SOME REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY IN 2-INNER PRODUCT SPACES

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Abstract. In this paper, some reverses of the Cauchy-Bunyakovsky-Schwarz inequality in 2-inner product spaces are given. Using this framework, some applications for determinantal integral inequalities are also provided.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3].

We recall here the basic definitions and the elementary properties of 2-inner product spaces that will be used in the sequel (see [4]).

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Let X be a linear space of dimension greater than 1 over the number filed K, where $K = R$ or $K = C$. Suppose that $(\cdot,\cdot|\cdot)$ is a K-valued function on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x, x|z) \geq 0$, $(x, x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x, x|z) = (z, z|x)$,
($2I_3$) $(x, y|z) = \overline{(y, x|z)}$,
($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in K$,
($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$,

where $x, x', y, z \in X$.

The functional $(\cdot,\cdot|\cdot)$ is called a *2-inner product* and $(X, (\cdot,\cdot|\cdot))$ a *2-inner product space* (or *2-pre-Hilbert space*).

Some basic properties of the 2-inner product spaces are as follows:

(1) If $K = R$, then ($2I_3$) reduces to $(x, y|z) = (y, x|z)$.
(2) From ($2I_3$) and ($2I_4$), we have $(0, y|z) = 0, (x, 0|y) = 0$ and also

$$(x, \alpha y|z) = \overline{\alpha}(x, y|z). \tag{1.1}$$

(3) Using ($2I_3$)$\sim$($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, z|z) + (y, y|z) \pm 2Re(x, y|z)$$

and

$$Re(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{1.2}$$

In the real case $K = R$, (1.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in R$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{1.4}$$

In the complex case $K = C$, using (1.1) and (1.2), we have

$$Im(x, y|z) = Re[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (1.2), yields

$$\begin{aligned} (x, y|z) = &\frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \\ &+ \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \end{aligned} \tag{1.5}$$

Using (1.5) and (1.1), we have, for any $\alpha \in C$, that

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{1.6}$$

However, for any $\alpha \in R$, (1.6) reduces to (1.4). Also, it follows from (1.6) that

$$(x, y|0) = 0.$$

(4) For any given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, x|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$. It is obvious that the inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

If $x = z$, then (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0, \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too.

Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.2), it follows the Cauchy-Bunyakovsky-Schwarz inequality (shortly, the CBS-inequality) for 2-inner products:

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to see that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot|\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$. It is easy to see that this function satisfies the following conditions:

$(2N_1)$ $\|x|z\| = 0$ if and only if x and z are linearly dependent,
$(2N_2)$ $\|x|z\| = \|z|x\|$,
$(2N_3)$ $\|\alpha x|z\| = |\alpha|\|x|z\|$ for all scalar $\alpha \in K$,
$(2N_4)$ $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot,\cdot\|$ satisfying the conditions $(2\mathrm{N}_1)\sim(2\mathrm{N}_4)$ is called a *2-norm* on X and $(X, \|\cdot|\cdot\|)$ a *linear 2-normed space*.

For a systematic presentation of the recent results related to the theory of linear 2-normed spaces, see the book [8].

In terms of the 2-norms, the (CBS)-inequality (1.9) can be written as

$$|(x, y|z)|^2 \leq \|x|z\|^2\|y|z\|^2. \tag{1.11}$$

The equality holds in (1.11) if and only if x, y and z are linearly dependent.

For recent inequalities in 2-inner product spaces, see the papers [1, 2], [4–7], [9–12] and the references therein.

2. Revereses of the CBS-Inequality

The following reverse of the CBS-inequality in 2-inner product spaces holds.

Theorem 2.1. *Let $A, a \in K(K = C, R)$ and $x, y, z \in X$, where, as above, $(X, (\cdot, \cdot|\cdot))$ is a 2-inner product space over K. If*

$$Re(Ay - x, x - ay|z) \geq 0 \tag{2.1}$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} y \middle| z \right\| \leq \frac{1}{2} |A - a| \|y|z\| \tag{2.2}$$

holds, then we have the inequality

$$0 \leq \|x|z\|^2 \|y|z\|^2 - |(x, y|z)|^2 \leq \frac{1}{4} |A - a|^2 \|y|z\|^4. \tag{2.3}$$

The constant $\frac{1}{4}$ is sharp in (2.3) *in the sense that it cannot be replaced by a smaller constant.*

Proof. Consider the vectors $x, u, U, z \in X$. We observe that

$$\begin{aligned} &\frac{1}{4} \|U - u|z\|^2 - \left\| x - \frac{U + u}{2} \middle| z \right\|^2 \\ &\quad = Re(U - x, x - u|z) \\ &\quad = Re[(x, u|z) + (U, x|z)] - Re(U, u|z) - \|x|z\|^2. \end{aligned}$$

Therefore, $Re(U - x, z - u|z) \geq 0$ if and only if

$$\left\| x - \frac{U + u}{2} \middle| z \right\| \leq \frac{1}{2} \|U - u|z\|.$$

If we apply this to the vectors $U = Ay$ and $u = ay$, then we deduce that the inequalities (2.1) and (2.2) are equivalent, as stated.

