ON SOME INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In the present paper we establish some new integral inequalities analogous to the well known Hadamard's inequality by using a fairly elementary analysis.

1. Introduction

The following inequality (see [3,p.49])

(H) $$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

Which holds for all convex functions f : [a,b] into R is known in the literature as Hadamard's inequality. Since its discovery in 1893, Hadamard's inequality [4] has proven to be one of the most useful inequalities in mathematical analysis. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications,see [3,8] and the references cited therein. The main purpose of this paper is to establish some new integral inequalities analogous to that of Hadamard's inequality given in (H) involving two convex functions. The analysis used in the proof is elementary and we believe that the inequalities established here are of independent interest.

1. **Statement of results**

We need the following Lemma proved in [8,p.104] which deals with the simple characterization of convex functions.

Lemma A. The following statements are equivalent for a mapping: f: [a,b] into R :

(I) f is convex on [a,b] ,
(II) for all x,y in [a,b] the mapping g : [0,1] into R, defined by
g(t) = f(tx+(1-t)y) is convex on [0,1].

For the proof of this Lemma, see [8].

Our main result is given in the following theorem.

Theorem 1. Let f and g be real-valued, nonnegative and convex functions on [a,b] . Then

$$(1) \qquad \frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

$$(2) \qquad 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

where M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a).

Remark 1. If we choose a = 0 and b = 1 and the convex functions f(x) = cx and g(x) = d(1-x) , where c, d are positive constants, then it is easy to observe that the inequalities obtained in (1) and (2) are sharp in the sense that equalities in (1) and (2) hold.

In the following theorem we shall give slight variants of the inequality (2) given in Theorem 1.

Theorem 2. Let f and g be real-valued, nonnegative and convex functions on [a,b]. Then

(3) $$\frac{3}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dtdydx$$

$$\leq \frac{1}{\text{b - a}}\int_a^b f(x)g(x)dx+\frac{1}{8}\left[\frac{M(a,b)+N(a,b)}{(b-a)^2}\right],$$

(4) $$\frac{3}{\text{b - a}}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx$$

$$\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{4}\left(\frac{1+b-a}{b-a}\right)[M(a,b)+N(a,b)],$$

Where M(a,b) and N(a,b) are as defined in Theorem 1.

Remark 2. We note that, if f and g be real-valued, nonnegative and concave functions on [a,b], then (1)-(4) hold with less than or equal to is replaced by greater than or equal to. For various other inequalities involving concave functions see [1,2,5,6,7].

2. **Proof of Theorem 1**

Since f and g are convex on [a,b], then for t in [0,1] we have

(5) $$\text{f(ta + (1 - t)b)} \leq \text{t f(a) + (1 - t)f(b)},$$

(6) $$\text{g(ta + (1 - t)b)} \leq \text{t g(a) + (1 - t)g(b)}.$$

From (5) and (6) we obtain

(7) $$\text{f}\left(\text{ta}+(1-\text{t})b\right)\text{g}\left(\text{ta}+(1-\text{t})b\right)$$

$$\leq t^2 f(a)g(a)+(1-t)^2 f(b)g(b)+t(1-t)[f(a)g(b)+f(b)g(a)].$$

By the Lemma A f(ta +(1-t)b) and g(ta + (1-t)b) are convex on [0,1], they are integrable on [0,1] and consequently f(ta + (1-t)b)g(ta +(1-t)b) is also integrable on [0,1]. Similarly since f and g are convex on [a,b], they are integrable on [a,b] and hence fg is also integrable on [a,b]. Integrating both sides of (7) over [0,1] we get

$$(8) \qquad \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)dt \le \frac{1}{3}\mathrm{M(a,b)} + \frac{1}{6}\mathrm{N(a,b)}.$$

By substituting ta + (1-t)b = x, it is easy to observe that

$$(9) \qquad \int_0^1 \mathrm{f}\left(ta+(1-t)b\right)g(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx \ .$$

Using (9) in (8) we get the desired inequality in (1).

Since f and g are convex on [a, b], then for $\mathrm{t} \in [\mathrm{a,b}]$ we observe that

$$(10) \qquad \mathrm{f}\left(\frac{\mathrm{a+b}}{2}\right)g\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)$$

$$.\mathrm{g}\left(\frac{\mathrm{ta+(1\text{-}t)b}}{2}+\frac{(1-t)a+tb}{2}\right)$$

$$\le \frac{1}{4}\left[f\left(ta+(1-t)b\right)+ f(1-t)a+tb)\right]$$

$$.\left[\mathrm{g}\left(\mathrm{ta+(1\text{-}t)b}\right)+g\left((1-t)a+tb\right)\right]$$

$$\le \frac{1}{4}\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\right.$$

$$\left.+\mathrm{f}\left((\mathrm{1\text{-}t)a+tb}\right)g\left((1-t)a+tb\right)\right]$$

$$+\frac{1}{4}\left[\left[tf(a)+(1-t)f(b)\right]\left[(1-t)g(a)+tg(b)\right]\right.$$

$$\left.+\left[(1\text{-}\mathrm{t})f(a)+tf(b)\right]\left[tg(a)+(1-t)g(b)\right]\right]$$

$$= \frac{1}{4}\left[f(ta+(1-t)b)g(ta+(1-t)b) \right.$$

$$\left. + f((1-t)a+tb)g((1-t)a+tb) \right]$$

$$+ \frac{1}{4}\left[2t(1-t)\left[f(a)g(a) + f(b)g(b) \right] \right.$$

$$\left. + \left[t^2 + (1-t)^2 \right]\left[f(a)g(b) + f(b)g(a) \right] \right].$$

