6}\right)-\left(\frac{a+5b}{6}-x\right)^2 f'_-\left(\frac{a+5b}{6}\right)\right] \\ \leq\ & \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f(x)\right]-\int_a^b f(t)\,dt \\ \leq\ & \frac{1}{2}\left[\left(x-\frac{5a+b}{6}\right)^2 f'_-(x)-\frac{(b-a)^2}{36}f'_+(a)\right. \\ & \left.+\frac{(b-a)^2}{36}f'_-(b)-\left(\frac{a+5b}{6}-x\right)^2 f'_+(x)\right]. \end{aligned} \tag{3.12}$$

If in (3.12) we choose $x = \frac{a+b}{2}$, then we get the Simpson's inequality

$$\begin{aligned} & \frac{1}{18}(b-a)^2\left[f'_+\left(\frac{5a+b}{6}\right)-\frac{1}{4}f'_-\left(\frac{5a+b}{6}\right)\right. \\ & \qquad\left.+\frac{1}{4}f'_+\left(\frac{a+5b}{6}\right)-f'_-\left(\frac{a+5b}{6}\right)\right] \\ \leq\ & \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt \\ \leq\ & \frac{1}{18}(b-a)^2\left[f'_-\left(\frac{a+b}{2}\right)-\frac{1}{4}f'_+(a)+\frac{1}{4}f'_-(b)-f'_+\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{3.13}$$

If the function is differentiable on (a,b), then we get

$$\begin{aligned} & -\frac{1}{24}(b-a)^2\left[f'\left(\frac{a+5b}{6}\right)-f'\left(\frac{5a+b}{6}\right)\right] \\ \leq\ & \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt \\ \leq\ & \frac{1}{72}(b-a)^2\left[f'_-(b)-f'_+(a)\right]. \end{aligned} \tag{3.14}$$

References

[1] G. ANASTASSIOU, Ostrowski type inequalities, *Proc. Amer. Math. Soc.,* **123**(12) (1995), 3775-3781.

[2] S.S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60** (1999), 495-508.

[3] S.S. DRAGOMIR, A generalisation of Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a,b]$, $1 < p < \infty$ and applications in numerical integration, *J. Math. Anal. Appl.,* **225** (2001), 605-626.

[4] S.S. DRAGOMIR, A generalisation of Ostrowski integral inequality for mappings whose derivatives belong to $L_1[a,b]$, and applications in numerical integration, *J. Computational Analysis and Appl.,* **3**(4) (2001), 343-360.

[5] S.S. DRAGOMIR and Th. M. RASSIAS (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht, 2002.

[6] S.S. DRAGOMIR, A generalised trapezoid type inequality for convex functions, (Preprint) *RGMIA Res. Rep. Coll.,* **5**(1) (2002), Article 9. [ONLINE] `http://rgmia.vu.edu.au/v5n1.html`

[7] S.S. DRAGOMIR, An Ostrowski type inequality for convex functions, (Preprint) *RGMIA Res. Rep. Coll.,* **5**(1) (2002), Article 5. [ONLINE] `http://rgmia.vu.edu.au/v5n1.html`

[8] A.M. FINK, Bounds on the derivation of a function from its averages, *Czech. Math. J.,* **42** (1992), 289-310.

[9] A. OSTROWSKI, Über die Absolutabweichung einer differentiienbaren Funcktion von ihren Integralwittelwert, *Comment. Math. Helv.,* **10** (1938), 220-227.

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