SOME GRÜSS' TYPE INEQUALITIES IN INNER PRODUCT SPACES

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Abstract. Some new Grüss type inequalities in inner product spaces and applications for integrals are given.

1. Introduction

In [1], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1. *Let $(H, \langle .,. \rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

Some particular cases of interest for integrable functions with real or complex values and the corresponding discrete versions are listed bellow.

Corollary 1. *Let $f, g : [a, b] \to \mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ be Lebesgue integrable and so that*

$$\operatorname{Re}\left[(\Phi - f(x))\left(\overline{f(x)} - \overline{\varphi}\right)\right] \geq 0, \quad \operatorname{Re}\left[(\Gamma - g(x))\left(\overline{g(x)} - \overline{\gamma}\right)\right] \geq 0 \tag{1.3}$$

for a.e. $x \in [a, b]$, where $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and $\bar{z}$ denotes the complex conjugate of z. Then we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx - \frac{1}{b-a} \int_a^b f(x)\, dx \cdot \frac{1}{b-a} \int_a^b \overline{g(x)} dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.4}$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is embodied in

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Corollary 2. *Let* $\mathbf{x},\mathbf{y}\in\mathbb{K}^n$ *and* $\varphi,\gamma,\Phi,\Gamma$ *are real or complex numbers so that*

$$\operatorname{Re}\left[\left(\Phi-x_i\right)\left(\overline{x_i}-\overline{\varphi}\right)\right]\geq 0,\operatorname{Re}\left[\left(\Gamma-y_i\right)\left(\overline{y_i}-\overline{\gamma}\right)\right]\geq 0 \tag{1.5}$$

for each $i\in\{1,...,n\}$. *Then we have the inequality*

$$\left|\frac{1}{n}\sum_{i=1}^{n}x_i\overline{y_i}-\frac{1}{n}\sum_{i=1}^{n}x_i\cdot\frac{1}{n}\sum_{i=1}^{n}\overline{y_i}\right|\leq\frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|. \tag{1.6}$$

The constant $\frac{1}{4}$ *is best possible.*

For other applications of Theorem 1, see the recent paper [2].

In the present paper we show that the condition (1.1) may be replaced by an equivalent but simpler assumption and a new proof of Theorem 1 is produced. A refinement of the Grüss' type inequality (1.2), some companions and applications for integrals are pointed out as well.

2. An Equivalent Assumption

The following lemma holds.

Lemma 1. *Let* a,x,A *be vectors in the inner product space* $(H,\langle.,.\rangle)$ *over* $\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ *with* $a\neq A$. *Then*

$$\operatorname{Re}\langle A-x,x-a\rangle\geq 0$$

if and only if

$$\left\|x-\frac{a+A}{2}\right\|\leq\frac{1}{2}\left\|A-a\right\|.$$

Proof. Define

$$I_1:=\operatorname{Re}\langle A-x,x-a\rangle,I_2:=\frac{1}{4}\left\|A-a\right\|^2-\left\|x-\frac{a+A}{2}\right\|^2.$$

A simple calculation shows that

$$I_1=I_2=\operatorname{Re}\left[\langle x,a\rangle+\langle A,x\rangle\right]-\operatorname{Re}\langle A,a\rangle-\left\|x\right\|^2$$

and thus, obviously, $I_1\geq 0$ iff $I_2\geq 0$ showing the required equivalence. ∎

The following corollary is obvious

Corollary 3. *Let* $x,e\in H$ *with* $\|e\|=1$ *and* $\delta,\Delta\in\mathbb{K}$ *with* $\delta\neq\Delta$. *Then*

$$\operatorname{Re}\langle\Delta e-x,x-\delta e\rangle\geq 0$$

iff

$$\left\|x-\frac{\delta+\Delta}{2}\cdot e\right\|\leq\frac{1}{2}\left|\Delta-\delta\right|.$$

Remark 1. *If* $H=\mathbb{C}$, *then*

$$\operatorname{Re}\left[(A-x)\left(\bar{x}-\bar{a}\right)\right]\geq 0$$

if and only if

$$\left|x-\frac{a+A}{2}\right|\leq\frac{1}{2}\left|A-a\right|$$

where $a,x,A\in\mathbb{C}$. *If* $H=\mathbb{R}$, *and* $A>a$ *then* $a\leq x\leq A$ *if and only if* $\left|x-\frac{a+A}{2}\right|\leq\frac{1}{2}\left|A-a\right|$.

