APPROXIMATION OF THE STIELTJES INTEGRAL AND APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. Some inequalities for the Stieltjes integral and applications in numerical integration are given. The Stieltjes integral is approximated by the product of the divided difference of the integrator and the Lebesgue integral of the integrand. Bounds on the approximation error are provided. Applications for the Fourier Sine and Cosine transforms on finite intervals are mentioned as well.

1. Introduction

The following definitions will be required subsequently.

A function $w : [a,b] \to \mathbb{R}$ is said to be of $r-H-$Hölder type if for $x, y \in [a,b]$ it satisfies the conditions

$$|w(x) - w(y)| \leq H|x-y|^r, \quad r \in (0,1] \text{ and } H > 0.$$

A $1-L-$Hölder type function is also said to be $L-$Lipschitzian. A function w is said to be of bounded variation if for any division I_n of $[a,b]$,

$$I_n : a = x_0 < x_1 < ... < x_n = b$$

the variation of w on I_n is finite, this means that

$$\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)| < \infty.$$

The total variation of w on $[a,b]$ is denoted by $\bigvee_a^b (w)$, where

$$\bigvee_a^b (w) := \sup \left\{ \sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)|, I_n \text{ is a division of } [a,b] \right\}.$$

In [4, 5], the authors considered the following functional

$$D(f;u) := \int_a^b f(x)\,du(x) - [u(b) - u(a)] \frac{1}{b-a} \int_a^b f(t)\,dt, \tag{1.1}$$

provided that the involved integrals exist.

It is the approximation of the functional $D(f;u)$ that forms the basis of this paper and for the sake of clarity, will be dealt with here.

The following result in estimating the above functional $D(f;u)$ has been obtained in [4].

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Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is $L-$Lipschitzian on $[a,b]$, and f is Riemann integrable on $[a,b]$. If $m,M\in\mathbb{R}$ are such that*

$$m\leq f(x)\leq M \quad \text{for any } x\in[a,b], \tag{1.2}$$

then we have the inequality

$$|D(f;u)|\leq\frac{1}{2}L(M-m)(b-a). \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Applications for special means were also provided in [4].

In [5], the following result complementing those above was obtained.

Theorem 2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian $(L>0)$. Then we have the inequality*

$$|D(f;u)|\leq\frac{1}{2}L(b-a)\bigvee_a^b(u). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the above sense.

Applications for approximating the Stieltjes integral were also provided in both [4] and [5]. In [2] general results for three-point approximations of the Stieltjes integral were investigated.

In this paper we point out other similar inequalities in an effort to complete the picture and apply them in the numerical approximation of the Stieltjes integral $\int_a^b f(x)\,du(x)$. Approximations for the Fourier Sine and Cosine transforms on finite intervals, with an application for electrical circuits are mentioned as well.

2. The Case of Lipschitzian Integrators

Throughout this section, the integrator $u:[a,b]\to\mathbb{R}$ in the Stieltjes integral $\int_a^b f(t)\,du(t)$ is assumed to be Lipschitzian with the constant L.

The following theorem holds.

Theorem 3. *Assume that $u:[a,b]\to\mathbb{R}$ is as above.*

(i) *If $f:[a,b]\to\mathbb{R}$ is of bounded variation, then*

$$|D(f;u)|\leq\frac{3}{4}L(b-a)\bigvee_a^b(f). \tag{2.1}$$

(ii) *If $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type, then*

$$|D(f;u)|\leq\frac{2HL(b-a)^{r+1}}{(r+1)(r+2)}. \tag{2.2}$$

(iii) *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$|D(f;u)|\leq\begin{cases}\dfrac{1}{3}L(b-a)^2\|f'\|_\infty, & \text{if } f'\in L_\infty[a,b];\\[2ex] \dfrac{2^{\frac{1}{q}}L(b-a)^{\frac{1}{q}+1}\|f'\|_p}{(q+1)^{\frac{1}{q}}(q+2)^{\frac{1}{q}}}, & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{3}{4}L(b-a)\|f'\|_1. & \end{cases} \tag{2.3}$$

