Generalizations of Weighted Trapezoidal Inequality for Mappings of Bounded Variation and Their Applications

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Abstract. In this paper, we establish some generalizations of weighted trapezoid inequality for mappings of bounded variation, and give several applications for $r-moment$, the expectation of a continuous random variable and the Beta mapping.

1. Introduction

The *trapezoid inequality* states that if f'' exists and is bounded on (a,b), then

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] \right| \le \frac{(b-a)^3}{12} \|f''\|_\infty , \tag{1.1}$$

where

$$\|f''\|_\infty := \sup_{x\in(a,b)} |f''| < \infty.$$

Now if we assume that $I_n : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *trapezoidal quadrature formula* $A_T(f, I_n)$, having an error given by $R_T(f, I_n)$, where

$$A_T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})]\, l_i,$$

and the remainder satisfies the estimation

$$|R_T(f, I_n)| \le \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} l_i^3,$$

with $l_i := x_{i+1} - x_i$ for $i = 0, 1, \cdots, n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [**2**]-[**9**].

Recently, Cerone-Dragomir-Pearce [**4**] proved the following two trapezoid type inequalities:

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.

Key words and phrases. Trapezoid Inequality, Mappings of bounded variation, Random Variable, r-moments, expectation, Beta function.

THEOREM 1. *Let $f:[a,b]\to R$ be a mapping of bounded variation. Then*

$$\left|\int_a^b f(t)dt-[(x-a)f(a)+(b-x)f(b)]\right| \leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]V_a^b(f), \tag{1.2}$$

for all $x\in[a,b]$, where $V_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Let I_n, l_i $(i=0,1,\cdots,n-1)$ be as above and let $\xi_i\in[x_i,x_{i+1}]$ $(i=0,1,\cdots,n-1)$ be intermediate points. Define the sum

$$T_P(f,I_n,\xi):=\sum_{i=0}^{n-1}[(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})].$$

We have the following result concerning the approximation of the integral $\int_a^b f(x)dx$ in terms of T_P.

THEOREM 2. *Let f be defined as in Theorem 1, then we have*

$$\int_a^b f(x)dx = T_P(f,I_n,\xi)+R_P(f,I_n,\xi).$$

The remainder term $R_P(f,I_n,\xi)$ satisfies the estimate

$$|R_P(f,I_n,\xi)| \leq \left[\frac{1}{2}\nu(l)+\max_{i=0,1,\cdots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|V_a^b(f)\right]\leq \nu(l)V_a^b(f), \tag{1.3}$$

where $\nu(l):=\max\{l_i\,|i=0,1,\cdots,n-1\}$. The constant $\frac{1}{2}$ is the best possible.

In this paper, we establish weighted generalizations of Theorems 1-2, and give several applications for $r-moment$, the expectation of a continuous random variable and the Beta mapping.

2. Some Integral Inequalities

We may state the following result.

THEOREM 3. *Let $g:[a,b]\to R$ be non-negative and continuous and let $h:[a,b]\to R$ be differentiable such that $h'(t)=g(t)$ on $[a,b]$. Suppose f is defined as in Theorem 1. Then*

$$\left|\int_a^b f(t)g(t)\,dt-[(x-h(a))f(a)+(h(b)-x)f(b)]\right| \leq \left[\frac{1}{2}\int_a^b g(t)\,dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]V_a^b(f), \tag{2.1}$$

for all $x\in[h(a),h(b)]$. The constant $\frac{1}{2}$ is the best possible.

PROOF. Let $x \in [h(a), h(b)]$. Using integration by parts, we have the following identity

$$\begin{aligned}(2.2)\qquad & \int_a^b (x - h(t))\, df(t) \\ = & \ (x - h(t))\, f(t)\,|_a^b + \int_a^b f(t) g(t)\, dt \\ = & \int_a^b f(t) g(t)\, dt - \left[(x - h(a))\, f(a) + (h(b) - x)\, f(b)\right].\end{aligned}$$

It is well known [**1**, p.159] that if $\mu, \nu : [a, b] \to R$ are such that μ is continuous on $[a, b]$ and ν is of bounded variation on $[a, b]$, then $\int_a^b \mu(t)\, d\nu(t)$ exists and [**1**, p.177]

$$(2.3)\qquad \left| \int_a^b \mu(t)\, d\nu(t) \right| \le \sup_{x \in [a,b]} |\mu(t)|\, V_a^b(\nu).$$

