APPROXIMATIONS FOR THE FIBER REFRACTIVE INDEX PROFILE

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Abstract. Two methods in approximating the fiber refractive index profile that have been recently obtained are reviewed. Two new methods based on the approximation of Stieltjes integral via mid-point and trapezoidal rule are also examined.

1. Introduction

Transversal interferometry of optical fibers has been established in the last twenty years as one of the most useful and accurate tools for refractive index profiling. As shown in [3], since introduction of accurate data reduction formulae by Sochacki [1], the measurement speed and precision depends mainly on the data acquisition techniques.

This, indeed, may be argued if an accurate numerical approximation of the refractive index representation formula obtained in [1]

$$n(u) := \exp\left[\frac{1}{\pi}\int_u^1 \frac{\psi(s)}{\sqrt{s^2-u^2}}ds\right], 0 \le u \le 1;$$

is assumed.

For some preliminary studies on the numerical approximation of the refractive index, see [4] and the references therein.

In [9], based on the Taylor's formula with integral remainder, the authors pointed out a numerical approach in approximating the refractive index in the assumption that the derivatives up to a certain order of the deflection function ψ are known. Since in experiments only discrete values of ψ may be obtained, we consider as the actual deflection function ψ to be the best fitting curve to data (polynomial etc...), which are differentiable up to a certain order and may be obtained, for example, by classical interpolative techniques [2]. A theoretical error analysis has been performed and the uniform convergence of the numerical procedure has been established. Some numerical examples showing a very good accuracy have been provided as well.

Another approximation of the refractive index profile has been obtained in [10] using the trapezoidal rule. This rule has been successfully implemented numerically when certain values of the deflection function are known [10].

Before we present our new approximation for the refractive index, based on the mid-point and trapezoidal rule for Stieltjes integral, some preliminary facts on the phase-stepping interferometry and optical fiber profiling are necessary. To briefly describe this, we follow [3].

1.1. **Basics of the Phase-Stepping Interferometry.** The intensity distribution, as observed in the exit plane of the interferometer of any kind, can be generally described as

$$I = I_0 + I_c \cos\left[\Delta(x,y) - \Phi_B\right] \tag{1.1}$$

or, equivalently:

$$I = I_0 + I_c \cos\Delta(x,y)\cos\Phi_B + I_c \sin\Delta(x,y)\sin\Phi_B \tag{1.2}$$

where I_0 is the background intensity, I_c is the interference pattern amplitude governing the image contrast, $\Delta(x,y)$ is the phase change introduced by investigated object, and Φ_B is the bias (phase difference between the interfering wave fronts, a quantity characteristic for the instrument). When Φ_B is constant, the observed field is an homogeneous one (in absence of $\Delta(x,y)$). When Φ_B is a linear function of space coordinates, a fringe pattern appears.

From (1.1) follows that the intensity I in any interferogram is a periodic function of the bias Φ_B with 2π period, so it can be alternately represented by a Fourier series in terms of Φ_B :

$$I = a_0 + \sum_{i=1}^{\infty}\left[a_i \cos\left(i\cdot\Phi_B\right) + b_i \sin\left(i\cdot\Phi_B\right)\right] \tag{1.3}$$

where the Fourier coefficients are, as usual, given by

$$\begin{aligned} a_i &= \frac{1}{2\pi}\int_0^{2\pi} I\left(\Phi_B\right)\cos\left(i\cdot\Phi_B\right)d\Phi_B, i = 1,2,... \\ b_i &= \frac{1}{2\pi}\int_0^{2\pi} I\left(\Phi_B\right)\sin\left(i\cdot\Phi_B\right)d\Phi_B, i = 1,2,.... \end{aligned} \tag{1.4}$$

If, for simplicity, we take into consideration only the constant and the first order terms of expansion (1.3) and relating them to the formula (1.2) the exit intensity I can be finally expressed as (see for example [3]):

$$I = a_0 + a_1\cos\left(\Phi_B\right) + b_1\sin\left(\Phi_B\right) \tag{1.5}$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} I\left(\Phi_B\right)d\Phi_B = I_0, \\ a_1 &= \frac{1}{2\pi}\int_0^{2\pi} I\left(\Phi_B\right)\cos\left(\Phi_B\right)d\Phi_B = I_1\cos\Delta(x,y), \\ b_1 &= \frac{1}{2\pi}\int_0^{2\pi} I\left(\Phi_B\right)\sin\left(\Phi_B\right)d\Phi_B = I_1\sin\Delta(x,y). \end{aligned} \tag{1.6}$$

From this comparison follows that the ratio of the first order coefficients of the Fourier expansion gives the tangent of the phase change $\Delta(x,y)$, hence it can be obtained as (see for example [3]):

$$\Delta(x,y) = \arctan\left(\frac{b_1}{a_1}\right). \tag{1.7}$$

To simplify the computation of coefficients a_1 and b_1 the integrals in (1.6) may be replaced by finite sums according to any quadrature rule. For periodic functions the

Bessel formulae are recommended. In such a case a_1 and b_1 may be approximated by:

$$
\begin{aligned}
a_1 &= \frac{1}{N}\sum_{k=0}^{2N-1} I_k \cos\left(\frac{k\pi}{N}\right), \\
b_1 &= \frac{1}{N}\sum_{k=0}^{2N-1} I_k \sin\left(\frac{k\pi}{N}\right),
\end{aligned} \tag{1.8}
$$

where I_k are the exit intensity distributions observed with bias $\Phi_B = \Phi_k = \frac{k\pi}{N}$, and $2N$ is the number of equidistant points Φ_k here the intensity samples are taken. They have to run over the entire period of I, so that $I(\Phi_0) = I(\Phi_{2N})$.

With the smallest possible number of samples $(2N = 4)$, the phase of interest can be retrieved from:

$$
\Delta(x, y) = \arctan\left[\frac{I_1(x, y) - I_3(x, y)}{I_0(x, y) - I_2(x, y)}\right] \tag{1.9}
$$

where I_0, I_1, I_2 and I_3 are intensity distributions as observed with $\Phi_B = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ respectively. This means that by introducing a $\frac{\pi}{2}$ step in the bias Φ_B the phase function $\Delta(x, y)$ can be unequivocally reconstructed from four interferograms.