Proof. Firstly, let us observe that $D(f,u)$ defined in (1.1) satisfies the identity

$$D(f;u)=\int_a^b\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)du(x). \tag{2.4}$$

It is well known that if $p:[c,d]\to\mathbb{R}$ is Riemann integrable and $v:[c,d]\to\mathbb{R}$ is $L-$Lipschitzian, then the Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and

$$\left|\int_c^d p(t)\,dv(t)\right|\le L\int_c^d |p(t)|\,dt. \tag{2.5}$$

Taking the modulus in (2.4) and using (2.5) we get

$$|D(f;u)|\le L\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|dx. \tag{2.6}$$

(i) In [3], the author proved the following Ostrowski type inequality for functions of bounded variation

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{b-a}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{2.7}$$

for any $x\in[a,b]$. Then, by (2.6), we may state that

$$\begin{aligned}|D(f;u)| &\le \frac{L}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]dx\bigvee_a^b(f)\\ &=\frac{L}{b-a}\cdot\left[\frac{1}{2}(b-a)^2+\frac{1}{4}(b-a)^2\right]\bigvee_a^b(f)\end{aligned}$$

and the inequality (2.1) is proved.

(ii) In [1], the following inequality of Ostrowski type for $r-H-$Hölder type functions, f has been pointed out

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\qquad\le\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r\end{aligned} \tag{2.8}$$

for any $x\in[a,b]$. Then, by (2.6), we have

$$\begin{aligned}&|D(f;u)|\\ &\le\frac{H}{r+1}(b-a)^r\left\{\frac{1}{(b-a)^{r+1}}\left[\int_a^b(b-x)^{r+1}dx+\int_a^b(x-a)^{r+1}dx\right]\right\}\\ &=\frac{2HL(b-a)^{r+1}}{(r+1)(r+2)}\end{aligned}$$

and the inequality (2.2) is proved.

(*iii*) Using the following set of inequalities of Ostrowski type for absolutely continuous functions [1]

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.9}$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty, & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p & \text{if } f'\in L_p[a,b]; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1, \end{cases}$$

for any $x\in[a,b]$, we have from (2.6)

$$|D(f;u)| \tag{2.10}$$

$$\leq \begin{cases} L\cdot(b-a)\left\|f'\right\|_\infty \int_a^b\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]dx, \\ \frac{L}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p\int_a^b\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}dx \\ L\cdot\left\|f'\right\|_1\int_a^b\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]dx. \end{cases}$$

Since

$$\int_a^b\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]dx=\frac{1}{3}(b-a),$$

then the first part of (2.3) is proved.

Using Hölder's integral inequality, we have

$$\int_a^b\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}dx$$

$$\leq\left(\int_a^b dx\right)^{\frac{1}{p}}\left(\int_a^b\left\{\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\right\}^q dx\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)\,2^{\frac{1}{q}}}{(q+2)^{\frac{1}{q}}}$$

and by (2.10) and (2.6) we deduce the second part of (2.3).

The last part of (2.3) is obvious and we omit the details.

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3. The Case of Integrators of Bounded Variation

Throughout this section, the integrator $u : [a,b] \to \mathbb{R}$ in the Stieltjes integral $\int_a^b f(t)\,du(t)$ is assumed to be of bounded variation. The following result holds.