Now, using identity (*2.2*) and inequality (*2.3*), we have

$$\begin{aligned}(2.4)\qquad & \left| \int_a^b f(t) g(t)\, dt - \left[(x - h(a))\, f(a) + (h(b) - x)\, f(b)\right] \right| \\ \le & \sup_{t \in [a,b]} |x - h(t)|\, V_a^b(f).\end{aligned}$$

Since $x - h(t)$ is decreasing on $[a, b]$, $h(a) \le x \le h(b)$ and $h'(t) = g(t)$ on $[a, b]$, we have

$$\begin{aligned}(2.5)\qquad & \sup_{t \in [a,b]} |x - h(t)| \\ = & \max\{x - h(a), h(b) - x\} \\ = & \frac{h(b) - h(a)}{2} + \left| x - \frac{h(a) + h(b)}{2} \right| \\ = & \frac{1}{2} \int_a^b g(t)\, dt + \left| x - \frac{h(a) + h(b)}{2} \right|.\end{aligned}$$

Thus, by (*2.4*) and (*2.5*), we obtain (*2.1*).

Let

$$\begin{aligned} g(t) &\equiv 1,\ t \in [a, b], \\ h(t) &= t,\ t \in [a, b], \\ f(t) &= \begin{cases} 0 & \text{as } t = a \\ 1 & \text{as } t \in (a, b) \\ 0 & \text{as } t = b, \end{cases}\end{aligned}$$

and $x = \frac{a+b}{2}$. Then, we can see that the constant $\frac{1}{2}$ is best possible. This completes the proof. ∎

REMARK 1. *(1) If we choose $g(t) \equiv 1, h(t) = t$ on $[a, b]$, then the inequality (2.1) reduces to (1.2).*

(2) If we choose $x = \frac{h(a)+h(b)}{2}$*, then we get*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \le \frac{1}{2}\int_a^b g(t)\,dt \cdot V_a^b(f), \tag{2.6}$$

which is the "weighted trapezoid" inequality.

Under the conditions of Theorem *3*, we have the following corollaries.

Corollary 1. *Let* $f \in C^{(1)}[a,b]$*. Then we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)g(t)\,dt - \left[(x-h(a))f(a) + (h(b)-x)f(b)\right]\right| \\ \le\ & \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right]\left\|f'\right\|_1,\end{aligned} \tag{2.7}$$

for all $x \in [h(a), h(b)]$*, where* $\|\cdot\|_1$ *is the* $L_1 -$ *norm, namely*

$$\left\|f'\right\|_1 := \int_a^b |f'(t)|\,dt.$$

Corollary 2. *Let* $f : [a,b] \to R$ *be a Lipschitzian mapping with the constant* $L > 0$*. Then we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)g(t)\,dt - \left[(x-h(a))f(a) + (h(b)-x)f(b)\right]\right| \\ \le\ & \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right](b-a)L,\end{aligned} \tag{2.8}$$

for all $x \in [h(a), h(b)]$*.*

Corollary 3. *Let* $f : [a,b] \to R$ *be a monotonic mapping. Then we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)g(t)\,dt - \left[(x-h(a))f(a) + (h(b)-x)f(b)\right]\right| \\ \le\ & \left[\frac{1}{2}\int_a^b g(t)\,dt + \left|x - \frac{h(a)+h(b)}{2}\right|\right] \cdot |f(b) - f(a)|,\end{aligned} \tag{2.9}$$

for all $x \in [h(a), h(b)]$*.*

Remark 2. *The following inequality is well-known in the literature as the Fejér inequality (see for example* [**10**]*):*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \le \int_a^b f(t)g(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt, \tag{2.10}$$

where $f : [a,b] \to R$ *is convex and* $g : [a,b] \to R$ *is non-negative integrable and symmetric to* $\frac{a+b}{2}$*. Using the above results and (2.6), we obtain the following error bound of the second inequality in (2.10),*

$$0 \le \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt \le \frac{1}{2}\int_a^b g(t)\,dt \cdot V_a^b(f), \tag{2.11}$$

provided that f *is of bounded variation on* $[a,b]$*.*

Remark 3. *If f is convex and Lipschitzian with the constant L on $[a,b]$, g is defined as in Remark 2 and $x=\frac{h(a)+h(b)}{2}$, then we get from (2.8) and (2.10) that*

$$0\leq\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\leq\frac{(b-a)L}{2}\int_a^b g(t)\,dt. \tag{2.12}$$