The technique briefly described above following [3], was first applied by Brunning et al. [5] to study optical surfaces in a Twyman-Green interferometer and is commonly called the phase stepping technique. It has quickly found applications in a number of wavefront measuring devices for optical elements control, surface profiling, recognition of shape and deformation, as well as measurement of vibration amplitudes and phases (see for example [6]). Many other modifications of the original algorithm were also published, see for example the review paper by Creath [7] and the references therein.

1.2. **Applications to Optical Fibers Profiling.** As shown in [3], the phase stepping technique can also be applied to retrieve the phase information necessary in the refractive index profiling of optical fibers.

We will briefly describe how this can be done.

Suppose that the basic interference formula (1.1) describes the interference pattern as observed in any transverse wavefront shearing interferometer, where the fiber is investigated. Choosing the coordinate system so that the axis of the fiber image is in the x-axis, the x dependence can be eliminated from formulae (1.1) – (1.9) due to the cylindrical symmetry of the object and its homogeneous along the axis. When the wavefront shear in direction perpendicular to the fiber axis exceeds its diameter, two fiber images can be observed and the function $\Phi(y) = \frac{\lambda}{2\pi}\Delta(y)$ represents the optical path length through the fiber. If the shear in the same direction is very small (comparing with fiber's diameter), then $\frac{\lambda}{2\pi}\Delta(y)$ is proportional to the derivative of the optical path length $\Phi(y)$ in the shear direction, where the image shear s is the proportionality constant (see for example [8]):

$$
\frac{d\Phi(y)}{dy} = \frac{1}{s} \cdot \frac{\lambda}{2\pi} \cdot \Delta(y). \tag{1.10}
$$

In this case the fiber is said to be observed in *differential interference contrast* (DIC).

In any case the derivative (with respect to the radial position in observation plane) of the optical path length $\Phi(y)$ through the fiber can be either directly obtained or numerically constructed from phase data $\Delta(y)$ retrieved from four intensity distribution measurements as required by formula (1.9). This derivative is essential in the refractive index profile retrieval algorithm proposed in [1]. In the first step of this algorithm it serves to calculate the *deflection function* $\psi(\widetilde{y})$

$$\psi(\widetilde{y}) = -\arcsin\left[\frac{d\Phi(y)}{dy}\right] \tag{1.11}$$

where $\widetilde{y}$ is the incident ray position in object plane (with illumination perpendicular to the fiber axis) and can be related to the image plane coordinate y by an appropriate mapping relation depending on the experiment conditions [1], [5].

The deflection function calculated in such of way is parametrically related to the fiber refractive index profile

$$n(u) := \exp\left[\frac{1}{\pi}\int_u^1 \frac{\psi(\widetilde{y})}{\sqrt{\widetilde{y}^2 - u^2}}d\widetilde{y}\right] \tag{1.12}$$

with parameter u defined as

$$u = r/n(u), 0 \le u \le 1 \tag{1.13}$$

and r being the radial position measured from the fiber axis, for simplicity normalized to 1 at its edge (as well as the refractive index).

From the above considerations it follows that the refractive index profile of optical fiber can be quite easily reconstructed by means of formulae (1.9) – (1.12) from four homogeneous field interferograms obtained with $\pi/2$ stepped bias Φ_B. Such experiment was successfully performed in [1], but we omit the details.

2. A Numerical Approximation Via Taylor's Expansion

It is obvious that, in practice, an accurate computation of the fiber refractive index profile provided by the analytic formula:

$$n(u) := \exp\left[\frac{1}{\pi}\int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}}ds\right], 0 \le u \le 1 \tag{2.1}$$

depends on the numerical accuracy in approximating the integral

$$I(\psi, u) := \frac{1}{\pi}\int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}}ds, 0 \le u \le 1. \tag{2.2}$$

A natural approach, if information on the derivatives of ψ are available, is to compute (2.2) by the use of Taylor's formula

$$\psi(s) = \psi(u) + \sum_{k=1}^{n} \frac{(s-u)^k}{k!}\psi^{(k)}(u) + \frac{1}{n!}\int_u^s (s-t)^n \psi^{(n+1)}(t)dt \tag{2.3}$$

for any $s \in [u,1]$, $u \in [0,1]$, where $n \geq 1$ is a natural number. Then we get [9]

$$\begin{aligned} I(\psi,u) &= \frac{1}{\pi}\psi(u)\int_u^1 \frac{ds}{\sqrt{s^2-u^2}} \\ &+\frac{1}{\pi}\sum_{k=1}^{n}\frac{\psi^{(k)}(u)}{k!}\int_u^1 (s-u)^{k-1}\sqrt{\frac{s-u}{s+u}}ds \\ &+\frac{1}{\pi n!}\int_u^1 \frac{ds}{\sqrt{s^2-u^2}}\left(\int_u^s (s-t)^n \psi^{(n+1)}(t)dt\right) \\ &= \frac{1}{\pi}M_0(u)\psi(u)+\frac{1}{\pi}\sum_{k=1}^{n} M_k(u)\psi^{(k)}(u)+R_n(\psi,u) \end{aligned} \tag{2.4}$$

where

$$M_0(u) := \int_u^1 \frac{ds}{\sqrt{s^2-u^2}}, \tag{2.5}$$

$$M_k(u) := \frac{1}{k!}\int_u^1 (s-u)^{k-1}\sqrt{\frac{s-u}{s+u}}ds \tag{2.6}$$

and $R_n(\psi,u)$ is the remainder in formula (2.4), *i.e.*,

$$R_n(\psi,u) := \frac{1}{\pi n!}\int_u^1 \frac{ds}{\sqrt{s^2-u^2}}\left(\int_u^s (s-t)^n \psi^{(n+1)}(t)dt\right).$$

We note, using Maple 6, that we may compute the functions $M_0(u)$, $M_k(u)$, for $k=1,2,...$ Since the expressions of $M_k(u)$, for $k=2,3,...$ are quite complicate we will not present them here explicitly. We mention only the expressions for $M_0(u)$ and $M_1(u)$:

$$M_0(u) = ln\left(1+\sqrt{1-u^2}\right) - ln(u),$$

$$\begin{aligned} M_1(u) &:= [\sqrt{(1-u)/(u+1)}\sqrt{1-u^2} \\ &-\sqrt{(1-u)/(u+1)}u ln(1+\sqrt{1-u^2}) \\ &+\sqrt{(1-u)/(u+1)}u\sqrt{1-u^2} \\ &-\sqrt{(1-u)/(u+1)}u^2 ln(1+\sqrt{1-u^2}) \\ &+\sqrt{1/u}u^{3/2}ln(u)\sqrt{1-u^2}]/(\sqrt{1-u^2}). \end{aligned}$$

The functions $M_k(u)$ for $k=2,3,...9$, which were computed by a Maple program have been used in the computer implementation of the approximation proposed for the refractive index profile $n(u)$ (see [9]).

