A GENERALISATION OF WAGNER'S INEQUALITY FOR SEQUENCES OF VECTORS IN INNER PRODUCT SPACES

SEVER S. DRAGOMIR

Abstract. An extension of the Cauchy-Buniakowski-Schwartz inequality due to Wagner for sequences of vectors in inner product spaces is given.

1. Introduction

In 1965, S.S. Wagner [3] (see also [1] or [2, p. 85]) pointed out the following generalisation of Cauchy-Buniakowski-Schwartz's inequality for real numbers.

Theorem 1. *Let* $\mathbf{a}=(a_1,...,a_n)$ *and* $\mathbf{b}=(b_1,...,b_n)$ *be two* n*-tuples of real numbers. Then for any* $x\in[0,1]$, *one has the inequality*

$$\left(\sum_{k=1}^{n}a_kb_k+x\cdot\sum_{1\leq i\neq j\leq n}a_ib_j\right)^2 \leq \left(\sum_{k=1}^{n}a_k^2+2x\cdot\sum_{1\leq i<j\leq n}a_ia_j\right)\cdot\left(\sum_{k=1}^{n}b_k^2+2x\cdot\sum_{1\leq i<j\leq n}b_ib_j\right). \tag{1.1}$$

For $x=0$, we recapture the Cauchy-Buniakowski-Schwartz's inequality, i.e., (see for example [2, p. 84])

$$\left(\sum_{k=1}^{n}a_kb_k\right)^2\leq\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_k^2,$$

with equality if and only if there exists a real number r so that $a_k=rb_k$ for each $k\in\{1,...,n\}$.

In this note we extend the above result for sequences of vectors in real or complex inner product spaces.

2. The Results

Let $(H;\langle .,.\rangle)$ be an inner product space over $\mathbf{K}$, where $\mathbf{K}=\mathbf{R}$ or $\mathbf{K}=\mathbf{C}$. The following result holds.

Date: January 21, 2003.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 46C10.

Key words and phrases. Cauchy-Buniakowski-Schwartz's Inequality, Wagner's Inequality, Inner products spaces.

Theorem 2. *Let* $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *be two* n*-tuples of vectors in* H*. Then for any* $\alpha \in [0,1]$ *one has the inequality*

$$\left[\sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \alpha \cdot \sum_{1\le i\ne j\le n} \operatorname{Re}\langle x_i, y_j\rangle\right]^2 \le \left[\sum_{k=1}^{n} \|x_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle x_i, x_j\rangle\right] \cdot \left[\sum_{k=1}^{n} \|y_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle y_i, y_j\rangle\right]. \tag{2.1}$$

Proof. Consider the function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(t) = (1-\alpha) \cdot \sum_{k=1}^{n} \|tx_k - y_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} (tx_k - y_k)\right\|^2. \tag{2.2}$$

Then

$$\begin{aligned} f(t) &= \left[(1-\alpha) \cdot \sum_{k=1}^{n} \|x_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} x_k\right\|^2\right] t^2 \\ &\quad +2\left[(1-\alpha) \cdot \sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \alpha \cdot \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} y_k\right\rangle\right] t \\ &\quad + \left[(1-\alpha) \cdot \sum_{k=1}^{n} \|y_k\|^2 + \alpha \cdot \left\|\sum_{k=1}^{n} y_k\right\|^2\right]. \end{aligned} \tag{2.3}$$

Observe that

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 + 2 \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle x_i, x_j\rangle \tag{2.4}$$

and

$$\left\|\sum_{k=1}^{n} y_k\right\|^2 = \sum_{k=1}^{n} \|y_k\|^2 + 2 \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle y_i, y_j\rangle. \tag{2.5}$$

Also

$$\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} y_k\right\rangle = \sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \sum_{1\le i\ne j\le n} \operatorname{Re}\langle x_i, y_j\rangle. \tag{2.6}$$

Using $(2.3) - (2.6)$, we deduce that

$$\begin{aligned} f(t) &= \left[\sum_{k=1}^{n} \|x_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle x_i, x_j\rangle\right] t^2 \\ &\quad +2\left[\sum_{k=1}^{n} \operatorname{Re}\langle x_k, y_k\rangle + \alpha \cdot \sum_{1\le i\ne j\le n} \operatorname{Re}\langle x_i, y_j\rangle\right] t \\ &\quad + \left[\sum_{k=1}^{n} \|y_k\|^2 + 2\alpha \cdot \sum_{1\le i<j\le n} \operatorname{Re}\langle y_i, y_j\rangle\right]. \end{aligned} \tag{2.7}$$

Since, by (2.2), $f(t) \geq 0$ for any $t \in \mathbf{R}$, it follows that the discriminant of the quadratic function given by (2.7) is negative, which is clearly equivalent with the desired inequality (2.1). ∎

One may obtain an interesting inequality if $\mathbf{x}$ and $\mathbf{y}$ are assumed to incorporate orthogonal vectors.