Remark 4. *If f is convex and monotonic on $[a,b]$, g is defined as in Remark 2 and $x=\frac{h(a)+h(b)}{2}$, then we get from (2.9) and (2.10) that*

$$\begin{aligned}0 &\leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\\ &\leq \frac{|f(b)-f(a)|}{2}\int_a^b g(t)\,dt.\end{aligned} \tag{2.13}$$

Remark 5. *If f is continuous, differentiable, convex on $[a,b]$ and $f'\in L_1(a,b)$, g is defined as in Remark 2 and $x=\frac{h(a)+h(b)}{2}$, then we get from (2.7) and (2.10) that*

$$0\leq\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\leq\frac{\|f'\|_1}{2}\int_a^b g(t)\,dt. \tag{2.14}$$

3. Applications for Quadrature Rules

Throughout this section, let g and h be defined as in Theorem 3.

Let $f:[a,b]\to R$, and let $I_n: a=x_0<x_1<\cdots<x_n=b$ be a partition of $[a,b]$ and $\xi_i\in[h(x_i),h(x_{i+1})]$ $(i=0,1,\cdots,n-1)$ be intermediate points. Put $l_i:=h(x_{i+1})-h(x_i)=\int_{x_i}^{x_{i+1}}g(t)\,dt$ and define the sum

$$T_P(f,g,h,I_n,\xi):=\sum_{i=0}^{n-1}\left[(\xi_i-h(x_i))f(x_i)+(h(x_{i+1})-\xi_i)f(x_{i+1})\right].$$

We have the following concerning the approximation of the integral $\int_a^b f(t)g(t)\,dt$ in terms of T_P.

Theorem 4. *Let f be defined as in Theorem 3 and let*

$$\int_a^b f(t)g(t)\,dt=T_P(f,g,h,I_n,\xi)+R_P(f,g,h,I_n,\xi). \tag{3.1}$$

Then, the remainder term $R_P(f,g,h,I_n,\xi)$ satisfies the estimate

$$\begin{aligned}&|R_P(f,g,h,I_n,\xi)|\\ \leq\ &\left[\frac{1}{2}\nu(l)+\max_{i=0,1,\cdots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\right]V_a^b(f)\\ \leq\ &\nu(l)V_a^b(f),\end{aligned} \tag{3.2}$$

where $\nu(l):=\max\{l_i\,|i=0,1,\cdots,n-1\}$. The constant $\frac{1}{2}$ is the best possible.

Proof. Apply Theorem 3 on the intervals $[x_i,x_{i+1}]$ $(i=0,1,\cdots,n-1)$ to get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}f(t)g(t)\,dt-\left[(\xi_i-h(x_i))f(x_i)+(h(x_{i+1})-\xi_i)f(x_{i+1})\right]\right|\\ \leq\ &\left[\frac{1}{2}l_i+\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\right]V_{x_i}^{x_{i+1}}(f),\end{aligned}$$

for all $i\in\{0,1,\cdots,n-1\}$.

Using this and the generalized triangle inequality, we have

$$
\begin{aligned}
& \left|R_P\left(f,g,h,I_n,\xi\right)\right| \\
\leq & \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)g\left(t\right)dt-\left[\left(\xi_i-h(x_i)\right)f\left(x_i\right)+\left(h(x_{i+1})-\xi_i\right)f\left(x_{i+1}\right)\right]\right| \\
\leq & \sum_{i=0}^{n-1}\left[\frac{1}{2}l_i+\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\right]V_{x_i}^{x_{i+1}}\left(f\right) \\
\leq & \max_{i=0,1,\cdots,n-1}\left[\frac{1}{2}l_i+\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\right]\sum_{i=0}^{n-1}V_{x_i}^{x_{i+1}}\left(f\right) \\
\leq & \left[\frac{1}{2}\nu\left(l\right)+\max_{i=0,1,\cdots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\right]V_a^b\left(f\right)
\end{aligned}
$$

and the first inequality in (*3.2*) is proved.