3. Error Analysis for Taylor's Expansion

Using the properties of the integral, we have

$$\begin{aligned} |R_n(\psi,u)| &\leq \frac{1}{\pi n!}\int_u^1 \left|\int_u^s (s-t)^n\psi^{(n+1)}(t)dt\right| \frac{ds}{\sqrt{s^2-u^2}} \\ &=: B_n(\psi,u). \end{aligned} \tag{3.1}$$

If $\psi^{(n+1)}$ is bounded in the interval $[0,1]$ and by denoting

$$\left\|\psi^{(n+1)}\right\|_{[u,s],\infty} := \sup_{t\in[u,s]} \left|\psi^{(n+1)}(t)\right| < \infty$$

then we have

$$\left|\int_u^s (s-t)^n \psi^{(n+1)}(t)dt\right| \leq \sup_{t\in[u,s]} \left|\psi^{(n+1)}(t)\right| \int_u^s (s-t)^n dt \leq \frac{(s-u)^{n+1}}{n+1} \left\|\psi^{(n+1)}\right\|_{[u,1],\infty}. \tag{3.2}$$

If $\psi^{(n+1)}$ is p-integrable $(p>1)$ in the interval $[0,1]$ and if we denote

$$\left\|\psi^{(n+1)}\right\|_{[u,s],p} := \left(\int_u^s \left|\psi^{(n+1)}(t)\right|^p dt\right)^{\frac{1}{p}}, p \geq 1;$$

then by Hölder's integral inequality for $p>1$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, the following relation can be deduced

$$\begin{aligned} &\left|\int_u^s (s-t)^n \psi^{(n+1)}(t)dt\right| \\ \leq &\left(\int_u^s (s-t)^{nq} dt\right)^{\frac{1}{q}} \times \left(\int_u^s \left|\psi^{(n+1)}(t)\right|^p dt\right)^{\frac{1}{p}} \\ \leq &\frac{(s-u)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} \left\|\psi^{(n+1)}\right\|_{[u,1],p}. \end{aligned} \tag{3.3}$$

If $\psi^{(n+1)}$ is integrable in the interval $[0,1]$ and if we denote

$$\left\|\psi^{(n+1)}\right\|_{[u,s],1} := \int_u^s \left|\psi^{(n+1)}(t)\right| dt;$$

then we have

$$\left|\int_u^s (s-t)^n \psi^{(n+1)}(t)dt\right| \leq \sup_{t\in[u,s]} (s-t)^n \int_u^s \left|\psi^{(n+1)}(t)\right| dt \leq (s-u)^n \left\|\psi^{(n+1)}\right\|_{[u,1],1}. \tag{3.4}$$

Using (3.2) – (3.4) the upper bounds for $B_n(\psi,u)$, can be written as, see [9],

$$\begin{aligned} &B_n(\psi,u) \\ \leq\ &\frac{1}{\pi} \times \begin{cases} \frac{1}{(n+1)!} \int_u^1 (s-u)^n \sqrt{\frac{s-u}{s+u}} \left\|\psi^{(n+1)}\right\|_{[u,s],\infty} dt \\ \frac{1}{n!(nq+1)^{1/q}} \int_u^1 (s-u)^{n+\frac{1}{q}-1} \sqrt{\frac{s-u}{s+u}} \left\|\psi^{(n+1)}\right\|_{[u,s],p} dt \\ \frac{1}{n!} \int_u^1 (s-u)^{n-1} \sqrt{\frac{s-u}{s+u}} \left\|\psi^{(n+1)}\right\|_{[u,s],1} dt \end{cases} \\ \leq\ &\frac{1}{\pi} \times \begin{cases} \frac{1}{(n+1)!} \left\|\psi^{(n+1)}\right\|_{[u,1],\infty} \int_u^1 (s-u)^n \sqrt{\frac{s-u}{s+u}} dt \\ \frac{1}{n!(nq+1)^{1/q}} \left\|\psi^{(n+1)}\right\|_{[u,1],p} \int_u^1 (s-u)^{n+\frac{1}{q}-1} \sqrt{\frac{s-u}{s+u}} dt \\ \frac{1}{n!} \left\|\psi^{(n+1)}\right\|_{[u,1],1} \int_u^1 (s-u)^{n-1} \sqrt{\frac{s-u}{s+u}} dt \end{cases}. \end{aligned} \tag{3.5}$$

Since, the following type of integrals occur

$$N(\alpha,u):=\int_u^1 (s-u)^\alpha \sqrt{\frac{s-u}{s+u}}dt, \alpha\geq 0, u\in[0,1]$$

and these type of integrals can not be exactly computed for general α (note, however, that for particular small α Maple 6 can provide the exact value), some evaluations of $N(\alpha,u)$ are highly desirable.