Corollary 1. *Assume that* $\{x_i\}_{i=1,\dots,n}$ *are orthogonal, i.e.,* $x_i \perp x_j$ *for any* $i, j \in \{1, ..., n\}, i \neq j$; *and* $\{y_i\}_{i=1,\dots,n}$ *are orthogonal as well in the real inner product space* $(H; \langle .,. \rangle)$. *Then*

$$\sup_{\alpha\in[0,1]} \left[\sum_{k=1}^{n} \langle x_k, y_k\rangle + \alpha \cdot \sum_{1\leq i\neq j\leq n} \langle x_i, y_j\rangle\right]^2 \leq \sum_{k=1}^{n} \|x_k\|^2 \sum_{k=1}^{n} \|y_k\|^2 . \tag{2.8}$$

3. Applications

1. If we assume that $H = \mathbf{C}$, with the inner product $\langle x, y\rangle = x \cdot \bar{y}$, then by (2.1) we may deduce the following Wagner's type inequality for complex numbers

$$\begin{aligned} &\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + \alpha \cdot \sum_{1\leq i\neq j\leq n} \operatorname{Re}\left(a_i \bar{b}_j\right)\right]^2 \\ \leq &\left[\sum_{k=1}^{n} |a_k|^2 + 2\alpha \cdot \sum_{1\leq i<j\leq n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right] \cdot \left[\sum_{k=1}^{n} |b_k|^2 + 2\alpha \cdot \sum_{1\leq i<j\leq n} \operatorname{Re}\left(b_i \bar{b}_j\right)\right], \end{aligned} \tag{3.1}$$

where $\alpha \in [0, 1]$ and $\mathbf{a} = (a_1, ..., a_n)$, $\mathbf{b} = (b_1, ..., b_n) \in \mathbf{C}^n$.

2. Consider the Hilbert space $L_2(\Omega, \mu) := \{f : \Omega \to \mathbf{C}, \int_\Omega |f(x)|^2 d\mu(x) < \infty\}$ where Ω is a μ-measurable space and $\mu : \Omega \to [0, \infty]$ is a positive measure on Ω. Then for $H = L_2(\Omega, \mu)$ and since the inner product generating the norm is given by

$$\langle f, g\rangle = \int_\Omega f(x)\, \bar{g}(x)\, d\mu(x),$$

we get the inequality

$$\begin{aligned} &\left[\sum_{k=1}^{n} \int_\Omega \operatorname{Re}\left(f_k(x)\, \bar{g}_k(x)\right) d\mu(x) + \alpha \cdot \sum_{1\leq i\neq j\leq n} \int_\Omega \operatorname{Re}\left(f_i(x)\, \bar{g}_j(x)\right) d\mu(x)\right]^2 \\ \leq &\left[\sum_{k=1}^{n} \int_\Omega |f_k(x)|^2 d\mu(x) + 2\alpha \cdot \sum_{1\leq i<j\leq n} \int_\Omega \operatorname{Re}\left(f_i(x)\, \bar{f}_j(x)\right) d\mu(x)\right] \\ &\times \left[\sum_{k=1}^{n} \int_\Omega |g_k(x)|^2 d\mu(x) + 2\alpha \cdot \sum_{1\leq i<j\leq n} \int_\Omega \operatorname{Re}\left(g_i(x)\, \bar{g}_j(x)\right) d\mu(x)\right] \end{aligned} \tag{3.2}$$

where $f_i, g_i \in L_2(\Omega, \mu)$, $i \in \{1, ..., n\}$ and $\alpha \in [0, 1]$.

References

[1] P. Flor, Über eine Ungleichung von S.S. Wagner, *Elemente Math.*, **20**(1965), 136.
[2] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[3] S.S. Wagner, *Amer. Math. Soc., Notices*, **12**(1965), 220.

School of Computer Science & Mathematics, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia
E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html