EVALUATIONS OF THE IMPROPER INTEGRALS $\int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}}\cos(bx)\,\mathrm{d}x$ AND $\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}}\cos(bx)\,\mathrm{d}x$

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Abstract. In this article, using L'Hospital rule, mathematical induction, the trigonometric power formulae and integration by parts, some integral formulae for improper integrals $\int_0^\infty \frac{\sin^{2m}(\alpha x)}{x^{2n}}\cos(bx)\,\mathrm{d}x$ and $\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}}\cos(bx)\,\mathrm{d}x$ are established, where $m \ge n$ are all positive integers and real numbers $\alpha \ne 0$ and $b \ge 0$.

1. Introduction

The following improper integral is well-known and is synonymous with names of Laplace and Dirichlet

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}. \tag{1}$$

In fact, in 1781, it was first obtained using the residue method by Euler. It can be found in standard textbooks for undergraduate students, for examples, [9, pp. 226–227] and [13, pp. 168–170].

Depending on the partial fraction decomposition

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{t-n\pi} + \frac{1}{t+n\pi}\right), \tag{2}$$

an elegant calculation of formula (1) is provided in [5, pp. 436–437] and [6, pp. 382–384], due to the noted geometrician N. I. Lobatscheuski. Another polished proof of identity (1) is given in [6, pp. 381–382].

As exercises in [10, p. 53, p. 147 and p. 335] and [12, p. 495], using the Laplace transform, the Parseval identities of sine and cosine Fourier transforms and the residue theorem, the following formulae are requested to compute:

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^4 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^3 t}{t^3}\,\mathrm{d}t = \frac{3\pi}{8}. \tag{3}$$

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In [14, pp. 74–75 and p. 84], using the Mellin transform and by approaches in theory of Fourier analysis or theory of residues, the following formulae are obtained:

$$\int_0^\infty \cos(tx)x^z \frac{\mathrm{d}x}{x} = \Gamma(z)t^{-z}\cos\frac{\pi z}{2}, \quad \operatorname{Re}(z)>0, \quad t>0; \tag{4}$$

$$\int_0^\infty \sin(tx)x^z \frac{\mathrm{d}x}{x} = \Gamma(z)t^{-z}\sin\frac{\pi z}{2}, \quad \operatorname{Re}(z)>-1, \quad t>0; \tag{5}$$

$$\int_0^\infty \frac{\sin x}{x^z}\,\mathrm{d}x = \Gamma(1-z)\cos\frac{\pi z}{2} = \frac{\pi}{2\Gamma(z)\sin\frac{\pi z}{2}}. \tag{6}$$

Especially, taking $t=1$ and $z\to 0$ in (5) or taking $z=1$ in (6) produces (1).

The following generalisation of formula (1) can be found in [2] and [3, p. 458, No. 3.836.5]:

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx)\,\mathrm{d}x = n(2^{n-1}n!)^{-1}\sum_{k=0}^{[r]}(-1)^k\binom{n}{k}(n-b-2k)^{n-1}, \tag{7}$$

where $0\le b<n$, $n\ge 1$, $r=\frac{n-b}{2}$, and $[r]$ is the largest integer contained in r.

In [11], some general results related to formulae (1) and (7) were obtained.

In [1] and [7, p. 606], the following inequality is given:

$$\int_{-\infty}^\infty \left|\frac{\sin t}{t}\right|^p \mathrm{d}t \le \pi\sqrt{\frac{2}{p}}, \quad p\ge 2; \tag{8}$$

Equality is valid only if $p=2$.

The integral (1) and other integral formulae stated above are useful and arising in research of damping vibration and other science or engineering. This was mentioned in [13, p. 170].