Having this in mind, we observe that, the simplest way to obtain an upper bound for $N(\alpha,u)$, is to consider that

$$0\leq \frac{s-u}{s+u}\leq 1, \text{ for } u\in[0,1], u\leq s\leq 1,$$

giving

$$N(\alpha,u)\leq \int_u^1 (s-u)^\alpha dt = \frac{(1-u)^{\alpha+1}}{(\alpha+1)}, \alpha\geq 0, u\in[0,1].$$

Using (3.4) and (3.5) we get the following simple estimate for the absolute value of the remainder $R_n(\psi,u)$ in approximating the integral $I(u)$ with the analytic expression [9]:

$$A_n(\psi,u):=\frac{1}{\pi}M_0(u)\psi(u)+\frac{1}{\pi}\sum_{k=1}^{n}M_k(u)\psi^{(k)}(u) \tag{3.6}$$

where $M_0(u), M_k(u)$ were defined in the previous section,

$$|R_n(\psi,u)|\leq\frac{1}{\pi}\times\begin{cases}\frac{(1-u)^{n+1}}{(n+1)!(n+1)}\left\|\psi^{(n+1)}\right\|_{[u,1],\infty}\\ \frac{(1-u)^{n+\frac{1}{q}}}{n!(nq+1)^{1/q+1}}\left\|\psi^{(n+1)}\right\|_{[u,1],p}\\ \frac{(1-u)^{n}}{n!n}\left\|\psi^{(n+1)}\right\|_{[u,1],1}\end{cases}. \tag{3.7}$$

The equation (3.7) shows that the remainder $R_n(\psi,u)$ is (rapidly) uniformly convergent to 0 as $n\to\infty$, meaning that the approximation of $I(u)$ by $A_n(\psi,u)$ is accurate for large enough n.

4. A Numerical Approximation Via Trapezoidal Rule

As noted before, an accurate computation of the fiber refractive index profile provided by the analytic formula

$$n(u):=\exp\left[\frac{1}{\pi}\int_u^1\frac{\psi(s)}{\sqrt{s^2-u^2}}ds\right], 0\leq u\leq 1 \tag{4.1}$$

depends on the numerical accuracy in approximating the integral

$$I(u):=\frac{1}{\pi}\int_u^1\frac{\psi(s)}{\sqrt{s^2-u^2}}ds, 0\leq u\leq 1. \tag{4.2}$$

If we change the variable such that $\tau=\sqrt{s^2-u^2}, s\in[u,1]$, then $\tau=0$ for $s=u$ and $\tau=\sqrt{1-u^2}$ for $s=1, s=\sqrt{\tau^2+u^2}$ and

$$ds=\frac{\tau d\tau}{\sqrt{\tau^2+u^2}},$$

and $I(u)$ becomes [10]

$$I(u)=\frac{1}{\pi}\int_0^{\sqrt{1-u^2}}\frac{\psi\left(\sqrt{\tau^2+u^2}\right)}{\sqrt{\tau^2+u^2}}d\tau, 0\le u\le 1. \tag{4.3}$$

This integral may be approximated in many different ways by the use of classical quadrature rules including the trapezoidal rule, mid-point rule, Simpson's rule etc...

It is well known that [2], if $f:[a,b]\to R$ is a twice differentiable function and the second derivative f'' is bounded, this means that $\|f''\|_\infty:=\sup_{t\in[a,b]}|f''(t)|<\infty$, then for any division

$$I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$$

of the interval $[a,b]$ with $h_i:=x_{i+1}-x_i (i=0,...,n-1)$ one has

$$\int_a^b f(x)\,dx=T_n(f;I_n)+R_n(f;I_n) \tag{4.4}$$

where

$$T_n(f;I_n):=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i$$

is the trapezoidal rule, and the error of approximating the integral in this way, denoted in (4.4) by $R_n(f;I_n)$, satisfies the estimate

$$\begin{aligned}|R_n(f;I_n)| &\le \frac{1}{12}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3 \\ &\le \frac{1}{12}\|f''\|_\infty[\nu(h)]^2(b-a),\end{aligned} \tag{4.5}$$

where $\nu(h):=\max\{h_i:=x_{i+1}-x_i; i=0,...,n-1\}$ is the norm of the division I_n. If $\nu(h)\to 0$ (as $n\to\infty$), then by (4.5) $R_n(f;I_n)\to 0$ with order two of accuracy.

Now, for a given $u\in(0,1)$, consider the division of the interval $\left[0,\sqrt{1-u^2}\right]$ given by

$$\Delta_n: \delta_i=\lambda_i\sqrt{1-u^2}, 0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1.$$

Using the trapezoidal rule, we can approximate $I(u)$ in the following manner [10]

$$I(u)=A_n(\psi,\lambda,u)+W_n(\psi,\lambda,u), u\in[0,1]; \tag{4.6}$$

where

$$\begin{aligned}&A_n(\psi,\lambda,u)\\ &:=\frac{\sqrt{1-u^2}}{2\pi}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\\ &\times\left[\frac{\psi\left(\sqrt{(1-u^2)\lambda_i^2+u^2}\right)}{\sqrt{(1-u^2)\lambda_i^2+u^2}}+\frac{\psi\left(\sqrt{(1-u^2)\lambda_{i+1}^2+u^2}\right)}{\sqrt{(1-u^2)\lambda_{i+1}^2+u^2}}\right]\end{aligned} \tag{4.7}$$

and $W_n(\psi,\lambda,u)$ is the remainder in approximating $I(u)$ by $A_n(\psi,\lambda,u)$.

5. Using Data Provided by Practical Experiments

In practice, performing accurate laboratory experiments, the function ψ may be measured in a number of $(m+1)$-points, say

$$0 = s_0 < s_1 < s_2 < ... < s_{m-1} < s_m = 1.$$

We will show now how the approximate value provided by formula (4.7) for the integral $I(u)$ may be computed by the use of the *assumed known values* $\psi(0), \psi(s_1), ..., \psi(s_{m-1})$, and $\psi(1)$.

If $u \in (0, s_1]$, we may choose the division

$$\begin{aligned} \lambda_0 &= 0, \lambda_1 = \sqrt{\frac{s_1^2 - u^2}{1-u^2}}, \lambda_2 = \sqrt{\frac{s_2^2 - u^2}{1-u^2}}, ..., \\ \lambda_{m-1} &= \sqrt{\frac{s_{m-1}^2 - u^2}{1-u^2}}, \lambda_m = 1; \end{aligned}$$

which obviously satisfies the condition

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{m-1} < \lambda_m = 1.$$

Then by formula (4.7), for $n = m$, we get

$$\begin{aligned} (5.1)\ A_n(\psi, \lambda, u) &= \frac{1}{2\pi}\left\{\sqrt{s_1^2 - u^2}\left[\frac{\psi(u)}{u} + \frac{\psi(s_1)}{s_1}\right]\right. \\ &+ \sum_{i=1}^{m-2}\left(\sqrt{s_{i+1}^2 - u^2} - \sqrt{s_i^2 - u^2}\right)\left[\frac{\psi(s_{i+1})}{s_{i+1}} + \frac{\psi(s_i)}{s_i}\right] \\ &+ \left.\left(\sqrt{1-u^2} - \sqrt{s_{m-1}^2 - u^2}\right)\left[\frac{\psi(s_{m-1})}{s_{m-1}} + \psi(1)\right]\right\}. \end{aligned}$$

Since $u \in (0, s_1]$, instead of $\frac{\psi(u)}{u}$ in formula (5.1) we may choose either $\lim_{u\to 0+} \frac{\psi(u)}{u}$ or $\frac{\psi(s_1)}{s_1}$.