The following lemma also holds.

Lemma 2. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2 . \tag{2.1}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned}\langle x - \lambda e, x - \langle x, e\rangle e\rangle &= \|x\|^2 - |\langle x, e\rangle|^2 - \lambda \left[\langle e, x\rangle - \langle e, x\rangle \|e\|^2\right] \\ &= \|x\|^2 - |\langle x, e\rangle|^2 .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2 - |\langle x, e\rangle|^2\right]^2 &= |\langle x - \lambda e, x - \langle x, e\rangle e\rangle|^2 \\ &\leq \|x - \lambda e\|^2 \|x - \langle x, e\rangle e\|^2 \\ &= \|x - \lambda e\|^2 \left[\|x\|^2 - |\langle x, e\rangle|^2\right]\end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \|x - \lambda e\|^2 , \lambda \in \mathbb{K}. \tag{2.2}$$

Taking the infimum in (2.2) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2 .$$

Since, for $\lambda_0 = \langle x, e\rangle$, we get $\|x - \lambda_0 e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2$, then the representation (2.1) is proved. ∎

We are able now to provide a different proof for the Grüss' type inequality in inner product spaced mentioned in Introduction, than the one from paper [1].

Theorem 2. *Let $(H, \langle ., .\rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions* (1.1) *hold, or, equivalently, the following assumptions*

$$\left\|x - \frac{\varphi + \Phi}{2} \cdot e\right\| \leq \frac{1}{2} |\Phi - \varphi| , \left\|y - \frac{\gamma + \Gamma}{2} \cdot e\right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{2.3}$$

are valid. Then one has the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{2.4}$$

The constant $\frac{1}{4}$ is best possible.

Proof. It can be easily shown (see for example the proof of Theorem 1 from [1]) that

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \left[\|x\|^2 - |\langle x, e\rangle|^2\right]^{\frac{1}{2}} \left[\|y\|^2 - |\langle y, e\rangle|^2\right]^{\frac{1}{2}} , \tag{2.5}$$

for any $x, y \in H$ and $e \in H, \|e\| = 1$. Using Lemma 2 and the conditions (2.3) we obviously have that

$$\left[\|x\|^2 - |\langle x, e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\| \leq \left\|x - \frac{\varphi + \Phi}{2} \cdot e\right\| \leq \frac{1}{2} |\Phi - \varphi|$$

and

$$\left[\|y\|^2 - |\langle y, e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|y - \lambda e\| \leq \left\|y - \frac{\gamma + \Gamma}{2} \cdot e\right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

and by (2.5) the desired inequality (2.4) is obtained.

The fact that $\frac{1}{4}$ is the best possible constant, has been shown in [1] and we omit the details. ∎

3. A Refinement of Grüss Inequality

The following result improving (1.1) holds

Theorem 3. *Let $(H, \langle ., . \rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions* (1.1)*, or, equivalently,* (2.3) *hold, then we have the inequality*

$$
\begin{aligned}
(3.1)\quad & |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\
& \le \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}} .
\end{aligned}
$$

Proof. As in [1], we have

$$
(3.2)\qquad |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \le \left[\|x\|^2 - |\langle x, e\rangle|^2\right] \left[\|y\|^2 - |\langle y, e\rangle|^2\right],
$$

$$
(3.3)\qquad \|x\|^2 - |\langle x, e\rangle|^2 = \operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] - \operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle
$$

and

$$
(3.4)\qquad \|y\|^2 - |\langle y, e\rangle|^2 = \operatorname{Re}\left[(\Gamma - \langle y, e\rangle)\left(\overline{\langle y, e\rangle} - \overline{\gamma}\right)\right] - \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle .
$$