Recently, Q.-M. Luo and B.-N. Guo in [8] obtained the following

Theorem A ([8])**.** *For a nonnegative integer $k\ge 0$ and $\alpha\ne 0$, we have*

$$\int_0^\infty \left(\frac{\sin(\alpha x)}{x}\right)^{2k+1}\mathrm{d}x = \frac{\operatorname{sgn}\alpha\sum_{i=0}^{k}(-1)^i(2k-2i+1)^{2k}C_{2k+1}^i}{4^k(2k)!}\cdot\alpha^{2k}\cdot\frac{\pi}{2}, \tag{9}$$

$$\int_0^\infty \left(\frac{\sin(\alpha x)}{x}\right)^{2k}\mathrm{d}x = \frac{\operatorname{sgn}\alpha\sum_{i=0}^{k-1}(-1)^i(k-i)^{2k-1}C_{2k}^i}{(2k-1)!}\cdot\alpha^{2k-1}\cdot\frac{\pi}{2}. \tag{10}$$

If taking $k=0$ in (9), the formula (1) follows.

In this article, using the L'Hospital rule, mathematical induction, trigonometric power formulae and integration by parts, we will establish integral formulae of the improper integrals $\int_0^\infty\frac{\sin^{2m}(\alpha x)}{x^{2n}}\cos(bx)\,\mathrm{d}x$ and $\int_0^\infty\frac{\sin^{2m+1}(\alpha x)}{x^{2n+1}}\cos(bx)\,\mathrm{d}x$, where $m\ge n$ are all positive integers and real numbers $\alpha\ne 0$ and $b\ge 0$. The following theorem holds.

Theorem 1. *Let m,n be nonnegatine integer , $m\ge n$, and $b\ge 0$. Then*

$$\int_0^\infty \frac{\sin^r x}{x^s}\cos(bx)\,\mathrm{d}x =$$

$$\begin{cases} \dfrac{(-1)^{m+n}\sum_{i=0}^{m}(-1)^i u(m,n,i,b)C_{2m+1}^i}{2^{2m+1}(2n)!}\cdot\dfrac{\pi}{2} \\ \qquad \textit{for } r=2m+1, s=2n+1, \\ \dfrac{(-1)^{m+n}\sum_{i=0}^{m-1}(-1)^i v(m,n,i,b)C_{2m}^i+(-1)^n C_{2m}^m b^{2n-1}}{2^{2m}(2n-1)!}\cdot\dfrac{\pi}{2} \\ \qquad \textit{for } r=2m,\ s=2n, \end{cases} \tag{11}$$

where

$$\begin{aligned} u(m,n,i,b) &= (2m-2i+b+1)^{2n} \\ &\quad + (2m-2i-b+1)^{2n}\operatorname{sgn}(2m-2i-b+1), \end{aligned} \tag{12}$$

$$\begin{aligned} v(m,n,i,b) &= (2m-2i+b)^{2n-1} \\ &\quad + (2m-2i-b)^{2n-1}\operatorname{sgn}(2m-2i-b). \end{aligned} \tag{13}$$

Theorem 2. *Let m,n be nonnegatine integer , $m \ge n$, and real numbers $\alpha \neq 0$ and $b \ge 0$. Then*

$$\int_0^\infty \frac{\sin^r(\alpha x)}{x^s}\cos(bx)\,\mathrm{d}x =$$

$$\begin{cases} \dfrac{(-1)^{m+n}\sum_{i=0}^{m}(-1)^i C_{2m+1}^i u(m,n,i,b,\alpha)}{2^{2m+1}(2n)!}\cdot\dfrac{\pi}{2}\operatorname{sgn}\alpha \\ \qquad \textit{if } r=2m+1, s=2n+1, \\ \dfrac{(-1)^{m+n}\sum_{i=0}^{m-1}(-1)^i C_{2m}^i v(m,n,i,b,\alpha)+(-1)^n C_{2m}^m b^{2n-1}}{2^{2m}(2n-1)!}\cdot\dfrac{\pi}{2}\operatorname{sgn}\alpha \\ \qquad \textit{if } r=2m, s=2n, \end{cases} \tag{14}$$

where

$$\begin{aligned} u(m,n,i,b,\alpha) &= (2m\alpha-2i\alpha+b+\alpha)^{2n} \\ &\quad + (2m\alpha-2i\alpha-b+\alpha)^{2n}\operatorname{sgn}(2m-2i-\frac{b}{\alpha}+1), \end{aligned} \tag{15}$$

$$\begin{aligned} v(m,n,i,b,\alpha) &= (2m\alpha-2i\alpha+b)^{2n-1} \\ &\quad + (2m\alpha-2i\alpha-b)^{2n-1}\operatorname{sgn}(2m-2i-\frac{b}{\alpha}). \end{aligned} \tag{16}$$

As direct consequences of Theorem 1 and Theorem 2, the following integral formulae hold.