Theorem 4. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation.*

(i) If $f : [a,b] \to \mathbb{R}$ is continuous and of bounded variation, then

$$|D(f;u)| \le \bigvee_a^b (f) \bigvee_a^b (u). \tag{3.1}$$

(ii) If $f : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type with $r \in (0,1]$ and $H > 0$, then

$$|D(f;u)| \le \frac{H}{r+1}(b-a)^r \bigvee_a^b (u). \tag{3.2}$$

(iii) If $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then

$$|D(f;u)| \le \begin{cases} \frac{1}{2}(b-a)\|f'\|_\infty \bigvee_a^b (u), & \text{if } f' \in L_\infty[a,b]; \\ \\ \frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p \bigvee_a^b (u), & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f'\|_1 \bigvee_a^b (u). \end{cases} \tag{3.3}$$

Proof. It is well known that if $p : [c,d] \to \mathbb{R}$ is continuous and $v : [a,b] \to \mathbb{R}$ of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and

$$\left|\int_c^d p(t)\,dv(t)\right| \le \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (v). \tag{3.4}$$

Using the identity (2.4) and taking the modulus, we get, via (3.4)

$$|D(f;u)| \le \sup_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \bigvee_a^b (u). \tag{3.5}$$

(i) Using the Ostrowski type inequality (2.7), we may state that

$$\begin{aligned} &\sup_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \sup_{x\in[a,b]} \left\{ \frac{1}{b-a}\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \right\} \\ &= \bigvee_a^b (f), \end{aligned}$$

proving the inequality (3.1).

(ii) By making use of the inequality (2.8) for Hölder continuous functions, we may state that

$$\begin{aligned}
&\sup_{x\in[a,b]}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \sup_{x\in[a,b]}\left\{\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right]\right\}(b-a)^r \\
&= \frac{H(b-a)^r}{r+1},
\end{aligned}$$

proving the inequality.

(iii) Finally, by the use of the inequality (2.9) for absolutely continuous functions, we may write that

$$\begin{aligned}
&\sup_{x\in[a,b]}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \begin{cases}
\sup_{x\in[a,b]}\left\{\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty\right\}, \\
\sup_{x\in[a,b]}\left\{\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p\right\}, \\
\sup_{x\in[a,b]}\left\{\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1\right\},
\end{cases} \\
&= \begin{cases}
\frac{1}{2}(b-a)\|f'\|_\infty, \\
\frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p, \\
\|f'\|_1,
\end{cases}
\end{aligned}$$

and then, by (3.5), we deduce (3.3).

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4. A Quadrature Formula

Consider the partition of the interval $[a,b]$ given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \tag{4.1}$$

Denote $h_i := x_{i+1} - x_i \quad (i = 0, \ldots, n-1)$ and define the quadrature

$$A_n(f,u;I_n) := \sum_{i=0}^{n-1}\frac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(t)\,dt. \tag{4.2}$$

In [5] (see also [7, p. 468]), the authors pointed out the following result in approximating the Riemann-Stieltjes integral $\int_a^b f(x)\,du(x)$ in terms of the quadrature rules defined by (4.2).

Theorem 5. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is $L-$Lipschitzian on $[a,b]$. Then we have*

$$\int_a^b f(x)\,du(x) = A_n(f,u;I_n) + R_n(f,u;I_n), \tag{4.3}$$

where $A_n(f,u;I_n)$ is the quadrature formula defined by (4.2), and the remainder $R_n(f,u;I_n)$ satisfies the bound

$$|R_n(f,u;I_n)| \le \frac{1}{2} L \nu_{I_n}(h) \bigvee_a^b (u), \tag{4.4}$$

where

$$\nu_{I_n}(h) = \max\{h_i,\ i = 0,\dots,n-1\}.$$

Another result for more general integrators f, is also valid [5] (see also [7, p. 471]).

Theorem 6. *Assume that u is of bounded variation and f is continuous on $[a,b]$. If I_n is a division such that $\nu_{I_n}(k) < \delta$, then we have the representation (4.3). The remainder satisfies the estimate*

$$|R_n(f,u;I_n)| \le w(f,\delta) \bigvee_a^b (u), \tag{4.5}$$

where $w(f,\delta)$ is the continuity modulus given by

$$w(f,\delta) = \sup_{|x-t|\le\delta} |f(x) - f(t)|.$$

Now, for a given division I_n, denote

$$M_i := \sup_{t\in[x_i,x_{i+1}]} f(t), \quad m_i := \inf_{t\in[x_i,x_{i+1}]} f(t),$$

and $h_i(f) := M_i - m_i$, $i = 0,\dots,n-1$.

