Ostrowski's Inequality in Complex Inner Product Spaces

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Abstract. A version of Ostrowski's inequality in complex inner product spaces is given. Applications for complex sequences and integrals are also provided.

1. Introduction

In 1951, A.M. Ostrowski [**2**, p. 289] proved the following result (see also [**1**, p. 92])

Theorem 1. *Suppose that* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{x}$ *are real n-tuples such that* $\mathbf{a} \neq \mathbf{0}$ *and*

$$\sum_{i=1}^{n} a_i x_i = 0 \text{ and } \sum_{i=1}^{n} b_i x_i = 1. \tag{1.1}$$

Then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \tag{1.2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}, \tag{1.3}$$

for $k \in \{1, ..., n\}$.

An integral version of this inequality was obtained by Pearce, Pečarić and Varošanec in 1998, [**3**].

H. Šikić and T. Šikić in 2001, [**4**], by the use of an argument based on orthogonal projection in inner product spaces have observed that Ostrowski's inequality may be naturally stated in this abstract setting as follows:

Theorem 2. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $a, b \in H$ *two linearly independent vectors. If* $x \in H$ *is so that*

$$\langle x, a \rangle = 0 \text{ and } \langle x, b \rangle = 1, \tag{1.4}$$

then we has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}, \tag{1.5}$$

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 46C05.
Key words and phrases. Ostrowski's Inequality, Inner Product Spaces.

with equality if and only if

$$x = \frac{\|a\|^2 b - \overline{\langle a, b\rangle} \cdot a}{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}. \tag{1.6}$$

In the present note, by the use of elementary arguments only and Schwarz's inequality in inner product spaces, we show that Ostrowski's inequality (1.5) holds true for a larger class of elements $x \in H$. The case of equality is analyzed. Applications for complex sequences and integrals are also provided.

2. The Results

The following theorem holds.

Theorem 3. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $a, b \in H$ *two linearly independent vectors. If* $x \in H$ *is so that*

$$\langle x, a\rangle = 0, \text{ and } |\langle x, b\rangle| = 1; \tag{2.1}$$

then one has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}. \tag{2.2}$$

The equality holds in (2.2) *if and only if*

$$x = \mu \left(b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a \right) \tag{2.3}$$

where $\mu \in \mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *is so that*

$$|\mu| = \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}. \tag{2.4}$$

Proof. We use Schwarz's inequality in the inner product space $(H; \langle \cdot, \cdot \rangle)$, i.e.,

$$\|u\|^2 \|v\|^2 \geq |\langle u, v\rangle|^2 ; u, v \in H \tag{2.5}$$

with equality iff there exists a scalar $\alpha \in \mathbb{K}$ so that $u = \alpha v$.

If we apply (2.5) for

$$u = z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c, v = d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c$$

where $c \neq 0$ and $c, d, z \in H$, we have

$$\left\| z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c \right\|^2 \left\| d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\|^2 \geq \left| \left\langle z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right\rangle \right|^2 \tag{2.6}$$

with equality iff there is a scalar $\beta \in \mathbb{K}$ so that

$$z = \frac{\langle z, c\rangle}{\|c\|^2} \cdot c + \beta \left(d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right). \tag{2.7}$$

Since simple calculation show that

$$\left\| z - \frac{\langle z, c\rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|z\|^2 \|c\|^2 - |\langle z, c\rangle|^2}{\|c\|^2},$$

$$\left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2}{\|c\|^2},$$

and

$$\left\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\rangle = \frac{\langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle}{\|c\|^2},$$

then, by (2.6), we deduce

$$\begin{aligned} & \left[\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \\ \geq & \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^2, \end{aligned} \tag{2.8}$$

with equality if and only if there is a $\beta \in \mathbb{K}$ so that (2.7) holds.

If a, x, b satisfy (2.1) then by (2.8) and (2.7) for the choices $z = x, c = a$ and $d = b$ we deduce the inequality (2.2) with equality iff there exists a $\mu \in \mathbb{K}$ so that

$$x = \mu \left(b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right)$$

and, by the second condition in (2.1) ,

$$\left| \mu \left\langle b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a, b \right\rangle \right| = 1. \tag{2.9}$$

Since (2.9) is clearly equivalent with (2.4), the theorem is completely proved. ∎

3. Applications

The following particular cases hold.

1. If $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \ell^2(\mathbb{K})$, where $\ell^2(\mathbb{K}) := \left\{ \mathbf{x} = (x_i)_{i \in \mathbb{N}}, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$, with $\mathbf{a}, \mathbf{b}$ linearly independent and

$$\sum_{i=1}^{\infty} x_i \overline{a_i} = 0 \text{ and } \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right| = 1,$$

then one has the inequality

$$\sum_{i=1}^{\infty} |x_i|^2 \geq \frac{\sum_{i=1}^{\infty} |a_i|^2}{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2} \tag{3.1}$$

with

$$x_i = \mu \left[b_i - \frac{\sum_{k=1}^{\infty} \overline{a_k} b_k}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], i \in \mathbb{N} \tag{3.2}$$

and $\mu \in \mathbb{K}$ with the property

$$|\mu| = \frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2}{\sum_{i=1}^{\infty} |a_i|^2}. \tag{3.3}$$

2. If $f,g,h\in L^{2}(\Omega,m)$, where Ω is a measurable space and $L^{2}(\Omega,m):=\left\{f:\Omega\to\mathbb{K},\ \int_{\Omega}|f(x)|^{2}dm(x)<\infty\right\}$, with f,g are linearly independent and

$$\int_{\Omega}h(x)\overline{f(x)}dm(x)=0,\quad\left|\int_{\Omega}h(x)\overline{g(x)}dm(x)\right|=1,$$

then one has the inequality

$$\int_{\Omega}|h(x)|^{2}dm(x)\geq\frac{\int_{\Omega}|f(x)|^{2}dm(x)}{\int_{\Omega}|f(x)|^{2}dm(x)\int_{\Omega}|g(x)|^{2}dm(x)-\left|\int_{\Omega}f(x)\overline{g(x)}dm(x)\right|^{2}}\tag{3.4}$$

with equality iff

$$h(x)=\nu\left[g(x)-\frac{\int_{\Omega}\overline{f(x)}g(x)\,dm(x)}{\int_{\Omega}|f(x)|^{2}dm(x)}\cdot f(x)\right]\tag{3.5}$$

for $m-a.e.x\in\Omega$, and $\nu\in\mathbb{K}$ with

$$|\nu|=\frac{\int_{\Omega}|f(x)|^{2}dm(x)}{\int_{\Omega}|f(x)|^{2}dm(x)\int_{\Omega}|g(x)|^{2}dm(x)-\left|\int_{\Omega}f(x)\overline{g(x)}dm(x)\right|}.\tag{3.6}$$

References

[1] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, Dordrecht, 1993.

[2] A.M. OSTROWSKI, *Vorlesungen über Differential und Integralrechnung II,* Birkhäuser, Basel, 1951.

[3] C.E.M. PEARCE, J.E. PEČARIĆ and S. VAROŠANEC, An integral analogue of the Ostrowski inequality, *J. Ineq. Appl.*, **2**(1998), 275-283.

[4] H. ŠIKIĆ and T. ŠIKIĆ, A note on Ostrowski's inequality, *Math. Ineq. Appl.*, **4**(2)(2001), 297-299.

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