For $u \in (s_1, s_2]$, we may choose the division

$$\begin{aligned} \lambda_0 &= 0, \lambda_1 = \sqrt{\frac{s_2^2 - u^2}{1-u^2}}, \lambda_2 = \sqrt{\frac{s_3^2 - u^2}{1-u^2}}, ..., \\ \lambda_{m-2} &= \sqrt{\frac{s_{m-1}^2 - u^2}{1-u^2}}, \lambda_{m-1} = 1; \end{aligned}$$

which obviously satisfies

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{m-2} < \lambda_{m-1} = 1$$

and for $n = m-1$, formula (4.7) will produce the quadrature [10]

$$\begin{aligned}(5.2)\quad & A_{m-1}(\psi,\lambda,u) \\ = & \frac{1}{2\pi}\left\{\sqrt{s_2^2-u^2}\left[\frac{\psi(u)}{u}+\frac{\psi(s_2)}{s_2}\right]\right. \\ & +\sum_{i=1}^{m-3}\left(\sqrt{s_{i+2}^2-u^2}-\sqrt{s_{i+1}^2-u^2}\right)\left[\frac{\psi(s_{i+2})}{s_{i+2}}+\frac{\psi(s_{i+1})}{s_{i+1}}\right] \\ & \left.+\left(\sqrt{1-u^2}-\sqrt{s_{m-1}^2-u^2}\right)\left[\frac{\psi(s_{m-1})}{s_{m-1}}+\psi(1)\right]\right\}.\end{aligned}$$

Here we may use instead of $\frac{\psi(u)}{u}$ in formula (5.2) the value $\frac{\psi(s_1)}{s_1}$.

In general, if we assume that $u\in(s_{k-1},s_k]$, $k=1,...,m-1$; we may choose the division

$$\begin{aligned}\lambda_0 &= 0,\lambda_1=\sqrt{\frac{s_k^2-u^2}{1-u^2}},\lambda_2=\sqrt{\frac{s_{k+1}^2-u^2}{1-u^2}},..., \\ \lambda_{m-k} &= \sqrt{\frac{s_{m-1}^2-u^2}{1-u^2}},\lambda_{m-k+1}=1;\end{aligned}$$

which satisfies the condition

$$0=\lambda_0<\lambda_1<...<\lambda_{m-k}<\lambda_{m-k+1}=1$$

and for which, by applying (4.7) we get the quadrature $(n=m-k+1)$ [10]:

$$\begin{aligned}(5.3)\quad & A_{m-k+1}(\psi,\lambda,u) \\ = & \frac{1}{2\pi}\left\{\sqrt{s_k^2-u^2}\left[\frac{\psi(u)}{u}+\frac{\psi(s_k)}{s_k}\right]\right. \\ & +\sum_{i=1}^{m-k-1}\left(\sqrt{s_{k+i}^2-u^2}-\sqrt{s_{k+i-1}^2-u^2}\right)\left[\frac{\psi(s_{k+i})}{s_{k+i}}+\frac{\psi(s_{k+i-1})}{s_{k+i-1}}\right] \\ & \left.+\left(\sqrt{1-u^2}-\sqrt{s_{m-1}^2-u^2}\right)\left[\frac{\psi(s_{m-1})}{s_{m-1}}+\psi(1)\right]\right\}.\end{aligned}$$

Here, instead of the term $\frac{\psi(u)}{u}$ we may take $\frac{\psi(s_{k-1})}{s_{k-1}}$.

Now, if $u\in(s_{m-2},s_{m-1}]$, then we may choose the division

$$\lambda_0=0,\lambda_1=\sqrt{\frac{s_{m-1}^2-u^2}{1-u^2}},\lambda_2=1$$

and the approximation formula will be (by (4.7))

$$\begin{aligned}(5.4)\quad A_3(\psi,\lambda,u) &= \frac{1}{2\pi}\left\{\sqrt{s_{m-1}^2-u^2}\left[\frac{\psi(u)}{u}+\frac{\psi(s_{m-1})}{s_{m-1}}\right]\right. \\ & \left.+\left(\sqrt{1-u^2}-\sqrt{s_{m-1}^2-u^2}\right)\left[\frac{\psi(s_{m-1})}{s_{m-1}}+\psi(1)\right]\right\}\end{aligned}$$

and the term $\frac{\psi(u)}{u}$ may be replaced by $\frac{\psi(s_{m-2})}{s_{m-2}}$.

Finally, for the last interval $(s_{m-1},1]$, we may take the division

$$\lambda_0=0,\lambda_1=1$$

and for $n=2$, we have the quadrature

$$A_2(\psi,\lambda,u)=\frac{\sqrt{1-u^2}}{2\pi}\left[\frac{\psi(u)}{u}+\psi(1)\right] \tag{5.5}$$

in which $\frac{\psi(u)}{u}$ may be replaced by $\frac{\psi(s_{m-1})}{s_{m-1}}$.

To numerically implement the above quadrature rule, an equidistant partitioning of the interval $[0,1]$ is natural to be considered; details of this implementations may be found in [10].

6. New Approximation Using Stieltjes Integrals

Let $f,g:[a,b]\to R$ be two mappings such that the Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists. It is well known that if, for instance f is continuous and g is of bounded variation, then the above Stieltjes integral exists and

$$\left|\int_a^b f(t)\,dg(t)\right|\le\max_{t\in[a,b]}|f(t)|\bigvee_a^b(g), \tag{6.1}$$

where $\bigvee_a^b(g)$ denotes the total variation of g. We also note that if f is Riemann integrable on $[a,b]$ and g is Lipschitzian on the same interval, then $\int_a^b f(t)\,dg(t)$ exists and

$$\left|\int_a^b f(t)\,dg(t)\right|\le L\int_a^b|f(t)|\,dt, \tag{6.2}$$

where L is the Lipschitzian constant of g on $[a,b]$.