The following result may be stated as well.

Theorem 7. *Assume that u is $L-$Lipschitzian on $[a,b]$ and f is Riemann integrable on $[a,b]$. If I_n is a division of the interval $[a,b]$ as defined by (4.1), then we have the representation (4.3). The remainder $R_n(f,u;I_n)$ satisfies the estimate*

$$|R_n(f,u;I_n)| \le \frac{1}{2} L(b-a) \max_{i=0,\dots,n-1} \{h_i(f)\}. \tag{4.6}$$

Proof. If we apply Theorem 1 on the interval $[x_i, x_{i+1}]$ $(i = 0,\dots,n-1)$ we have

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,du(x) - \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(t)\,dt \right| \le \frac{1}{2} L(M_i - m_i)(x_{i+1} - x_i). \tag{4.7}$$

Summing (4.7) over i from 0 to $n-1$, and using the generalised triangle inequality, we deduce the estimate (4.6). ∎

The following result for Lipschitzian integrators holds.

Theorem 8. *Assume that I_n is a division of the interval $[a,b]$ as defined in (4.1) and $u : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant L.*

(*i*) *If $f:[a,b]\to\mathbb{R}$ is of bounded variation, then the remainder in the representation (4.3) satisfies the estimate*

$$|R_n(f,u;I_n)|\le\frac{3}{4}L\nu_{I_n}(h)\bigvee_a^b(f). \tag{4.8}$$

(*ii*) *If $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type, then we have the estimate*

$$\begin{aligned}|R_n(f,u;I_n)| &\le \frac{2HL}{(r+1)(r+2)}\sum_{i=0}^{n-1}h_i^{r+1}\\ &\le \frac{2HL(b-a)(\nu_{I_n}(h))^r}{(r+1)(r+2)}.\end{aligned} \tag{4.9}$$

(*iii*) *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$|R_n(f,u;I_n)| \le \begin{cases}\dfrac{1}{3}L\|f'\|_{\infty,[a,b]}\displaystyle\sum_{i=0}^{n-1}h_i^2, & \text{if } f'\in L_\infty[a,b];\\[2ex] \dfrac{2^{\frac{1}{q}}L\|f'\|_{p,[a,b]}}{(q+1)^{\frac{1}{q}}(q+2)^{\frac{1}{q}}}\left(\displaystyle\sum_{i=0}^{n-1}h_i^{1+q}\right)^{\frac{1}{q}} & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \dfrac{3}{4}L\nu_{I_n}(h)\|f'\|_{1,[a,b]}. \end{cases} \tag{4.10}$$

The proof follows by Theorem 3 and we omit the details.

Finally, by the use of Theorem 4, we may point out the following result for integrators of bounded variation.

Theorem 9. *Assume that I_n is a division of the interval $[a,b]$ as defined by (4.1) and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$.*

(*i*) *If $f:[a,b]\to\mathbb{R}$ is continuous and of bounded variation on $[a,b]$, then the remainder in (4.3) satisfies the estimate*

$$|R_n(f,u;I_n)|\le\max_{i=0,\dots,n-1}\left\{\bigvee_{x_i}^{x_{i+1}}(f)\right\}\bigvee_a^b(u) \tag{4.11}$$

(*ii*) *If $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type with $r\in(0,1]$, $H>0$, then we have the estimate*

$$|R_n(f,u;I_n)|\le\frac{H}{r+1}[\nu_{I_n}(h)]^r\bigvee_a^b(u). \tag{4.12}$$

(iii) If $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then

$$|R_n(f,u;I_n)| \le \begin{cases} \frac{1}{2}\nu_{I_n}(h)\|f'\|_{\infty,[a,b]} \bigvee_a^b(u), & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}[\nu_{I_n}(h)]^{\frac{1}{q}}\|f'\|_{p,[a,b]}\bigvee_a^b(u) & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max_{i=0,\dots,n-1}\left\{\int_{x_i}^{x_{i+1}}|f'(t)|\,dt\right\}\bigvee_a^b(u). \end{cases} \tag{4.13}$$