Using the elementary inequality

$$
4 \operatorname{Re}\left(a\overline{b}\right) \le |a + b|^2 ; a, b \in \mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})
$$

we may state that

$$
(3.5)\qquad \operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] \le \frac{1}{4} |\Phi - \varphi|^2
$$

and

$$
(3.6)\qquad \operatorname{Re}\left[(\Gamma - \langle y, e\rangle)\left(\overline{\langle y, e\rangle} - \overline{\gamma}\right)\right] \le \frac{1}{4} |\Gamma - \gamma|^2 .
$$

Consequently, by (3.2) – (3.6) we may state that

$$
\begin{aligned}
(3.7)\quad & |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \\
& \le \left[\frac{1}{4} |\Phi - \varphi|^2 - \left([\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}}\right)^2\right] \\
& \qquad \times \left[\frac{1}{4} |\Gamma - \gamma|^2 - \left([\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\right)^2\right].
\end{aligned}
$$

Finally, using the elementary inequality for positive real numbers

$$
\left(m^2 - n^2\right)\left(p^2 - q^2\right) \le (mp - nq)^2
$$

we have

$$\left[\frac{1}{4}\left|\Phi-\varphi\right|^{2}-\left(\left[\operatorname{Re}\left\langle \Phi e-x,x-\varphi e\right\rangle\right]^{\frac{1}{2}}\right)^{2}\right]$$
$$\times\left[\frac{1}{4}\left|\Gamma-\gamma\right|^{2}-\left(\left[\operatorname{Re}\left\langle \Gamma e-y,y-\gamma e\right\rangle\right]^{\frac{1}{2}}\right)^{2}\right]$$
$$\leq\left(\frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|-\left[\operatorname{Re}\left\langle \Phi e-x,x-\varphi e\right\rangle\right]^{\frac{1}{2}}\left[\operatorname{Re}\left\langle \Gamma e-y,y-\gamma e\right\rangle\right]^{\frac{1}{2}}\right)^{2}$$

giving the desired inequality (3.1). ∎

4. Some Companion Inequalities

The following companion of Grüss inequality in inner product spaces holds.

Theorem 4. *Let $(H,\langle .,.\rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e\in H$, $\|e\|=1$. If $\gamma,\Gamma\in\mathbb{K}$ and $x,y\in H$ are so that*

$$\operatorname{Re}\left\langle \Gamma e-\frac{x+y}{2},\frac{x+y}{2}-\gamma e\right\rangle\geq 0 \tag{4.1}$$

or, equivalently,

$$\left\|\frac{x+y}{2}-\frac{\gamma+\Gamma}{2}\cdot e\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|, \tag{4.2}$$

then we have the inequality

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Start with the well known inequality

$$\operatorname{Re}\langle z,u\rangle\leq\frac{1}{4}\left\|z+u\right\|^{2};\, z,u\in H. \tag{4.4}$$

Since

$$\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle=\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle$$

then using (4.4) we may write

$$\begin{aligned}\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]&=\operatorname{Re}\left[\langle x-\langle x,e\rangle e,y-\langle y,e\rangle e\rangle\right]\\&\leq\frac{1}{4}\left\|x-\langle x,e\rangle e+y-\langle y,e\rangle e\right\|^{2}\\&=\left\|\frac{x+y}{2}-\left\langle\frac{x+y}{2},e\right\rangle\cdot e\right\|^{2}\\&=\left\|\frac{x+y}{2}\right\|^{2}-\left|\left\langle\frac{x+y}{2},e\right\rangle\right|^{2}.\end{aligned} \tag{4.5}$$

If we apply Grüss' inequality in inner product spaces for, say, $a=b=\frac{x+y}{2}$, we get

$$\left\|\frac{x+y}{2}\right\|^{2}-\left|\left\langle\frac{x+y}{2},e\right\rangle\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{4.6}$$

Making use of (4.5) and (4.6) we deduce (4.3).