Corollary 1. *Let m,n be nonnegatine integer , $m \ge n$, and $\alpha \neq 0$. Then*

$$\int_0^\infty \frac{\sin^r(\alpha x)}{x^s}\,\mathrm{d}x =$$

$$\begin{cases} \dfrac{(-1)^{m+n}\operatorname{sgn}\alpha\sum_{i=0}^{m}(-1)^i(2m-2i+1)^{2n}C_{2m+1}^i}{4^m(2n)!}\cdot\alpha^{2n}\cdot\dfrac{\pi}{2} \\ \qquad \textit{if } r=2m+1, s=2n+1, \\ \dfrac{(-1)^{m+n}\operatorname{sgn}\alpha\sum_{i=0}^{m-1}(-1)^i(m-i)^{2n-1}C_{2m}^i}{4^{m-n}(2n-1)!}\cdot\alpha^{2n-1}\cdot\dfrac{\pi}{2} \\ \qquad \textit{if } r=2m, s=2n. \end{cases} \tag{17}$$

Corollary 2. *For nonnegative integer m and n, we have*

$$\int_0^\infty \frac{\sin^r x}{x^s}\,\mathrm{d}x = \begin{cases} \dfrac{(-1)^{m+n}\sum_{i=0}^{m}(-1)^i(2m-2i+1)^{2n}C_{2m+1}^i}{4^m(2n)!}\cdot\dfrac{\pi}{2} \\ \qquad \text{if } r=2m+1, s=2n+1, \\ \dfrac{(-1)^{m+n}\sum_{i=0}^{m-1}(-1)^i(m-i)^{2n-1}C_{2m}^i}{4^{m-n}(2n-1)!}\cdot\dfrac{\pi}{2} \\ \qquad \text{if } r=2m, s=2n. \end{cases} \tag{18}$$

Corollary 3. *Let m be a nonnegative integer, $\alpha\in\mathbb{R}$, then we have*

$$\int_0^\infty \frac{\sin^{2m+1}(\alpha x)}{x}\,\mathrm{d}x = \operatorname{sgn}\alpha\cdot\frac{(2m)!}{4^m(m!)^2}\cdot\frac{\pi}{2}. \tag{19}$$

2. Lemmae

The following trigonometric power formulae are the basis and key of our proof for Theorem 1.

Lemma 1 ([4, p. 41 and p. 280] and [15])**.** *For $\alpha\in\mathbb{R}$ and $k\in\mathbb{N}$, we have*

$$\int_0^\infty \frac{\sin(\alpha x)}{x}\,\mathrm{d}x = \frac{\pi}{2}\operatorname{sgn}(\alpha), \tag{20}$$

$$\sin^{2k+1} x = \frac{1}{2^{2k}}\sum_{i=0}^{k}(-1)^{k+i}C_{2k+1}^i\sin[(2k-2i+1)x], \tag{21}$$

$$\sin^{2k} x = \frac{1}{2^{2k-1}}\left[\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^i\cos[2(k-i)x]+\frac{1}{2}C_{2k}^k\right], \tag{22}$$

where $C_n^k=\frac{n!}{(n-k)!k!}$.

The following three combinatorial identities can be regarded as by-products, enabling us to employ the L'Hospital rule in the proof of Theorem 1. They can also be found in [8].

Lemma 2. *For $1\le m\le k$, $k\in\mathbb{N}$ and real number $b\ge 0$, we have*

$$\sum_{i=0}^{k}(-1)^iC_{2k+1}^i\left[(2k-2i+b+1)^{2m-1}+(2k-2i-b+1)^{2m-1}\right]=0, \tag{23}$$

$$\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^i+\frac{1}{2}C_{2k}^k=0. \tag{24}$$

For $1\le\ell\le k-1$, $2\le k\in\mathbb{N}$ and real number $b\ge 0$, we have

$$\sum_{i=0}^{k-1}(-1)^iC_{2k}^i\left[(2k-2i+b)^{2\ell}+(2k-2i-b)^{2\ell}\right]+C_{2k}^kb^{2\ell}=0. \tag{25}$$