Now, let us consider the real numbers

$$I_1 = Re[(A\|y|z\|^2 - (x, y|z))(\overline{(x, y|z)} - \overline{a}\|y|z\|^2)]$$

and

$$I_2 = \|y|z\|^2 Re(Ay - x, x - ay|z).$$

A simple calculation shows that

$$I_1 = \|y|z\|^2 Re[A(x, y|z) + \overline{a}(x, y|z)] - |(x, y|z)|^2 - \|y|z\|^4 Re(A\overline{a})$$

and

$$\begin{aligned} I_2 = {} & \|y|z\|^2 Re[A\overline{(x, y|z)} + \overline{a}(x, y|z)] \\ & - \|x|z\|^2 \|y, z\|^2 - \|y, z\|^4 Re(A\overline{a}), \end{aligned}$$

which give

$$I_1 - I_2 = \|x|z\|^2 \|y|z\|^2 - |(x, y|z)|^2 \tag{2.4}$$

for any $x, y, z \in X$ and $a, A \in K$. If (2.2) holds, then $I_2 \geq 0$ and thus

$$\begin{aligned} & \|x|z\|^2 \|y|z\|^2 - |(x, y|z)|^2 \\ & \leq Re[(A\|y|z\|^2 - (x, y|z))(\overline{(x, y|z)} - \overline{a}\|y|z\|^2). \end{aligned} \tag{2.5}$$

If we use the elementary inequality $Re(\alpha\overline{\beta}) \leq \frac{1}{4}|\alpha + \beta|^2$ for any $\alpha, \beta \in K(K = R, C)$, then we have that

$$\begin{aligned} & Re[(A\|y|z\|^2 - (x, y|z))(\overline{(x, y|z)} - \overline{a}\|y, z\|^2)] \\ & \leq \frac{1}{4}|A - a|^2 \|y|z\|^4. \end{aligned} \tag{2.6}$$

Making use of the inequalities (2.5) and (2.6), we deduce the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{4}$, assume that (2.4) holds with a constant $C > 0$, i.e.,

$$\|x|z\|^2 \|y|z\|^2 - |(x, y|z)|^2 \leq C|A - a|^2 \|y|z\|^4, \tag{2.7}$$

where x, y, z, A and a satisfy the hypothesis of the theorem.

Consider $y \in X$ with $\|y|z\| = 1$, $a \neq A$, $m \in X$ with $\|m|z\| = 1$ and $(y, m|z) = 0$ and define

$$x = \frac{A+a}{2} y + \frac{A-a}{2} m.$$

Then we have

$$(Ay - x, x - ay|z) = \frac{|A-a|^2}{4} (y - m, y + m|z) = 0$$

and then the condition (2.1) is fulfilled. From (2.7), we deduce

$$\begin{aligned} &\left\| \frac{A+a}{2} y + \frac{A-a}{2} m \middle| z \right\|^2 - \left| \left(\frac{A+a}{2} y + \frac{A-a}{2} m, y \middle| z \right) \right|^2 \\ &\leq C|A-a|^2 \end{aligned} \tag{2.8}$$

and, since

$$\left\| \frac{A+a}{2} y + \frac{A-a}{2} m \middle| z \right\|^2 = \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2$$

and

$$\left| \left(\frac{A+a}{2} y + \frac{A-a}{2} m, y \middle| z \right) \right|^2 = \left| \frac{A+a}{2} \right|^2,$$

by (2.8), we have

$$\left| \frac{A-a}{2} \right|^2 \leq C|A-a|^2$$

for any $A, a \in K$ with $a \neq A$, which implies $C \geq \frac{1}{4}$. This completes the proof.