For the second inequality in (*3.2*), we observe that

$$\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\leq\frac{1}{2}l_i\ \left(i=0,1,\cdots,n-1\right);$$

and then

$$\max_{i=0,1,\cdots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\leq\frac{1}{2}\nu\left(l\right).$$

Thus the theorem is proved. ∎

Remark 6. *If we choose* $g\left(t\right)\equiv 1, h\left(t\right)=t$ *on* $[a,b]$*, then the inequality* (*3.2*) *reduces to* (*1.3*).

The following corollaries can be useful in practice.

Corollary 4. *Let* $f:[a,b]\to R$ *be a Lipschitzian mapping with the constant* $L>0$, I_n *be defined as above and choose* $\xi_i=\frac{h(x_i)+h(x_{i+1})}{2}$ $\left(i=0,1,\cdots,n-1\right)$. *Then we have the formula*

$$
\begin{aligned}
\int_a^b f(t)g\left(t\right)dt &= T_P\left(f,g,h,I_n,\xi\right)+R_P\left(f,g,h,I_n,\xi\right) \qquad (3.3)\\
&= \sum_{i=0}^{n-1}\left[\frac{f(x_i)+f(x_{i+1})}{2}\int_{x_i}^{x_{i+1}}g\left(t\right)dt\right]+R_P\left(f,g,h,I_n,\xi\right),
\end{aligned}
$$

and the remainder satisfies the estimate

$$\left|R_P\left(f,g,h,I_n,\xi\right)\right|\leq\frac{\nu\left(l\right)\cdot L\cdot\left(b-a\right)}{2}. \qquad (3.4)$$

Corollary 5. *Let* $f:[a,b]\to R$ *be a monotonic mapping and let* ξ_i $\left(i=0,1,\cdots,n-1\right)$ *be defined as in Corollary 4. Then we have the formula* (*3.3*) *and the remainder satisfies the estimate*

$$\left|R_P\left(f,g,h,I_n,\xi\right)\right|\leq\frac{\nu\left(l\right)}{2}\cdot\left|f\left(b\right)-f\left(a\right)\right|. \qquad (3.5)$$

The case of equidistant division is embodied in the following corollary and remark:

COROLLARY 6. *Suppose that* $g(t) > 0$,

$$x_i = h^{-1}\left[h(a) + \frac{i(h(b)-h(a))}{n}\right] \quad (i = 0, 1, \cdots, n)$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b)-h(a)}{n} = \frac{1}{n}\int_a^b g(t)\,dt \quad (i = 0, 1, \cdots, n-1).$$

Let f *be defined as in Theorem 4 and choose* $\xi_i = \frac{h(x_i)+h(x_{i+1})}{2}$ $(i = 0, 1, \cdots, n-1)$. *Then we have the formula*

$$\int_a^b f(t)g(t)\,dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi) \tag{3.6}$$
$$= \frac{1}{2n}\int_a^b g(t)\,dt \cdot \sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})] + R_P(f, g, h, I_n, \xi),$$

and the remainder satisfies the estimate

$$|R_P(f, g, h, I_n, \xi)| \le \frac{1}{2n}\int_a^b g(t)\,dt \cdot V_a^b(f). \tag{3.7}$$

REMARK 7. *If we want to approximate the integral* $\int_a^b f(t)g(t)\,dt$ *by* $T_P(f, g, h, I_n, \xi)$ *with an accuracy less that* $\varepsilon > 0$, *we need at least* $n_\varepsilon \in N$ *points for the partition* I_n, *where*

$$n_\varepsilon := \left[\frac{1}{2\varepsilon}\int_a^b g(t)\,dt \cdot V_a^b(f)\right] + 1,$$

and $[r]$ *denotes the Gaussian integer of* $r \in R$.

4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$, $r \in R$, and let X be a continuous random variable having the continuous probability density function $g : [a, b] \to R$ and the $r-moment$

$$E_r(X) := \int_a^b t^r g(t)\,dt,$$

which is assumed to be finite.