5. Approximating Fourier Sine and Cosine Transforms

For a function $f : [0,\infty) \to \mathbb{R}$ and $0 \le a < b < \infty$, consider the *Fourier Sine* and *Cosine transforms* on the finite interval $[a,b]$:

$$F_S(s) := \int_a^b f(x)\sin(sx)\,dx, \qquad s \in [0,\infty), \tag{5.1}$$

$$F_C(s) := \int_a^b f(x)\cos(sx)\,dx, \qquad s \in [0,\infty). \tag{5.2}$$

To point out the dependence on the interval, if necessary, we may write

$$F_S(s;a,b) := F_S(s) \quad \text{and} \quad F_C(s;a,b) := F_C(s).$$

We also need the following trigonometric means, for $p,q \in \mathbb{R}$,

$$SIN(p,q) := \begin{cases} \frac{\sin p - \sin q}{p-q}, & \text{if } p \ne q; \\ \cos q, & \text{if } p = q \end{cases} \tag{5.3}$$

and

$$COS(p,q) := \begin{cases} \frac{\cos p - \cos q}{p-q}, & \text{if } p \ne q; \\ -\sin q, & \text{if } p = q. \end{cases} \tag{5.4}$$

For $s \ne 0$, observe that

$$F_S(s;a,b) = -\frac{1}{s}\int_a^b f(x)\,d(\cos(sx)),$$

and

$$F_C(s;a,b) = \frac{1}{s}\int_a^b f(x)\,d(\sin(sx)),$$

and thus (5.1) and (5.2) may be viewed as Stieltjes integrals with continuous integrators $u(x) := \cos(sx)$ and $u(x) = \sin(sx)$, respectively. Here $x \in [a,b]$ and $s > 0$.

If we consider the division (see (4.1))

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

then the quadrature formula (4.2) may be written for these particular choices as

$$A_{S_n}(f,I_n,s) := -\frac{1}{s}\sum_{i=0}^{n-1}\frac{\cos(sx_{i+1})-\cos(sx_i)}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(t)\,dt = -\sum_{i=0}^{n-1} COS(sx_{i+1},sx_i)\int_{x_i}^{x_{i+1}} f(t)\,dt \tag{5.5}$$

and

$$A_{C_n}(f,I_n,s) := \frac{1}{s}\sum_{i=0}^{n-1}\frac{\sin(sx_{i+1})-\sin(sx_i)}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(t)\,dt = \sum_{i=0}^{n-1} SIN(sx_{i+1},sx_i)\int_{x_i}^{x_{i+1}} f(t)\,dt. \tag{5.6}$$

The following proposition holds.

Proposition 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function on $[0,\infty)$ and $L-$Lipschitzian on $[a,b]$. Then for any I_n a division of the interval $[a,b]$, we have*

$$F_S(s;a,b) = A_{S_n}(f,I_n,s) + R_{S_n}(f,I_n,s),\quad s>0 \tag{5.7}$$

and

$$F_C(s;a,b) = A_{C_n}(f,I_n,s) + R_{C_n}(f,I_n,s),\quad s>0, \tag{5.8}$$

where $A_{S_n}(f,I_n,s)$ and $A_{C_n}(f,I_n,s)$ are the quadrature rules provided in (5.5) and (5.6). The remainders $R_{S_n}(f,I_n,s)$ and $R_{C_n}(f,I_n,s)$ satisfy the estimates

$$|R_{S_n}(f,I_n,s)| \le \frac{1}{2}L\nu_{I_n}(h)\int_a^b|\sin(sx)|\,dx \le \frac{1}{4}sL\nu_{I_n}(h)\left(b^2-a^2\right) \tag{5.9}$$

and

$$|R_{C_n}(f,I_n,s)| \le \frac{1}{2}L\nu_{I_n}(h)\int_a^b|\cos(sx)|\,dx \le \frac{1}{2}L\nu_{I_n}(h)(b-a), \tag{5.10}$$

where $\nu_{I_n}(h) := \max\{h_i|i=0,\dots,n-1\}$ and $h_i := x_{i+1}-x_i$, $i=0,\dots,n-1$.