Again as explained in the proof of inequality (1) given above we integrate both sides of (10) over [0,1] and obtain

$$(11) \qquad f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{4}\int_0^1 \left[f(ta+(1-t)b)g(ta+(1-t)b) \right.$$

$$\left. + f((1-t)a+tb)g((1-t)a+tb) \right] dt$$

$$+ \frac{1}{12}M(a,b) + \frac{1}{6}N(a,b).$$

From (11) it is easy to observe that

$$(12) \qquad f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{2}\int_0^1 \left[f(ta+(1-t)b)g(ta+(1-t)b) \right] dt$$

$$+ \frac{1}{12}M(a,b) + \frac{1}{6}N(a,b).$$

Now multiplying both sides of (12) by 2 and using (9) we get the required inequality in (2). The proof is complete.

3. **Proof of Theorem 2**

Since f and g are convex on [a,b], then for x,y in [a,b] and t in [0,1] we have

$$(13) \qquad f(tx+(1-t)y) \le tf(x) + (1-t)f(y),$$

(14) $$g(t\,x+(1-t)y)\le tg(x)+(1\text{-}\mathrm{t})g(y).$$

From (13) and (14) we obtain

(15) $$f(tx+(1-t)y)\,g(tx+(1-t)y)$$

$$\le t^2 f(x)g(x)+(1-t)^2 f(y)g(y)+t(1-t)[f(x)g(y)+f(y)g(x)].$$

As explained in the proof of inequality (1) given above, we integrate both sides of (15) over [0,1] and obtain

(16) $$\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dt$$

$$\le \frac{1}{3}[f(x)g(x)+f(y)g(y)]+\frac{1}{6}[f(x)g(y)+f(y)g(x)].$$

Integrating both sides of (16) on $[a,b]\times[a,b]$ we obtain

(17) $$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dtdydx$$

$$\le \frac{1}{3}(b-a)\left[\int_a^b f(x)g(x)dx+\int_a^b f(y)g(y)dy\right]$$

$$+\frac{1}{6}\left[\left(\int_a^b f(x)dx\right)\left(\int_a^b g(y)dy\right)+\left(\int_a^b f(y)dy\right)\left(\int_a^b g(x)dx\right)\right].$$

By using the right half of the Hadamard’s inequality given in (H) on the right side of (17) we have

(18) $$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dtdydx$$

$$\le \frac{2}{3}(b-a)\int_a^b f(x)g(x)dx+\frac{1}{12}[M(a,b)+N(a,b)].$$

Now dividing both sides of (18) by $\frac{2}{3}(b-a)^2$ we get the desired inequality in (3).

Since f and g are convex on $[a,b]$ we have

$$(19)\qquad f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\le tf(x)+(1-t)f\left(\frac{a+b}{2}\right),$$

$$(20)\qquad g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\le tg(x)+(1-t)g\left(\frac{a+b}{2}\right),$$

for x in [a,b] and t in [0,1] . From (19) and (20) we have

$$(21)\qquad f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)$$

$$\le t^2 f(x)g(x)+(1-t)^2 f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$+t(1-t)\left[f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right].$$

As explained in the proof of inequality (1) given above , we integrate both sides of (21) over [0,1] and obtain

$$(22)\qquad \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dt$$

$$\le\frac{1}{3}\left[f(x)g(x)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right]$$

$$+\frac{1}{6}\left[f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right].$$

As explained in the proof of inequality (3) given above, fg is integrable on $[a,b]$. Now integrating both sides of (22) on $[a,b]$, using the right half of the Hadamard's inequality given in (H) and the convexity of f and g we observe that

$$(23)\quad \int_a^b \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx$$

$$\le \frac{1}{3}\int_a^b f(x)g(x)dx+\frac{1}{3}(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$+\frac{1}{6}\left[g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx+f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right]$$

$$\le \frac{1}{3}\int_a^b f(x)g(x)dx+\frac{1}{12}(b-a)(f(a)+f(b))(g(a)+g(b))$$

$$+\frac{1}{6}\left[\left(\frac{g(a)+g(b)}{2}\right)\left(\frac{f(a)+f(b)}{2}\right)+\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{g(a)+g(b)}{2}\right)\right]$$

$$=\frac{1}{3}\int_a^b f(x)g(x)dx+\frac{1}{12}(b-a)[M(a,b)+N(a,b)]$$

$$+\frac{1}{12}[M(a,b)+N(a,b)].$$

Now multiplying both sides of (23) by $\frac{3}{b-a}$, we get the required inequality in (4). The proof is complete.

Remark 3. We note that the inequalities of the type obtained here involving two concave functions have been studied by many authors in the literature by using various techniques. For other results related to such inequalities, see [1,2,5-7] where further references are given.

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