Now, consider $I_n:a=x_0<x_1<...<x_{n-1}<x_n=b$ to be a division of the interval $[a,b]$, $h_i:=x_{i+1}-x_i \quad (i=0,...,n-1)$ and, $v(h):=\max\{h_i|i=0,...,n-1\}$. Define the general Riemann-Stieltjes sum

$$S(f,g,I_n,\boldsymbol{\xi}):=\sum_{i=0}^{n-1}f(\xi_i)\left[g(x_{i+1})-g(x_i)\right], \tag{6.3}$$

where $\boldsymbol{\xi}=(\xi_0,...,\xi_{n-1})$ is a sequence of intermediate points in I_n, i.e. $\xi_i\in[x_i,x_{i+1}]$, $\quad i=0,...,n-1$.

If we consider the approximation

$$\int_a^b f(t)\,dg(t)=S(f,g,I_n,\boldsymbol{\xi})+R(f,g,I_n,\boldsymbol{\xi}), \tag{6.4}$$

where $S(f,g,I_n,\boldsymbol{\xi})$ is the Riemann-Stieltjes sum given by (6.3) and $R(f,g,I_n,\boldsymbol{\xi})$ is the remainder, then it has been shown in [11] that for f Lipschitzian with the constant L and g of bounded variation, the error estimate satisfies the bound

$$|R(f,g,I_n,\boldsymbol{\xi})|\le L\left[\frac{1}{2}v(h)+\max_{i=0,\dots,n-1}\left|\xi_i-\frac{x_i+x_{i+i}}{2}\right|\right]\bigvee_a^b(g). \tag{6.5}$$

It is obvious that the best inequality one can have in (6.5) is for $\xi_i=\frac{x_i+x_{i+1}}{2}; \quad i=0,...,n-1$ obtaining the approximation

$$\int_a^b f(t)\,dg(t)=Y(f,g,I_n)+R(f,g,I_n) \tag{6.6}$$

where $Y(f,g,I_n)$ is the mid-point formula:

$$Y(f,g,I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)[g(x_{i+1}) - g(x_i)] \tag{6.7}$$

and the remainder satisfies the estimate:

$$R(f,g,I_n) \le \frac{1}{2} L v(h) \bigvee_a^b (g). \tag{6.8}$$

Here the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Now, let consider the integral

$$I(\Psi,u) := \frac{1}{\pi}\int_u^1 \frac{\Psi(s)}{\sqrt{s^2-u^2}}ds, \quad 0 \le u \le 1 \tag{6.9}$$

which is connected to the fiber refractive index profile by

$$n(u) = \exp[I(\Psi,u)], \quad 0 \le u \le 1. \tag{6.10}$$

One may observe that

$$I(\Psi,u) = \frac{1}{\pi}\int_u^1 f(s)\, dg(s), \tag{6.11}$$

where $f(s) = \frac{\Psi(s)}{s}$ and $g(s) = \sqrt{s^2-u^2}$, $s \in [u,1], 0 < u \le 1$.

Now is we assume that Ψ is differentiable, then

$$f'(s) = \frac{s\Psi'(s) - \Psi(s)}{s^2}, \quad s \in [u,1], 0 < u \le 1 \tag{6.12}$$

and if there exists a constant $M > 0$ such that

$$|s\Psi'(s) - \Psi(s)| \le Ms^2, \quad s \in [u,1], 0 < u \le 1, \tag{6.13}$$

then we may conclude that f is Lipschitzian with the constant M mentioned above.

Since $g'(s) = \frac{s}{\sqrt{s^2-u^2}}$, $s \in [u,1], 0 < u \le 1$, then

$$\bigvee_u^1 (g) = \int_u^1 |g'(s)|\, ds = \int_u^1 \frac{sds}{\sqrt{s^2-u^2}} = \sqrt{1-u^2}. \tag{6.14}$$

For a given $u \in (0,1)$, consider now the division

$$(A_n) : u = \delta_0 < \delta_1 < \delta_2 < ... < \delta_{n-1} < \delta_n = 1. \tag{6.15}$$

For this division and the above choices of the function f and g, we may consider the mid-point rule

$$\begin{aligned} Y(\Psi,u,A_n) &= \sum_{i=0}^{n-1} \frac{\Psi\left(\frac{\delta_i+\delta_{i+1}}{2}\right)}{\frac{\delta_i+\delta_{i+1}}{2}} \cdot \left[\sqrt{\delta_{i+1}^2 - u^2} - \sqrt{\delta_i^2 - u^2}\right] \\ &= 2\sum_{i=0}^{n-1} \Psi\left(\frac{\delta_i+\delta_{i+1}}{2}\right) \cdot \frac{\delta_{i+1}-\delta_i}{\sqrt{\delta_{i+1}^2-u^2} + \sqrt{\delta_i^2-u^2}}. \end{aligned} \tag{6.16}$$

Using the approximation (6.6), we may conclude that the desired integral $I(\Psi,u)$ may be approximated in the following way

$$I(\Psi,u) = \frac{1}{\pi} Y(\Psi,u,A_n) + Q(\Psi,u,A_n), \tag{6.17}$$

where the error $Q(\Psi, u, A_n)$, by (6.8), satisfies the estimate

$$|Q(\Psi, u, A_n)| \le \frac{1}{2\pi} M v(\delta) \sqrt{1-u^2}, 0 < u < 1. \tag{6.18}$$

Here M is the constant defined in (6.13) and $v(\delta) = \max_{i=0,\dots,n-1} (\delta_{i+1} - \delta_i)$.

