SOME OSTROWSKI TYPE INEQUALITES VIA CAUCHY'S MEAN VALUE THEOREM

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Abstract. Some Ostrowski type inequalities via Cauchy's mean value theorem and applications for certain particular instances of functions are given.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\leq M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [2], the author has proved the following Ostrowski type inequality.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ with $a>0$ and differentiable on (a,b). Let $p\in\mathbb{R}\setminus\{0\}$ and assume that*

$$K_p(f'):=\sup_{u\in(a,b)}\left\{u^{1-p}|f'(u)|\right\}<\infty.$$

Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{K_p(f')}{|p|(b-a)} \tag{1.2}$$

$$\times\begin{cases}2x^p(x-A)+(b-x)L_p^p(b,x)-(x-a)L_p^p(x,a) & \text{if } p\in(0,\infty);\\[2mm] (x-a)L_p^p(x,a)-(b-x)L_p^p(b,x)-2x^p(x-A) & \text{if } p\in(-\infty,-1)\cup(-1,0)\\[2mm] (x-a)L^{-1}(x,a)-(b-x)L^{-1}(b,x)-\frac{2}{x}(x-A) & \text{if } p=-1,\end{cases}$$

for any $x\in(a,b)$, where for $a\neq b$,

$$A=A(a,b):=\frac{a+b}{2},\quad \textit{is the arithmetic mean,}$$

$$L_p=L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\ \textit{is the } p-\textit{logarithmic mean } p\in\mathbb{R}\setminus\{-1,0\},$$

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and

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \quad \textit{is the logarithmic mean.}$$

Another result of this type obtained in the same paper is:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ (with $a > 0$) and differentiable on (a,b). If*

$$P(f') := \sup_{u \in (a,b)} |uf'(x)| < \infty,$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{P(f')}{b-a}\left[\ln\left[\frac{[I(x,b)]^{b-x}}{[I(a,x)]^{x-a}} \right] + 2(x-A)\ln x \right] \tag{1.3}$$

for any $x \in (a,b)$, where for $a \neq b$

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad \textit{is the identric mean.}$$

If some local information around the point $x \in (a,b)$ is available, then we may state the following result as well [2].

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Let $p \in (0,\infty)$ and assume, for a given $x \in (a,b)$, we have that*

$$M_p(x) := \sup_{u \in (a,b)} \left\{ |x-u|^{1-p} |f'(u)| \right\} < \infty.$$

Then we have the inequality

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1} + (b-x)^{p+1} \right] M_p(x). \end{aligned} \tag{1.4}$$

For recent results in connection to Ostrowski's inequality see the papers [3],[4] and the monograph [5].

The main aim of this paper is to point out some generalizations of the results incorporated in Theorems 2-4 by the use of Cauchy mean value theorem. Applications for other particular instances of functions are given as well.

2. The Results

We may state the following theorem.

Theorem 5. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $g'(t) \neq 0$ for each $t \in (a,b)$ and*

$$\left\| \frac{f'}{g'} \right\|_\infty := \sup_{t \in (a,b)} \left| \frac{f'(t)}{g'(t)} \right| < \infty, \tag{2.1}$$

then for any $x \in [a,b]$ one has the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left|2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)g(x)+\frac{\int_x^b g(t)\,dt-\int_a^x g(t)\,dt}{b-a}\right|\cdot\left\|\frac{f'}{g'}\right\|_\infty . \tag{2.2}$$

Proof. Let $x,t \in [a,b]$ with $t \neq x$. Applying Cauchy's mean value theorem, there exists a η between t and x such that

$$(f(x)-f(t)) = \frac{f'(\eta)}{g'(\eta)}(g(x)-g(t))$$

from where we get

$$|f(x)-f(t)| = \left|\frac{f'(\eta)}{g'(\eta)}\right||g(x)-g(t)| \leq \left\|\frac{f'}{g'}\right\|_\infty |g(x)-g(t)|, \tag{2.3}$$

for any $t,x \in [a,b]$.

Using the properties of the integral, we deduce by (2.3), that

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{1}{b-a}\int_a^b |f(x)-f(t)|\,dt \\ &\leq \left\|\frac{f'}{g'}\right\|_\infty \frac{1}{b-a}\int_a^b |g(x)-g(t)|\,dt.\end{aligned} \tag{2.4}$$

Since $g'(t) \neq 0$ on (a,b), it follows that either $g'(t) > 0$ or $g'(t) < 0$ for any $t \in (a,b)$.