Another reverse of the CBS-inequality in 2-inner product spaces is incorporated in the following theorem:

Theorem 2.2. *Assume that x, y, z, a and A are the same as in Theorem 2.1. If $Re(\overline{a}, A) > 0$, then we have the inequalities*

$$\begin{aligned} \|x|z\| \|y|z\| &\leq \frac{1}{2} \cdot \frac{Re[(\overline{A} + \overline{a})(x, y|z)]}{[Re(\overline{a}A)]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|A+a|}{[Re(\overline{a}A)]^{\frac{1}{2}}} |(x, y|z)|. \end{aligned} \tag{2.9}$$

The constant $\frac{1}{2}$ *is best possible in both inequalities in the sense that it cannot be replaced by a smaller constant.*

Proof. Define

$$\begin{aligned} I &= Re(Ay - x, x - ay|z) \\ &= Re[A\overline{(x, y|z)} + \overline{a}(x, y|z)] - \|x|z\|^2 - Re(\overline{a}A)\|y|z\|^2. \end{aligned}$$

We know that, for a complex number $\alpha \in C, Re(\alpha) = Re(\overline{\alpha})$ and thus

$$Re[A\ \overline{(x, y|z)}] = Re[\overline{A}(x, y|z)],$$

which implies

$$I = Re[(\overline{A} + \overline{a})(x, y|z)] - \|x|z\|^2 - Re(\overline{a}A)\|y|z\|^2. \tag{2.10}$$

Since x, y, z, α, A are assumed to satisfy the condition (2.1), by (2.10), we deduce the inequality

$$\|x|z\|^2 + Re(\overline{a}A)\|y|z\|^2 \leq Re[(\overline{A} + \overline{a})(x, y|z)],$$

which gives

$$\begin{aligned} &\frac{1}{[Re(\overline{a}A)]^{\frac{1}{2}}}\|x|z\|^2 + [Re(\overline{a}A)]^{\frac{1}{2}}\|y|z\|^2 \\ &\leq \frac{Re[(\overline{A} + \overline{a})(x, y|z)]}{[Re(\overline{a}A)]^{\frac{1}{2}}} \end{aligned} \tag{2.11}$$

since $Re(\overline{a}A) > 0$.

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha}q^2 \geq 2pq$$

for $p, q \geq 0$ and $\alpha > 0$, we have

$$2\|x|z\|\|y|z\| \leq \frac{1}{[Re(\overline{a}A)]^{\frac{1}{2}}}\|x|z\|^2 + [Re(\overline{a}A)]^{\frac{1}{2}}\|y|z\|^2. \tag{2.12}$$

Using (2.11) and (2.12), we deduce the first inequality in (2.9). The last part is obvious by the fact that, for $z \in C, |Re(z)| \leq |z|$.

To prove the sharpness of the constant $\frac{1}{2}$ in the first inequality in (2.9), we assume that (2.9) holds with a constant $C > 0$, i.e.,

$$\|x|z\|\|y|z\| \leq C\frac{Re[(\overline{A}+\overline{a})(x,y|z)]}{[Re(\overline{a}A)]^{\frac{1}{2}}}, \tag{2.13}$$

provided x, y, z, a and A satisfy (2.1). If we choose $a = A = 1$, $y = x \neq 0$, then obviously (2.1) holds and, from (2.13), we obtain

$$\|x|z\|^2 \leq 2C\|x|z\|^2 \tag{2.14}$$

for any linearly independent vectors $x, z \in X$, which implies $C \geq \frac{1}{2}$. This completes the proof.

When the constants involved are assumed to be positive, then we may state the following result:

Corollary 2.3. *Let $M \geq m > 0$ and assume that, for $x, y, z \in X$, we have*

$$Re(My - x, x - my|z) \geq 0$$

or, equivalently,

$$\left\|x - \frac{m+M}{2}\middle|z\right\| \leq \frac{1}{2}(M-m)\|y, z\|.$$

Then we have the following reverse of the CBS-inequality

$$\|x|z\|\|y|z\| \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}Re(x,y|z) \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}|(x,y|z)|. \tag{2.15}$$

The constant $\frac{1}{2}$ is sharp in (2.15).

Some additive versions of the above are obtained in the following:

Corollary 2.4. *With the assumptions of Theorem 2.2, we have the inequalities*

$$\begin{aligned} 0 &\leq \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 \\ &\leq \frac{1}{4}\cdot\frac{|A-a|^2}{Re(\overline{a}A)}|(x,y|z)|^2. \end{aligned} \tag{2.16}$$

The constant $\frac{1}{4}$ *is best possible in* (2.16).