Proof. By the trigonometric power formula (21), it is easy to see that

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sum_{i=0}^{k}(-1)^{k+i}C_{2k+1}^{i}\big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\big]}{x^{2k}}\\
&=2\lim_{x\to 0}\frac{\sum_{i=0}^{k}(-1)^{k+i}C_{2k+1}^{i}\sin[(2k-2i+1)x]\cos(bx)}{x^{2k}} \qquad (26)\\
&=2^{2k+1}\lim_{x\to 0}\frac{\sin^{2k+1}x}{x^{2k}}\cos(bx)=0,
\end{aligned}$$

this means that the function $\sum_{i=0}^{k}(-1)^{k+i}C_{2k+1}^{i}\big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\big]$ tends to zero at higher speed than x^{2k} as $x\to 0$, that is

$$\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\big]=o(x^{2k}) \text{ as } x\to 0,$$

then, for $0\le j\le 2k$, by L'Hospital rule, from (26), it follows that

$$\lim_{x\to 0}\frac{\Big(\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\big]\Big)^{(j)}}{x^{2k-j}}=0,$$

which is equivalent to

$$\left(\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\big]\right)^{(j)}=o(x^{2k-j})$$

as $x\to 0$. Therefore, for any natural number $1\le m\le k$, we have

$$\begin{aligned}
0&=\lim_{x\to 0}\left(\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\Big[\sin[(2k-2i+b+1)x]+\sin[(2k-2i-b+1)x]\Big]\right)^{(2m-1)}\\
&=\lim_{x\to 0}\Bigg((-1)^{m-1}\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\Big[(2k-2i+b+1)^{2m-1}\cos[(2k-2i+b+1)x]\\
&\quad+(2k-2i-b+1)^{2m-1}\cos[(2k-2i-b+1)x]\Big]\Bigg)\\
&=(-1)^{m-1}\sum_{i=0}^{k}(-1)^{i}C_{2k+1}^{i}\big[(2k-2i+b+1)^{2m-1}+(2k-2i-b+1)^{2m-1}\big].
\end{aligned} \qquad (27)$$

Identity (23) follows.

By the trigonometric power formula (22), it is not difficult to obtain

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\Big[\cos[(2k-2i+b)x]+\cos[(2k-2i-b)x]\Big]+C_{2k}^{k}\cos(bx)}{x^{2k-1}}\\
&=2\lim_{x\to 0}\frac{\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\cos[2(k-i)x]\cos(bx)+\frac{1}{2}C_{2k}^{k}\cos(bx)}{x^{2k-1}}\\
&=2^{2k}\lim_{x\to 0}\frac{\sin^{2k}x}{x^{2k-1}}\cos(bx)=0,
\end{aligned}$$

hence

$$\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\Big[\cos[(2k-2i+b)x]+\cos[(2k-2i-b)x]\Big]+C_{2k}^{k}\cos(bx)=o(x^{2k-1})$$

as $x\to0$. Consequently

$$\begin{aligned}0&=\lim_{x\to0}\left\{\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\cos[2(k-i)x]\cos(bx)+\frac{1}{2}C_{2k}^{k}\cos(bx)\right\}\\&=\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}+\frac{1}{2}C_{2k}^{k}\end{aligned}$$

and, for $1\le j\le 2k-1$,

$$\begin{aligned}&\left(\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\Big[\cos[(2k-2i+b)x]+\cos[(2k-2i-b)x]\Big]+C_{2k}^{k}\cos(bx)\right)^{(j)}\\&=o(x^{2k-j-1})\text{ as }x\to0,\end{aligned}$$

then, for any positive integer ℓ such that $1\le\ell\le k-1$, we have

$$\begin{aligned}0&=\lim_{x\to0}\left[\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\Big[\cos[(2k-2i+b)x]+\cos[(2k-2i-b)x]\Big]+C_{2k}^{k}\cos(bx)\right]^{(2\ell)}\\&=(-1)^{\ell}\lim_{x\to0}\left(\sum_{i=0}^{k-1}(-1)^{k+i}C_{2k}^{i}\Big[(2k-2i+b)^{2\ell}\cos[(2k-2i+b)x]\right.\\&\quad\left.+(2k-2i-b)^{2\ell}\cos[(2k-2i-b)x]\Big]+C_{2k}^{k}b^{2\ell}\cos(bx)\right)\\&=(-1)^{k+\ell}\left[\sum_{i=0}^{k-1}(-1)^{i}C_{2k}^{i}\Big[(2k-2i+b)^{2\ell}+(2k-2i-b)^{2\ell}\Big]+C_{2k}^{k}b^{2\ell}\right].\end{aligned}$$