THEOREM 5. *The inequality*

$$\left|E_r(X) - \frac{a^r + b^r}{2}\right| \le \frac{1}{2}|b^r - a^r|, \tag{4.1}$$

holds.

PROOF. If we put $f(t) = t^r$, $h(t) = \int_a^t g(x)\,dx$ $(t \in [a, b])$ and $x = \frac{h(a)+h(b)}{2}$ in Corollary 3, we obtain the inequality

$$\left|\int_a^b f(t)g(t)\,dt - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt\right| \tag{4.2}$$
$$\le \frac{1}{2}\int_a^b g(t)\,dt \cdot |f(b) - f(a)|.$$

Since

$$\begin{aligned}\int_a^b f(t)g(t)\,dt &= E_r(X), \qquad \int_a^b g(t)\,dt = 1, \\ \frac{f(a)+f(b)}{2} &= \frac{a^r+b^r}{2}, \text{ and } |f(b)-f(a)| = |b^r-a^r|,\end{aligned}$$

(4.1) follows from (4.2), immediately. This completes the proof. ∎

If we choose $r = 1$ in Theorem 5, then we have the following remark:

Remark 8. *The inequality*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{b-a}{2}, \tag{4.3}$$

where $E(X)$ is the expectation of the random variable X.

5. Inequality for the Beta Mapping

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad p>0,\ q>0.$$

Theorem 6. *Let p, $q \geq 1$. Then the inequality*

$$\begin{aligned}&\left|\beta(p,q) - \frac{1}{2np}\sum_{i=0}^{n-1}\left\{\left[1-\left(\frac{i}{n}\right)^{\frac{1}{p}}\right]^{q-1} + \left[1-\left(\frac{i+1}{n}\right)^{\frac{1}{p}}\right]^{q-1}\right\}\right| \tag{5.1}\\ &\leq \frac{1}{2np},\end{aligned}$$

holds for any positive integer n.

Proof. Let p, $q \geq 1$. If we put $a = 0$, $b = 1$, $f(t) = (1-t)^{q-1}$, $g(t) = t^{p-1}$ and $h(t) = \frac{t^p}{p}$ $(t \in [0,1])$ in Corollary 6, we obtain the inequality (5.1). This completes the proof. ∎

References

[1] T. M. Apostol, *Mathematical Analysis*, Second Edition, Addision-Wesley Publishing Company, 1975.

[2] N. S. Barnett, S. S. Dragomir and C. E. M. Pearce, A quasi- trapezoid inequality for double integrals, submitted.

[3] P. Cerone, Weighted three point identities and their bounds, *The SUT J. of Math.*, **38**(1), 17-37.

[4] P. Cerone, S. S. Dragomir and C. E. M. Pearce, Generalizations of the trapezoid inequality for mappings of bounded variation and applications, *Turkish Journal of Mathematics*, **24**(2) (2000), 147-163.

[5] P. Cerone and S.S. Dragomir, Trapezoidal-type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics*, Editor: G. Anastassiou, CRC Press, N.Y., (2000), 65-134.

[6] P. Cerone, J. Roumeliotis and G. Hanna, On weighted three point quadrature rules, *ANZIAM J.*, **42**(E) (2000), C340-361.

[7] S. S. Dragomir, On the trapezoid formula for mappings of bounded variation and applications, *Kragujevac J. Math.*, **23** (2001), 25-36.

[8] S. S. Dragomir, P. Cerone and A. Sofo, Some remarks on the trapezoid rule in numerical integration, *Indian J. of Pure and appl. Math.*, **31**(5) (2000), 475-494.. *RGMIA Res. Rep. Coll.*, **5**, 2(1999), Article 1.

[9] S. S. Dragomir and T. C. Peachey, New estimation of the remainder in the trapezoidal formula with applications, *Studia Universitatis Babes-Bolyai Mathematica*, **XLV**(4) (2000), 31-42 (http://rgmia.vu.edu.au/v1n2/trap1.dvi.)

[10] L. Fejér, Uberdie Fourierreihen, II, Math. Natur. Ungar. Akad Wiss. 24 (1906), 369-390. [In Hungarian]

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