If $g'(t) > 0$ for all $t \in (a,b)$, then g is strictly monotonic increasing on (a,b) and

$$\begin{aligned}\int_a^b |g(x)-g(t)|\,dt &= \int_a^x (g(x)-g(t))\,dt + \int_x^b (g(t)-g(x))\,dt \\ &= 2\left(x-\frac{a+b}{2}\right)g(x)+\int_x^b g(t)\,dt-\int_a^x g(t)\,dt.\end{aligned}$$

If $g'(t) < 0$ for all $t \in (a,b)$, then

$$\int_a^b |g(x)-g(t)|\,dt = -\left[2\left(x-\frac{a+b}{2}\right)g(x)+\int_x^b g(t)\,dt-\int_a^x g(t)\,dt\right]$$

and the inequality (2.2) is proved. ∎

The following midpoint inequality is a natural consequence of the above result.

Corollary 1. *With the above assumptions for f and g, one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{b-a}\left|\int_{\frac{a+b}{2}}^b g(t)\,dt-\int_a^{\frac{a+b}{2}} g(t)\,dt\right|\left\|\frac{f'}{g'}\right\|_\infty . \tag{2.5}$$

Remark 1. (1) *If in the above theorem, we choose $g(t) = t$, then from (2.2) we recapture Ostrowski's inequality (1.1).*

(2) *If in Theorem 5 we choose* $g(t) = t^p$, $p \in \mathbb{R}\setminus\{0\}$, *or* $g(t) = \ln t$ *with* $t \in (a,b) \subset (0,\infty)$, *then we obtain Theorem 2 and Theorem 3 respectively.*

One may obtain many inequalities from Theorem 5 on choosing different instances of functions g.

Proposition 1. *Let* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *If there exists a constant* $\Gamma < \infty$ *such that*

$$|f'(t)| \leq \Gamma e^{-t} \quad \textit{for any} \quad t \in (a,b), \tag{2.6}$$

then one has the inequality:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \Gamma\left[2\left(\frac{x - A(a,b)}{b-a}\right)e^x + \frac{(b-x)E(x,b) - (x-a)E(a,x)}{b-a}\right] \tag{2.7}$$

for any $x \in (a,b)$, *where* $A = A(a,b) = \frac{a+b}{2}$ *and* E *is the exponential men, i.e.,*

$$E(x,y) := \begin{cases} \dfrac{e^x - e^y}{x-y} & \textit{if } x \neq y \\ e^y & \textit{if } x = y \end{cases}, \quad x, y \in \mathbb{R}.$$

In particular, we have

$$\left| f(A) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2}\left[E(A,b) - E(a,A)\right]\Gamma. \tag{2.8}$$

The proof is obvious by Theorem 5 on choosing $g(t) = e^t$ and we omit the details.

Another example is considered in the following proposition.

Proposition 2. *Let* $f : [a,b] \subset \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b).

(*i*) *If there exists a constant* $\Gamma_1 < \infty$ *such that*

$$|f'(t)| \leq \Gamma_1 \cos t, \ t \in (a,b), \tag{2.9}$$

then one has the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \Gamma_1\left[2\left(\frac{x - A(a,b)}{b-a}\right)\sin x + \frac{(x-a)C(a,x) - (b-x)C(x,b)}{b-a}\right] \tag{2.10}$$

for any $x \in (a,b)$, *where* C *is the cos-mean value, i.e.,*

$$C(x,y) := \begin{cases} \dfrac{\cos x - \cos y}{x-y} & \textit{if } x \neq y \\ -\sin y & \textit{if } x = y \end{cases}.$$

In particular we have

$$\left|f(A)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left[C(a,A)-C(A,b)\right]\Gamma_1. \tag{2.11}$$

(*ii*) *If there exists a constant* $\Gamma_2<\infty$ *such that*

$$|f'(t)| \leq \Gamma_1 \sin t,\ t\in(a,b), \tag{2.12}$$

then one has the inequality

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \Gamma_2\left[2\left(\frac{x-A(a,b)}{b-a}\right)\cos x+\frac{(b-x)S(x,b)-(x-a)S(a,x)}{b-a}\right],\end{aligned} \tag{2.13}$$

for any $x\in(a,b)$, *where* S *is the* $\sin -$*mean value, i.e.,*

$$S(x,y):=\begin{cases}\dfrac{\sin x-\sin y}{x-y} & \text{if } x\neq y \\ \cos y & \text{if } x=y\end{cases}.$$

In particular, we have

$$\left|f(A)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left[S(A,b)-S(a,A)\right]\Gamma_2. \tag{2.14}$$

The following result also holds.