The fact that $\frac{1}{4}$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x = y$, then it becomes $\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0$, implying $0 \leq \|x\|^2 - |\langle x, e\rangle|^2 \leq \frac{1}{4}|\Gamma - \gamma|^2$, for which, by Grüss' inequality in inner product spaces, we know that the constant $\frac{1}{4}$ is best possible. ∎

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right].$$

Corollary 4. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H, \|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are so that*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \gamma e \right\rangle \geq 0 \tag{4.7}$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2}|\Gamma - \gamma|, \tag{4.8}$$

then we have the inequality

$$\left|\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right]\right| \leq \frac{1}{4}|\Gamma - \gamma|^2. \tag{4.9}$$

If the inner product space H is real, then (for $m, M \in \mathbb{R}$, $M > m$)

$$\left\langle Me - \frac{x \pm y}{2}, \frac{x \pm y}{2} - me \right\rangle \geq 0 \tag{4.10}$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{m + M}{2} \cdot e \right\| \leq \frac{1}{2}(M - m), \tag{4.11}$$

implies

$$\left|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right| \leq \frac{1}{4}(M - m)^2. \tag{4.12}$$

In both inequalities (4.9) and (4.12), the constant $\frac{1}{4}$ is best possible.

Proof. We only remark that, if

$$\operatorname{Re}\left\langle \Gamma e - \frac{x - y}{2}, \frac{x - y}{2} - \gamma e \right\rangle \geq 0$$

holds, then by Theorem 4, we get

$$\operatorname{Re}\left[-\langle x, y\rangle + \langle x, e\rangle \langle e, y\rangle\right] \leq \frac{1}{4}|\Gamma - \gamma|^2$$

showing that

$$\operatorname{Re}\left[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle\right] \geq -\frac{1}{4}|\Gamma - \gamma|^2. \tag{4.13}$$

Making use of (4.3) and (4.13) we deduce the desired result (4.9). ∎

Finally, we may state and proof the following dual result as well

Proposition 1. *Let* $(H, \langle .,. \rangle)$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ *and* $e \in H, \|e\| = 1$. *If* $\varphi, \Phi \in \mathbb{K}$ *and* $x, y \in H$ *are so that*

$$\operatorname{Re}\left[(\Phi - \langle x,e\rangle)\left(\overline{\langle x,e\rangle} - \overline{\varphi}\right)\right] \leq 0, \tag{4.14}$$

then we have the inequalities

$$\begin{aligned} \|x - \langle x,e\rangle e\| &\leq \left[\operatorname{Re}\langle x - \Phi e, x - \varphi e\rangle\right]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{2}\left[\|x - \Phi e\|^2 + \|x - \varphi e\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.15}$$

Proof. We know that the following identity holds true (see (3.3))

$$\|x\|^2 - |\langle x,e\rangle|^2 = \operatorname{Re}\left[(\Phi - \langle x,e\rangle)\left(\overline{\langle x,e\rangle} - \overline{\varphi}\right)\right] + \operatorname{Re}\langle x - \Phi e, x - \varphi e\rangle. \tag{4.16}$$

Using the assumption (4.14) and the fact that

$$\|x\|^2 - |\langle x,e\rangle|^2 = \|x - \langle x,e\rangle e\|^2$$

by (4.16) we deduce the first inequality in (4.15).