Proof. We use Theorem 5, for which

$$\bigvee_a^b(u) = s\int_a^b|\sin(sx)|\,dx \le s\int_a^b|sx|\,dx = s^2\cdot\frac{b^2-a^2}{2},$$

and

$$\bigvee_a^b(u) = s\int_a^b|\cos(sx)|\,dx \le s(b-a)$$

respectively.

We omit the details. ∎

Consider now $u:[a,b]\subset(0,\infty)\to\mathbb{R}$, $u(x)=\cos(sx)$, $s>0$. Obviously

$$|u'(x)| = s|\sin(sx)| \le s^2|x|$$

giving

$$\|u'\|_{\infty,[a,b]} = \sup_{x\in[a,b]}|u'(x)| \le s^2(b-a).$$

Consequently, for a given s, u as defined above, is Lipschitzian on $[a,b]$ with the constant $L = s^2(b-a)$.

If $u : [a,b] \subset (0,\infty) \to \mathbb{R}$, $u(x) = \sin(sx)$, $s > 0$, then

$$|u'(x)| = s|\cos(sx)| \leq s$$

giving

$$\|u'\|_{\infty,[a,b]} = \sup_{x\in[a,b]} |u'(x)| \leq s.$$

Using Theorem 8, we may state the following result in approximating the Sine and Cosine transforms.

Proposition 2. *Let I_n be a division of the interval $[a,b]$.*

(*i*) *If $f : [a,b] \to \mathbb{R}$ is of bounded variation, then we have (5.7) and (5.8). The remainders $R_{S_n}(f, I_n, s)$ and $R_{C_n}(f, I_n, s)$ satisfy the estimates:*

$$|R_{S_n}(f, I_n, s)| \leq \frac{3}{4} s(b-a)\nu_{I_n}(h) \bigvee_a^b (f) \tag{5.11}$$

and

$$|R_{C_n}(f, I_n, s)| \leq \frac{3}{4} \nu_{I_n}(h) \bigvee_a^b (f). \tag{5.12}$$

(*ii*) *If $f : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type, then the remainders satisfy the bounds*

$$\begin{aligned} |R_{S_n}(f, I_n, s)| &\leq \frac{2Hs(b-a)}{(r+1)(r+2)} \sum_{i=0}^{n-1} h_i^{r+1} \\ &\leq \frac{2Hs(b-a)^2}{(r+1)(r+2)} [\nu_{I_n}(h)]^r \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} |R_{C_n}(f, I_n, s)| &\leq \frac{2H}{(r+1)(r+2)} \sum_{i=0}^{n-1} h_i^{r+1} \\ &\leq \frac{2H(b-a)}{(r+1)(r+2)} [\nu_{I_n}(h)]^r. \end{aligned} \tag{5.14}$$