Theorem 6. *Let* $f,g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* $(a,b)\setminus\{x\}$, $x\in(a,b)$. *If* $g'(t)\neq 0$ *for* $t\in(a,x)\cup(x,b)$, *then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left|g(x)(x-a)-\int_a^x g(t)\,dt\right|\cdot\left\|\frac{f'}{g'}\right\|_{(a,x),\infty} \\ &\quad+\frac{1}{b-a}\left|g(x)(b-x)-\int_x^b g(t)\,dt\right|\cdot\left\|\frac{f'}{g'}\right\|_{(x,b),\infty}.\end{aligned} \tag{2.15}$$

Proof. We obviously have:

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &=\left|\frac{1}{b-a}\int_a^b (f(x)-f(t))\,dt\right| \\ &\leq \frac{1}{b-a}\int_a^b |f(x)-f(t)|\,dt \\ &=\frac{1}{b-a}\left[\int_a^x |f(x)-f(t)|\,dt+\int_x^b |f(x)-f(t)|\,dt\right].\end{aligned} \tag{2.16}$$

Applying Cauchy's mean value theorem on the interval (a,x), we deduce (see the proof of Theorem 5) that

$$|f(x)-f(t)| \leq \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} |g(x)-g(t)| \tag{2.17}$$

for any $t \in (a,x)$, and, similarly

$$|f(x)-f(t)| \leq \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} |g(x)-g(t)| \tag{2.18}$$

for any $t \in (x,b)$.

Consequently

$$\int_a^x |f(x)-f(t)|\,dt \leq \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} \int_a^x |g(x)-g(t)|\,dt$$

and

$$\int_x^b |f(x)-f(t)|\,dt \leq \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} \int_x^b |g(x)-g(t)|\,dt.$$

Since g' has a constant sign in either (a,x) or (x,b), it follows that g is strictly increasing or strictly decreasing in (a,x) and (x,b).

Thus

$$\begin{aligned} \int_a^x |g(x)-g(t)|\,dt &= \begin{cases} g(x)(x-a) - \int_a^x g(t)\,dt & \text{if } g \text{ is increasing on } [a,x] \\ \int_a^x g(t)\,dt - g(x)(x-a) & \text{if } g \text{ is decreasing} \end{cases} \\ &= \left| g(x)(x-a) - \int_a^x g(t)\,dt \right| \end{aligned}$$

and, in a similar way

$$\int_x^b |g(x)-g(t)|\,dt = \left| g(x)(b-x) - \int_x^b g(t)\,dt \right|.$$

Consequently, by the use of (2.16), we deduce the desired inequality (2.15). ∎

The following particular case may be of interest.

Corollary 2. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\left\{\frac{a+b}{2}\right\}$. If $g'(t)\neq 0$ on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$, then we have the inequality*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{2}\left\{ \left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)\,dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{\left(a,\frac{a+b}{2}\right),\infty} \right. \\ &\quad \left. + \left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_{\frac{a+b}{2}}^b g(t)\,dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{\left(\frac{a+b}{2},b\right),\infty} \right\}. \end{aligned} \tag{2.19}$$

The following result also holds.

Proposition 3. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\{x\}$, $x\in(a,b)$. Assume that, for $p>0$, we have*

$$|f'(t)|\leq\begin{cases}M_{1,p}(x)(x-t)^{1-p} & \text{for any } t\in(a,x),\\ \\ M_{2,p}(x)(t-x)^{1-p} & \text{for any } t\in(x,b).\end{cases}\tag{2.20}$$

Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{p(p+1)}(b-a)\left[M_{1,p}(x)(x-a)^{p+1}+M_{2,p}(x)(b-x)^{p+1}\right].\tag{2.21}$$

The proof follows by Theorem 6 applied for $g(x)=|x-t|^p$, $p>0$. We omit the details.

Remark 2. *If f is as in Proposition 3 and*

$$|f'(t)|\leq\begin{cases}M_1\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}-t\right)^{1-p} & \text{for any } t\in\left(a,\frac{a+b}{2}\right),\\ \\ M_2\left(\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)^{1-p} & \text{for any } t\in\left(\frac{a+b}{2},b\right),\end{cases}\tag{2.22}$$

then, by (2.21), we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^{p+1}}{2^{p+1}p(p+1)}\left[M_1\left(\frac{a+b}{2}\right)+M_2\left(\frac{a+b}{2}\right)\right].\tag{2.23}$$

Remark 3. *If f is as in Proposition 3 and*

$$|f'(t)|\leq M_p(x)|x-t|^{1-p}\quad t\in(a,b),$$

then, by (2.21), we get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]M_p(x),\tag{2.24}$$

which is the result obtained in (1.4).