In practical situations, it is useful if we consider the equidistant partitioning

$$\delta_i := u + i \cdot \frac{1-u}{n}, i = 0, ..., n.$$

In this case the quadrature (6.16) becomes

$$\begin{aligned} & Y_n(\Psi, u) \\ = & \frac{2(1-u)}{n} \cdot \sum_{i=0}^{n-1} \Psi\left[u + \left(i + \frac{1}{2}\right) \cdot \frac{1-u}{n}\right] \\ & \times \frac{1}{\sqrt{\frac{(i+1)^2(1-u)^2}{n^2} + 2(i+1)u\frac{1-u}{n}} + \sqrt{\frac{i^2(1-u)^2}{n^2} + 2iu\frac{1-u}{n}}}. \end{aligned} \tag{6.19}$$

Consequently we have the approximation formula

$$I(\Psi, u) = \frac{1}{\pi} Y_n(\Psi, u) + Q_n(\Psi, u), \tag{6.20}$$

where $Y_n(\Psi, u)$ is defined by (6.19) and the error of approximation $Q_n(\Psi, u)$ satisfies the estimate

$$|Q_n(\Psi, u)| \le \frac{1-u}{2\pi n} M \sqrt{1-u^2} \le \frac{M}{2\pi n}, 0 < u < 1, n \ge 1. \tag{6.21}$$

It is obvious that $Q_n(\Psi, u) \to 0$ uniformly over $u \in [0, 1]$ as $n \to \infty$.

The above inequality (6.21) provides an *a priori* estimate for the error in approximating $I(\Psi, u)$ by $Y_n(\Psi, u), u \in (0, 1)$ when the constant M is known.

For the division $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$, consider the trapezoidal rule for Stieltjes integral

$$T(f, g, I_n) := \sum_{i=0}^{n-1} \frac{f(x) + f(x_{i+1})}{2} [g(x_{i+1}) - g(x_i)]. \tag{6.22}$$

It has been shown that for appropriate choices of f and g one can approximate the Stieltjes integral $\int_a^b f(t)\, dg(t)$ in the following way [12, Chapter 8]

$$\int_a^b f(t)\, dg(t) = T(f, g, I_n) + W(f, g, I_n), \tag{6.23}$$

where $W(f, g, I_n)$ denotes the error of the approximation.

If $f, g : [a, b] \to R$ is Lipschitzian with the constant L and $u : [a, b] \to R$ is a bounded variation on $[a, b]$ then (see [12, Chapter 8])

$$|W(f, g, I_n)| \le \frac{1}{2} L v(h) \bigvee_a^b (u), \tag{6.24}$$

where $v(h) := \max_{i=0,\dots,n-1} \{h_i\}$, and $h_i := x_{i+1} - x_i, i \in \{0, ..., n-1\}$.

Consider now the integral from (6.11)

$$I(\Psi, u) = \frac{1}{\pi} \int_u^1 f(s)\, dg(s), \tag{6.25}$$

where $f(s) = \frac{\Psi(s)}{s}, g(s) = \sqrt{s^2 - u^2}, s \in [u, 1], 0 < u \leq 1$, and assume that there exists a constant $M > 0$ such that (6.13) holds. Further, assume that $u \in (0, 1)$ and utilizing the division of the interval $[0, 1]$ as defined by (6.15) for the functions f and g, one may consider the trapezoidal rule

$$\begin{aligned} T(\Psi, u, A_n) &= \sum_{i=0}^{n-1} \frac{\frac{\Psi(\delta_{i+1})}{\delta_{i+1}} + \frac{\Psi(\delta_i)}{\delta_i}}{2} \left[\sqrt{\delta_{i+1}^2 - u^2} - \sqrt{\delta_i^2 - u^2}\right] \\ &= \frac{1}{2} \sum_{i=0}^{n-1} \frac{\delta_i \Psi(\delta_{i+1}) + \delta_{i+1} \Psi(\delta_i)}{\delta_i \delta_{i+1}} \left[\sqrt{\delta_{i+1}^2 - u^2} - \sqrt{\delta_i^2 - u^2}\right]. \end{aligned} \tag{6.26}$$

Using the approximation (6.23), the integral in (6.25) may be computed in the following way

$$I(\Psi, u) = \frac{1}{\pi} T(\Psi, u, A_n) + S(\Psi, u, A_n), \tag{6.27}$$

where the error $S(\Psi, u, A_n)$ defined by (6.24) satisfies the estimate

$$|S(\Psi, u, A_n)| \leq \frac{1}{2\pi} M v(\delta) \sqrt{1 - u^2}, 0 < u < 1. \tag{6.28}$$

The constant M in (6.28) is defined by (6.13) and $v(\delta) = \max_{i=0,\ldots,n-1} (\delta_{i+1} - \delta_i)$.

Usually an equidistant partitioning of the interval $[u, 1]$ is preferred, i.e. $\delta_i = u + i \cdot \frac{1-u}{n}, i = 0, ..., n$. As such, the quadrature $T(\Psi, u, A_n)$ becomes

$$\begin{aligned} T_n(\Psi, u) &= \frac{1}{2} \sum_{i=0}^{n-1} \frac{\left(u + i \cdot \frac{1-u}{n}\right) \Psi\left[u + (i+1)\frac{1-u}{n}\right] + \left[u + (i+1)\frac{1-u}{n}\right] \Psi\left(u + i \cdot \frac{1-u}{n}\right)}{\left(u + i \cdot \frac{1-u}{n}\right)\left[u + (i+1) \cdot \frac{1-u}{n}\right]} \\ &\times \left[\sqrt{\frac{(i+1)^2 (1-u)^2}{n^2} + 2(i+1) u \frac{1-u}{n}} - \sqrt{\frac{i^2 (1-u)^2}{n^2} + 2iu\frac{1-u}{n}}\right]. \end{aligned} \tag{6.29}$$

In conclusion, we have the approximation formula

$$I(\Psi, u) = \frac{1}{\pi} T_n(\Psi, u) + S_n(\Psi, u), \tag{6.30}$$

where $T_n(\Psi, u)$ is defined in (6.29), with an error of approximation $S_n(\Psi, u)$ satisfying the bound

$$|S_n(\Psi, u)| \leq \frac{1-u}{2\pi n} M \sqrt{1 - u^2} \leq \frac{M}{2\pi n}, 0 < u < 1, n \geq 1.$$

Also note that $S_n(\Psi, u) \to 0$ uniformly over $u \in [0, 1]$ as $n \to \infty$.