Corollary 2.5. *With the assumptions of Corollary 2.3, we have*

$$\begin{aligned} 0 &\leq \|x|z\|\|y|z\| - |(x,y|z)| \leq \|x|z\|\|y|z\| - Re(x,y|z) \\ &\leq \frac{1}{2} \cdot \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} Re(x,y|z) \leq \frac{1}{2} \cdot \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} |(x,y|z)| \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} 0 &\leq \|x|z\|^2\|y|z\|^2 - |(x,y|z)|^2 \leq \|x|z\|^2\|y|z\|^2 - [Re(x,y|z)]^2 \\ &\leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} [Re(x,y|z)]^2 \leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} |(x,y|z)|^2. \end{aligned} \tag{2.18}$$

The constant $\frac{1}{2}$ *in* (2.17) *and the constant* $\frac{1}{4}$ *in* (2.18) *are best possible.*

The third inequality in (2.17) may be used to point out a reverse of the triangle inequality in 2-inner product spaces.

Corollary 2.6. *Assume that* x, y, z, m *and* M *are the same as in Corollary 2.3. Then we have the following reverse of the triangle inequality*

$$0 \leq \|x|z\| + \|y|z\| - \|x+y|z\| \leq \frac{\sqrt{M}-\sqrt{m}}{(mM)^{\frac{1}{4}}} \sqrt{Re(x,y|z)}. \tag{2.19}$$

Proof. It is easy to see that

$$0 \leq (\|x|z\| + \|y|z\|)^2 - \|x+y|z\|^2 = 2[\|x|z\|\|y\| - Re(x,y|z)] \tag{2.20}$$

for any $x, y, z \in X$. If the assumptions of Corollary 2.3 hold, then (2.17) is valid and, by (2.20), we deduce

$$0 \leq (\|x|z\| + \|y|z\|)^2 - \|x+y|z\|^2 \leq \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} Re(x,y|z),$$

which gives

$$(\|x|z\| + \|y|z\|)^2 \leq \|x+y|z\|^2 + \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} Re(x,y|z). \tag{2.21}$$

Taking the square root in (2.21), we have

$$\begin{aligned}\|x|z\| + \|y|z\| &\leq \sqrt{\|x+y|z\|^2 + \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} Re(x,y|z)} \\ &\leq \|x+y|z\| + \frac{\sqrt{M}-\sqrt{m}}{(mM)^{\frac{1}{4}}} \sqrt{Re(x,y|z)},\end{aligned}$$

from where we deduce the desired inequality (2.21). This completes the proof.

3. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with valued in $R \cup \{\infty\}$. Denote by $L^2_\phi(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2-ϕ-integrable on Ω, i.e., $\int_\Omega \phi(s)|\phi(s)|^2 d\mu(s) < \infty$, where $\phi : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\phi(\Omega)$

$$\begin{aligned}&(f,g|h)_\phi \\ &= \frac{1}{2}\int_\Omega\int_\Omega \phi(x)\phi(y) \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} \cdot \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} d\mu(x)d\mu(y),\end{aligned} \tag{3.1}$$

where by $\begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix}$ we understand the determinant of the matrix $\begin{bmatrix} f(x) & f(y) \\ h(x) & h(y) \end{bmatrix}$, generating the 2-norm

$$\|f|h\|_\phi = \left(\frac{1}{2}\int_\Omega\int_\Omega \phi(x)\phi(y) \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix}^2 d\mu(x)d\mu(y)\right)^{\frac{1}{2}}. \tag{3.2}$$

A simple computation with integrals shows that

$$(f,g|h)_\phi = \begin{vmatrix} \int_\Omega \phi(x)f(x)g(x)d\mu(x) & \int_\Omega \phi(x)f(x)h(x)d\mu(x) \\ \int_\Omega \phi(x)g(x)h(x)d\mu(x) & \int_\Omega \phi(x)h^2(x)d\mu(x) \end{vmatrix}$$

and

$$\|f|h\|_\phi = \begin{vmatrix} \int_\Omega \phi(x)f^2(x)d\mu(x) & \int_\Omega \phi(x)f(x)h(x)d\mu(x) \\ \int_\Omega \phi(x)f(x)h(x)d\mu(x) & \int_\Omega \phi(x)h^2(x)d\mu(x) \end{vmatrix}^{\frac{1}{2}}.$$

We recall that the pair of function $(q,p) \in L^2_\phi(\Omega) \times L^2_\phi(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for almost every $x, y \in \Omega$. If $\Omega = [a,b]$ is an interval of real numbers, then a sufficient condition of synchronicity for (p,q) is that they are monotonic in the same sense, i.e., both of them are increasing or decreasing on $[a,b]$.