3. Some Inequalities of Midpoint Type

(1) Let $0<a<b$. Consider the function $g:[a,b]\to\mathbb{R}$, $g(t)=t^p$, $t\in\mathbb{R}\setminus\{0,-1\}$. Then $g'(t)=pt^{p-1}$, $g\left(\frac{a+b}{2}\right)=A^p(a,b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}g(t)\,dt=L_p^p(a,A(a,b)),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b g(t)\,dt=L_p^p(A(a,b),b),$$

and by Corollary 2, we may state the following proposition.

Proposition 4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)|\leq\begin{cases}M_1\left(\frac{a+b}{2}\right)t^p, & t\in\left(a,\frac{a+b}{2}\right),\\ M_2\left(\frac{a+b}{2}\right)t^p, & t\in\left(\frac{a+b}{2},b\right),\end{cases}\tag{3.1}$$

then we have the inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2p}\left\{M_1\left(\frac{a+b}{2}\right)\left|A^p(a,b)-L_p^p(a,A(a,b))\right|+M_2\left(\frac{a+b}{2}\right)\left|L_p^p(A(a,b),b)-A^p(a,b)\right|\right\}.\tag{3.2}$$

The particular case $p=1$ is of interest and so we may state the following corollary.

Corollary 3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)|\leq\begin{cases}N_1\left(\frac{a+b}{2}\right)t, & t\in\left(a,\frac{a+b}{2}\right),\\ N_2\left(\frac{a+b}{2}\right)t, & t\in\left(\frac{a+b}{2},b\right),\end{cases}\tag{3.3}$$

then we have the inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}\left[N_1\left(\frac{a+b}{2}\right)+N_2\left(\frac{a+b}{2}\right)\right](b-a).\tag{3.4}$$

(2) Let $0<a<b$. Consider the function $g:[a,b]\to\mathbb{R}$, $g(t)=\frac{1}{t}$. Then $g'(t)=-\frac{1}{t^2}$, $g\left(\frac{a+b}{2}\right)=A^{-1}(a,b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}g(t)\,dt=L^{-1}(a,A(a,b)),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b g(t)\,dt=L^{-1}(A(a,b),b),$$

and by Corollary 2 we may state the following Proposition.

Proposition 5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)|\leq\begin{cases}M_1\left(\frac{a+b}{2}\right)t^{-2}, & t\in\left(a,\frac{a+b}{2}\right),\\ M_2\left(\frac{a+b}{2}\right)t^{-2}, & t\in\left(\frac{a+b}{2},b\right),\end{cases}\tag{3.5}$$

then we have the inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.6}$$
$$\leq \frac{1}{2}\left[M_1\left(\frac{a+b}{2}\right)\cdot\frac{[A(a,b)-L(a,A(a,b))]}{L(a,A(a,b))\,A(a,b)}\right.$$
$$\left.+\,M_2\left(\frac{a+b}{2}\right)\cdot\frac{[L(A(a,b),b)-A(a,b)]}{L(A(a,b),b)\,A(a,b)}\right].$$

(3) Let $0<a<b$. Consider the function $g:[a,b]\to\mathbb{R},\ g(t)=\ln t$. Then $g'(t)=\frac{1}{t},\ g\left(\frac{a+b}{2}\right)=\ln A(a,b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)\,dt=\ln I(a,A(a,b)),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b g(t)\,dt=\ln I(A(a,b),b),$$

and by Corollary 2 we may state the following proposition.

Proposition 6. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\setminus\left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)|\leq\begin{cases} M_1\left(\frac{a+b}{2}\right)t^{-1}, & t\in\left(a,\frac{a+b}{2}\right),\\ \\ M_2\left(\frac{a+b}{2}\right)t^{-1}, & t\in\left(\frac{a+b}{2},b\right),\end{cases} \tag{3.7}$$

then we have the inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.8}$$
$$\leq \ln\left\{G\left(\left[\frac{A(a,b)}{I(a,A(a,b))}\right]^{M_1\left(\frac{a+b}{2}\right)},\left[\frac{I(A(a,b),b)}{A(a,b)}\right]^{M_2\left(\frac{a+b}{2}\right)}\right)\right\}.$$

4. The Case of Weighed Integrals

We may state the following theorem.