(*iii*) *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then*

$$|R_{S_n}(f, I_n, s)| \tag{5.15}$$
$$\leq \begin{cases} \frac{1}{3} s(b-a)\|f'\|_{\infty,[a,b]} \sum_{i=0}^{n-1} h_i^2, & \text{if } f' \in L_\infty[a,b]; \\ \frac{2^{\frac{1}{q}} s(b-a)\|f'\|_{p,[a,b]}}{(q+1)^{\frac{1}{q}}(q+2)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}}, & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4} s(b-a)\nu_{I_n}(h)\|f'\|_{1,[a,b]}, & \end{cases}$$

and

$$(5.16)\quad |R_{C_n}(f,I_n,s)| \le \begin{cases} \dfrac{1}{3}\|f'\|_{\infty,[a,b]}\sum\limits_{i=0}^{n-1}h_i^2, & \text{if } f'\in L_\infty[a,b]; \\ \dfrac{2^{\frac{1}{q}}\|f'\|_{p,[a,b]}}{(q+1)^{\frac{1}{q}}(q+2)^{\frac{1}{q}}}\left(\sum\limits_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}}, & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4}\nu_{I_n}(h)\|f'\|_{1,[a,b]}. \end{cases}$$

Similar bounds may be obtained from Theorem 9, but we omit the details.

6. An Application for Electrical Circuits

Consider the electrical oscillation in a circuit containing a resistance R, an inductance L, a condenser of capacity C, and a source of electromotive force $E_0P(t)$, where E_0 is a constant and $P(t)$ is a known function of the time t.

If the charge on the plates of the condenser is q, then the potential difference across the plates is $\frac{q}{c}$. Similarly, if i is the current flowing through the resistance and the inductance, the differences of potential between their ends are Ri and $L\left(\frac{di}{dt}\right)$, respectively. By the equation of continuity

$$i=\frac{dq}{dt} \tag{6.1}$$

so that these potential differences may be written as $R\frac{dq}{dt}$ and $L\frac{d^2q}{dt^2}$ respectively. Thus we obtain the ordinary differential equation [8, p. 93]

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E_0P(t) \tag{6.2}$$

for the determination of the charge q which accumulates on the plates of the condenser.

If we assume that initially this charge is Q and that a current I is flowing in the circuit, then we obtain the initial conditions

$$\begin{cases} q(0)=Q, \\ \dfrac{dq(0)}{dt}=I. \end{cases} \tag{6.3}$$

It is well known that if the resistance of the circuit is zero, i.e., $R=0$, then the solution of (1.2) with the initial conditions (6.3) is given by (see for example [8, p. 95])

$$q(t)=Q\cos(\omega t)+\frac{I}{\omega}\sin(\omega t)+\frac{E_0}{\omega L}\int_0^t P(s)\sin[\omega(t-s)]\,ds, \tag{6.4}$$

where $\omega^2=\frac{1}{LC}$.

Consequently, there is a practical need in computing the following quasi Fourier Sine Transform:

$$A(0,t;P,\omega,t):=\int_0^t P(s)\sin[\omega(t-s)]\,ds,$$

which may be a difficult task if P is complicated enough. In this case, a numerical approach would satisfy the user if the accuracy is good.

In [6], by the use of pre-Grüss type inequalities, some numerical approximations for $A(0,t;P,\omega,t)$ were given. Here we show how the transforms F_S and F_C defined in Section 5 may be used to produce different approximations.

Since

$$\sin[\omega(t-s)] = \sin(\omega t)\cos(\omega s) - \cos(\omega t)\sin(\omega s)$$

we have

$$\begin{aligned} A(0,t;P,\omega,t) &= \sin(\omega t)\int_0^t P(s)\cos(\omega s)\,ds - \cos(\omega t)\int_0^t P(s)\sin(\omega s)\,ds \\ &= \sin(\omega t)\,F_C(P)(\omega;0,t) - \cos(\omega t)\,F_S(P)(\omega;0,t) \end{aligned}$$

where, as in Section 5,

$$F_S(f)(s;a,b) := F_S(s;a,b) = F_S(s) := \int_a^b f(x)\sin(sx)\,dx,$$

is the Fourier Sine transform and

$$F_C(f)(s;a,b) := F_C(s;a,b) = F_C(s) := \int_a^b f(x)\cos(sx)\,dx$$

is the Cosine transform.

Consequently, by the use of approximations provided in Section 5, we may produce the corresponding results for $A(0,t;P,\omega,t)$. We omit the details.

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