Now, suppose that $h \in L^2_\phi(\Omega)$ is such that $h(x) \neq 0$ for almost every $x \in \Omega$. Then, by (3.1), we have

$$\begin{aligned} &(f,g|h)_\phi \\ &= \frac{1}{2}\int_\Omega\int_\Omega \phi(x)\phi(y)h^2(x)h^2(y)\left(\frac{f(x)}{h(x)} - \frac{f(y)}{h(y)}\right) \\ &\quad \times \left(\frac{g(x)}{h(x)} - \frac{g(y)}{h(y)}\right)d\mu(x)d\mu(y) \end{aligned} \tag{3.3}$$

and thus a sufficient condition for the inequality

$$(f,g|h)_\phi \geq 0 \tag{3.4}$$

to hold is that the pair of function $(\frac{f}{h}, \frac{g}{h})$ to be synchronous.

We are able now to state some integral inequalities that can be derived using the general framework presented above.

Proposition 3.1. *Let $M > m > 0$ and $f, g, h \in L^2_\phi(\Omega)$ so that the functions*

$$M \cdot \frac{g}{h} - \frac{f}{h}, \qquad \frac{f}{h} - m \cdot \frac{g}{h} \tag{3.5}$$

are synchronous. Then we have the inequalities

$$
(3.6)\qquad
\begin{aligned}
0 &\le \begin{vmatrix} \int_\Omega \phi f^2 & \int_\Omega \phi fh \\ \int_\Omega \phi fh & \int_\Omega \phi h^2 \end{vmatrix} \cdot \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix} \\
&\quad - \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^2 \\
&\le \frac{1}{4}(M-m)^2 \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^2 .
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best possible in* (3.6)*.*

The proof is obvious by Theorem 2.1 and we omit the details.

Proposition 3.2. *With the assumptions of Proposition 3.1, we have the inequality*

$$
(3.7)\qquad
\begin{aligned}
0 &\le \begin{vmatrix} \int_\Omega \phi f^2 & \int_\Omega \phi fh \\ \int_\Omega \phi fh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \\
&\le \frac{1}{2} \cdot \frac{(M-m)}{\sqrt{mM}} \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix} .
\end{aligned}
$$

The constant $\frac{1}{2}$ *is best possible in* (3.7)*.*

The following counterpart of the (CBS)-inequality for determinants also holds.

Proposition 3.3. *With the assumptions of Proposition 3.1, we have the inequalities*

$$
(3.8)\qquad
\begin{aligned}
0 &\le \begin{vmatrix} \int_\Omega \phi f^2 & \int_\Omega \phi fh \\ \int_\Omega \phi fh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \\
&\quad - \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix} \\
&\le \frac{1}{2} \frac{(\sqrt{M}-\sqrt{m})^2}{\sqrt{mM}} \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}
\end{aligned}
$$

and

$$\begin{aligned} 0 &\leq \begin{vmatrix} \int_\Omega \phi f^2 & \int_\Omega \phi fh \\ \int_\Omega \phi fh & \int_\Omega \phi h^2 \end{vmatrix} \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix} \\ &\quad - \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^2 \\ &\leq \frac{1}{4} \frac{(M-m)^2}{\sqrt{mM}} \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^2 . \end{aligned} \tag{3.9}$$

The constants $\frac{1}{2}$ *in* (3.8) *and* $\frac{1}{4}$ *in* (3.9) *are best possible.*

Finally, we may state the following reverse of the triangle inequality for determinants:

Proposition 3.4. *With the assumptions of Proposition 3.1, we have the inequalities*

$$\begin{aligned} 0 &\leq \begin{vmatrix} \int_\Omega \phi f^2 & \int_\Omega \phi fh \\ \int_\Omega \phi fh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} \int_\Omega \phi g^2 & \int_\Omega \phi gh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \\ &\quad - \begin{vmatrix} \int_\Omega \phi (f+g)^2 & \int_\Omega \phi (f+g)h \\ \int_\Omega \phi (f+g)h & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} \\ &\leq \frac{\sqrt{M}-\sqrt{m})}{(mM)^{\frac{1}{4}}} \begin{vmatrix} \int_\Omega \phi fg & \int_\Omega \phi fh \\ \int_\Omega \phi gh & \int_\Omega \phi h^2 \end{vmatrix}^{\frac{1}{2}} . \end{aligned} \tag{3.10}$$

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