7. Numerical experiments

Consider the function $\Psi : [0, 1] \to [0, \infty), \Psi(s) = s^3$. The exact transform of Ψ is given by

$$I(\Psi, u) = (1/3\sqrt{(1 - u^2)} + 2/3\sqrt{(1 - u^2)}u^2)/\pi, u \in [0, 1]. \tag{7.1}$$

The plot of this function is incorporated in Figure 1. The plots of the errors $Q_n(\Psi, u)$ (for Stieltjes mid-point rule) and $S_n(\Psi, u)$ (for Stieltjes trapezoidal rule)

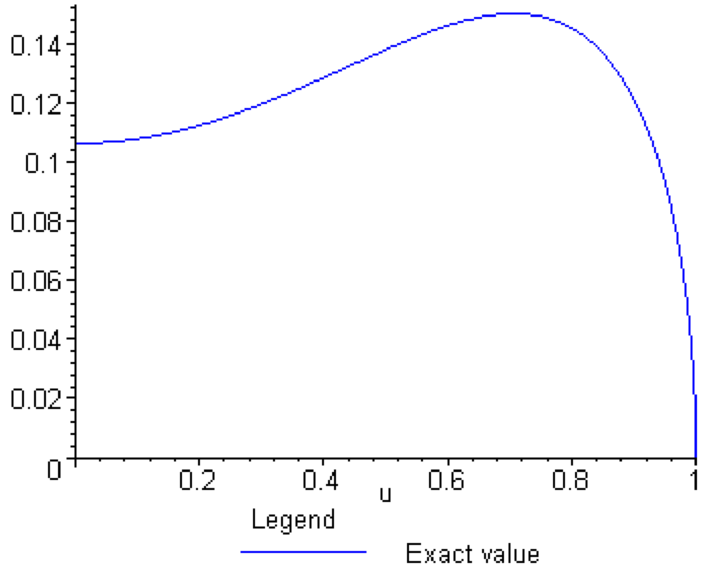

FIGURE 1. Exact value for $I(\Psi, u)$, where $\Psi(s) = s^3$.

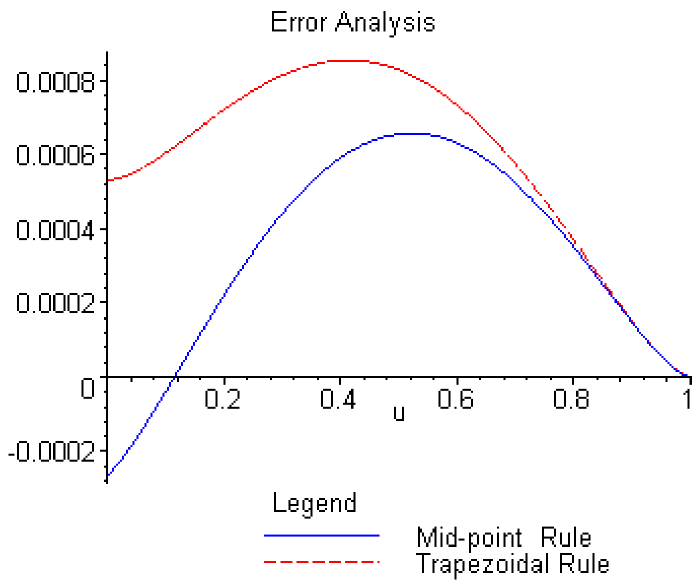

FIGURE 2. Comparisson of the errors as computed by the mid-point rule and trapezoidal rule for $n = 10$.

computed by Maple 6, for $n = 10, 10^2$ and 10^3 are incorporated in Figure 2, 3 and 4, respectively.

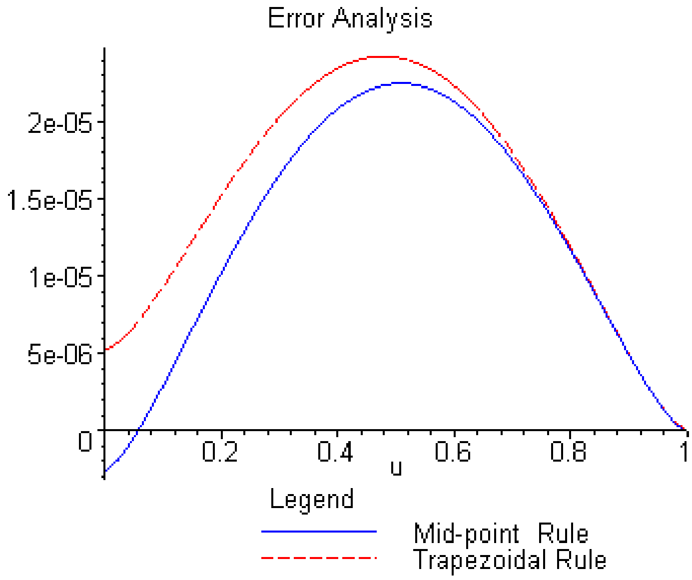

FIGURE 3. Comparisson of the errors as computed by the mid-point rule and trapezoidal rule for $n = 10^2$.

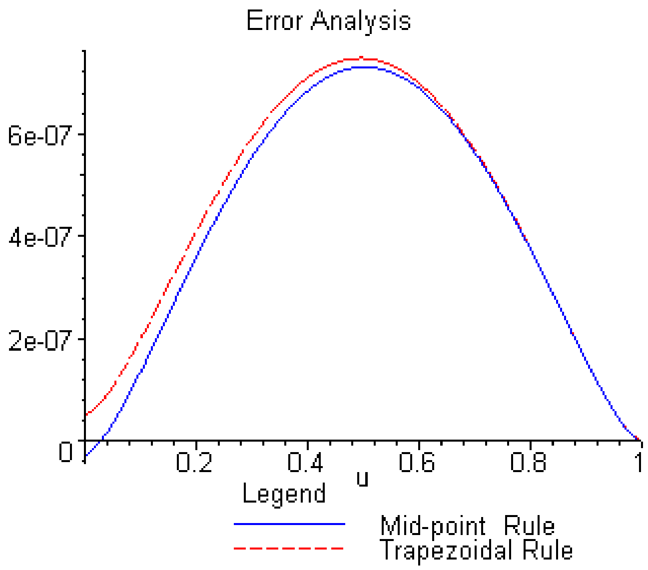

FIGURE 4. Comparisson of the errors as computed by the mid-point rule and trapezoidal rule for $n = 10^3$.

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