The second inequality in (4.15) follows by the fact that for any $v, w \in H$ one has

$$\operatorname{Re}\langle w,v\rangle \leq \frac{1}{2}\left(\|w\|^2 + \|v\|^2\right).$$

The proposition is thus proved. ∎

5. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L^2(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and $2-$integrable on Ω, i.e.,

$$\int_\Omega |f(s)|^2\, d\mu(s) < \infty.$$

The following proposition holds

Proposition 2. *If* $f, g, h \in L^2(\Omega,\mathbb{K})$ *and* $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, *are so that* $\int_\Omega |h(s)|^2\, d\mu(s) = 1$ *and*

$$\begin{aligned} &\int_\Omega \operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \varphi\overline{h(s)}\right)\right] d\mu(s) \geq 0 \\ &\int_\Omega \operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \gamma\overline{h(s)}\right)\right] d\mu(s) \geq 0 \end{aligned} \tag{5.1}$$

or, equivalently

$$\begin{aligned} \left(\int_\Omega \left|f(s) - \frac{\Phi+\varphi}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} &\leq \frac{1}{2}|\Phi - \varphi|, \\ \left(\int_\Omega \left|g(s) - \frac{\Gamma+\gamma}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} &\leq \frac{1}{2}|\Gamma - \gamma|, \end{aligned} \tag{5.2}$$

then we have the following refinement of Grüss integral inequality

$$\left|\int_{\Omega} f(s)\overline{g(s)}d\mu(s) - \int_{\Omega} f(s)\overline{h(s)}d\mu(s)\int_{\Omega} h(s)\overline{g(s)}d\mu(s)\right| \tag{5.3}$$
$$\leq \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma| - \left[\int_{\Omega}\operatorname{Re}\left[(\Phi h(s)-f(s))\left(\overline{f(s)}-\overline{\varphi h(s)}\right)\right]d\mu(s)\right.$$
$$\left.\times\int_{\Omega}\operatorname{Re}\left[(\Gamma h(s)-g(s))\left(\overline{g(s)}-\overline{\gamma h(s)}\right)\right]d\mu(s)\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ *is best possible.*

The proof follows by Theorem 3 on choosing $H = L^2(\Omega,\mathbb{K})$ with the inner product

$$\langle f,g\rangle := \int_{\Omega} f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 2. *It is obvious that a sufficient condition for* (5.1) *to hold is*

$$\operatorname{Re}\left[(\Phi h(s)-f(s))\left(\overline{f(s)}-\overline{\varphi h(s)}\right)\right]\geq 0,$$

and

$$\operatorname{Re}\left[(\Gamma h(s)-g(s))\left(\overline{g(s)}-\overline{\gamma h(s)}\right)\right]\geq 0,$$

for $\mu-$*a.e.* $s\in\Omega$, *or equivalently,*

$$\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|\leq\frac{1}{2}|\Phi-\varphi||h(s)| \quad \textit{and}$$
$$\left|g(s)-\frac{\Gamma+\gamma}{2}h(s)\right|\leq\frac{1}{2}|\Gamma-\gamma||h(s)|,$$

for $\mu-$*a.e.* $s\in\Omega$.

The following result may be stated as well.

Corollary 5. *If* $z,Z,t,T\in\mathbb{K}$, $\rho\in L(\Omega,\mathbb{R})$, $\mu(\Omega)<\infty$ *and* $f,g\in L^2(\Omega,\mathbb{K})$ *are such that:*

$$\operatorname{Re}\left[(Z-f(s))\left(\overline{f(s)}-\bar{z}\right)\right]\geq 0, \tag{5.4}$$
$$\operatorname{Re}\left[(T-g(s))\left(\overline{g(s)}-\bar{t}\right)\right]\geq 0 \quad \textit{for a.e. } s\in\Omega$$

or, equivalently

$$\left|f(s)-\frac{z+Z}{2}\right|\leq\frac{1}{2}|Z-z|, \tag{5.5}$$
$$\left|g(s)-\frac{t+T}{2}\right|\leq\frac{1}{2}|T-t| \quad \textit{for a.e. } s\in\Omega$$

then we have the inequality

$$\left|\frac{1}{\mu(\Omega)}\int_\Omega f(s)\overline{g(s)}d\mu(s)-\frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_\Omega\overline{g(s)}d\mu(s)\right| \tag{5.6}$$
$$\leq\frac{1}{4}|Z-z|\,|T-t|-\frac{1}{\mu(\Omega)}\left[\int_\Omega \operatorname{Re}\left[(Z-f(s))\left(\overline{f(s)}-\bar{z}\right)\right]d\mu(s)\right.$$
$$\left.\times\int_\Omega \operatorname{Re}\left[(T-g(s))\left(\overline{g(s)}-\bar{t}\right)\right]d\mu(s)\right]^{\frac{1}{2}}.$$