Theorem 7. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and $w:[a,b]\to[0,\infty)$ an integrable function such that $\int_a^b w(s)\,ds>0$. If $g'(t)\neq 0$ for each $t\in(a,b)$ and*

$$\left\|\frac{f'}{g'}\right\|_\infty:=\sup_{t\in(a,b)}\left|\frac{f'(t)}{g'(t)}\right|<\infty, \tag{4.1}$$

then for any $x\in(a,b)$ one has the inequality

$$\left|f(x)-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,dt\right| \tag{4.2}$$
$$\leq\left|g(x)\cdot\frac{\int_a^x w(t)\,dt-\int_x^b w(t)\,dt}{\int_a^b w(t)\,dt}+\frac{\int_x^b w(t)\,g(t)\,dt-\int_a^x g(t)\,w(t)\,dt}{\int_a^b w(t)\,dt}\right|\cdot\left\|\frac{f'}{g'}\right\|_\infty.$$

Proof. Let $x,t\in[a,b]$ with $t\neq x$. Applying Cauchy's mean value theorem, there exists a η between t and x such that

$$f(x)-f(t)=\frac{f'(\eta)}{g'(\eta)}[g(x)-g(t)],$$

from where we get

$$|f(x)-f(t)|=\left|\frac{f'(\eta)}{g'(\eta)}\right||g(x)-g(t)|\leq\left\|\frac{f'}{g'}\right\|_{\infty}|g(x)-g(t)| \tag{4.3}$$

for any $t,x\in[a,b]$.

Using the properties of the integral, we deduce by (4.3), that

$$\begin{aligned}&\left|f(x)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)f(s)\,ds\right| \\ &\leq\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)|f(x)-f(t)|\,dt \\ &\leq\left\|\frac{f'}{g'}\right\|_{\infty}\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)|g(x)-g(t)|\,dt.\end{aligned} \tag{4.4}$$

Since $g'(t)\neq 0$ on (a,b), it follows that either $g'(t)>0$ or $g'(t)<0$ for any $t\in(a,b)$.

If $g'(t)>0$ for all $t\in(a,b)$, then g is strictly monotonic increasing on (a,b) and

$$\begin{aligned}&\int_a^b w(t)|g(x)-g(t)|\,dt \\ &=\int_a^x w(t)(g(x)-g(t))\,dt+\int_x^b w(t)(g(t)-g(x))\,dt \\ &=g(x)\int_a^x w(t)\,dt-\int_a^x w(t)g(t)\,dt+\int_x^b w(t)g(t)\,dt-g(x)\int_x^b w(t)\,dt \\ &=g(x)\left[\int_a^x w(t)\,dt-\int_x^b w(t)\,dt\right]+\int_x^b w(t)g(t)\,dt-\int_a^x w(t)g(t)\,dt.\end{aligned}$$

If $g'(t)<0$ for all $t\in(a,b)$, then

$$\begin{aligned}&\int_a^b w(t)|g(x)-g(t)|\,dt \\ &=-\left[g(x)\left[\int_a^x w(t)\,dt-\int_x^b w(t)\,dt\right]+\int_x^b w(t)g(t)\,dt-\int_a^x w(t)g(t)\,dt\right],\end{aligned}$$

and the inequality (2.2) is proved. ∎

Corollary 4. *If $x_0\in[a,b]$ is a point for which*

$$\int_a^{x_0} w(t)\,dt=\int_{x_0}^b w(t)\,dt, \tag{4.5}$$

and f, g, w are as in Theorem 7, then we have the inequality

$$\left| f(x_0) - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt \right| \leq \frac{\left| \int_{x_0}^b w(t) g(t)\,dt - \int_a^{x_0} g(t) w(t)\,dt \right|}{\int_a^b w(t)\,dt} \cdot \left\| \frac{f'}{g'} \right\|_\infty . \tag{4.6}$$

In a similar manner, we may deduce the following result as well.

Theorem 8. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b) \setminus \{x\}$, $x \in (a, b)$. If $w : [a, b] \to [0, \infty)$ is integrable and $\int_a^b w(s)\,ds > 0$ and $g'(t) \neq 0$ for $t \in (a, x) \cup (x, b)$, then we have the inequality*

$$\begin{aligned} \left| f(x) - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt \right| &\leq \left| g(x) \cdot \frac{\int_a^x w(t)\,dt}{\int_a^b w(t)\,dt} - \frac{\int_a^x w(t) g(t)\,dt}{\int_a^b w(t)\,dt} \right| \cdot \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} \\ &\quad + \left| g(x) \cdot \frac{\int_x^b w(t)\,dt}{\int_a^b w(t)\,dt} - \frac{\int_x^b w(t) g(t)\,dt}{\int_a^b w(t)\,dt} \right| \cdot \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} . \end{aligned} \tag{4.7}$$

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