Using Theorem 4 we may state the following result as well.

Proposition 3. *If $f,g,h\in L^2(\Omega,\mathbb{K})$ and $\gamma,\Gamma\in\mathbb{K}$ are such that $\int_\Omega|h(s)|^2\,d\mu(s)=1$ and*

$$\int_\Omega \operatorname{Re}\left\{\left[\Gamma h(s)-\frac{f(s)+g(s)}{2}\right]\cdot\left[\frac{\overline{f(s)}+\overline{g(s)}}{2}-\bar{\gamma}\bar{h}(s)\right]\right\}d\mu(s)\geq 0 \tag{5.7}$$

or, equivalently,

$$\left(\int_\Omega\left|\frac{f(s)+g(s)}{2}-\frac{\gamma+\Gamma}{2}h(s)\right|^2d\mu(s)\right)^{\frac{1}{2}}\leq\frac{1}{2}|\Gamma-\gamma|, \tag{5.8}$$

then we have the inequality

$$I:=\int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s) \tag{5.9}$$
$$-\operatorname{Re}\left[\int_\Omega f(s)\overline{h(s)}d\mu(s)\cdot\int_\Omega h(s)\overline{g(s)}d\mu(s)\right]$$
$$\leq\frac{1}{4}|\Gamma-\gamma|^2.$$

If (5.7) and (5.8) hold with " $\pm$ " instead of " $+$ ", then

$$|I|\leq\frac{1}{4}|\Gamma-\gamma|^2. \tag{5.10}$$

Remark 3. *It is obvious that a sufficient condition for (5.7) to hold is*

$$\operatorname{Re}\left\{\left[\Gamma h(s)-\frac{f(s)+g(s)}{2}\right]\cdot\left[\frac{\overline{f(s)}+\overline{g(s)}}{2}-\bar{\gamma}\bar{h}(s)\right]\right\}\geq 0 \tag{5.11}$$

for a.e. $s\in\Omega$, or equivalently

$$\left|\frac{f(s)+g(s)}{2}-\frac{\gamma+\Gamma}{2}h(s)\right|\leq\frac{1}{2}|\Gamma-\gamma|\,|h(s)|\quad\text{for a.e. } s\in\Omega. \tag{5.12}$$

Finally, the following corollary holds.

Corollary 6. *If $Z,z\in\mathbb{K}$, $\mu(\Omega)<\infty$ and $f,g\in L^2(\Omega,\mathbb{K})$ are such that*

$$\operatorname{Re}\left[\left(Z-\frac{f(s)+g(s)}{2}\right)\left(\frac{\overline{f(s)}+\overline{g(s)}}{2}-z\right)\right]\geq 0\quad\text{for a.e. } s\in\Omega \tag{5.13}$$

or, equivalently

$$\left|\frac{f(s)+g(s)}{2}-\frac{z+Z}{2}\right|\leq\frac{1}{2}|Z-z|\quad\text{for a.e. } s\in\Omega, \tag{5.14}$$

then we have the inequality

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)}\int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s) \\ &\qquad - \operatorname{Re}\left[\frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)}d\mu(s)\right] \\ &\le \frac{1}{4}|Z-z|^2 . \end{aligned}$$

If (5.13) and (5.14) hold with " ± " instead of " + ", then

$$|J| \le \frac{1}{4}|Z-z|^2 . \tag{5.15}$$

Remark 4. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

References

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