Identities (24) and (25) follow. The proof is complete. □

3. Proofs of theorems

Proof of Theorem 1. From Lemma 1 and formula (23) in Lemma 2, using the L'Hospital rule and integration by parts yields

$$\begin{aligned}&\int_0^\infty\frac{\sin^{2m+1}x}{x^{2n+1}}\cos(bx)\,\mathrm{d}x=\frac{1}{2^{2m}}\int_0^\infty\frac{\sum_{i=0}^{m}(-1)^{m+i}C_{2m+1}^{i}\sin[(2m-2i+1)x]\cos(bx)}{x^{2n+1}}\,\mathrm{d}x\\&=\frac{1}{2^{2m+1}}\int_0^\infty\frac{\sum_{i=0}^{m}(-1)^{m+i}C_{2m+1}^{i}\Big[\sin[(2m-2i+b+1)x]+\sin[(2m-2i-b+1)x]\Big]}{x^{2n+1}}\,\mathrm{d}x\\&=\frac{(-1)^{2j-1}(2n-j)!}{2^{2m+1}(2n)!}\\&\quad\times\left\{\Bigg[\sum_{i=0}^{m}(-1)^{m+i}C_{2m+1}^{i}\Big[(2m-2i+b+1)^{j-1}\sin\Big[(2m-2i+b+1)x+\frac{(j-1)\pi}{2}\Big]\right.\\&\quad+(2m-2i-b+1)^{j-1}\sin\Big[(2m-2i-b+1)x+\frac{(j-1)\pi}{2}\Big]\Big]\Bigg]\frac{1}{x^{2n-j+1}}\Bigg|_0^\infty\end{aligned}$$

$$
\begin{aligned}
&- \int_0^\infty \Bigg[\sum_{i=0}^{m}(-1)^{m+i}C_{2m+1}^{i}\Big[(2m-2i+b+1)^j \sin\big[(2m-2i+b+1)x+\frac{j\pi}{2}\big]\\
&+ (2m-2i-b+1)^j \sin\big[(2m-2i-b+1)x+\frac{j\pi}{2}\big]\Big]\Bigg]\frac{1}{x^{2n-j+1}}\,\mathrm{d}x\Bigg\}\\
&= \frac{(-1)^n}{2^{2m+1}(2n)!}\int_0^\infty \Bigg[\sum_{i=0}^{m}(-1)^{m+i}C_{2m+1}^{i}\Big[(2m-2i+b+1)^{2n}\sin\big[(2m-2i+b+1)x\big]\\
&+ (2m-2i-b+1)^{2n}\sin\big[(2m-2i-b+1)x\big]\Big]\Bigg]\frac{1}{x}\,\mathrm{d}x\\
&= \frac{(-1)^{m+n}}{2^{2m+1}(2n)!}\sum_{i=0}^{m}(-1)^{i}C_{2m+1}^{i}\Bigg\{(2m-2i+b+1)^{2n}\int_0^\infty \frac{\sin[(2m-2i+b+1)x]}{x}\,\mathrm{d}x\\
&+ (2m-2i-b+1)^{2n}\int_0^\infty \frac{\sin[(2m-2i-b+1)x]}{x}\,\mathrm{d}x\Bigg\}\\
&= \frac{(-1)^{m+n}\sum_{i=0}^{m}(-1)^{i}C_{2m+1}^{i}u(m,n,i,b)}{2^{2m+1}(2n)!}\cdot\frac{\pi}{2},
\end{aligned}
$$

where $u(m,n,i,b) = (2m-2i+b+1)^{2n} + (2m-2i-b+1)^{2n}\operatorname{sgn}(2m-2i-b+1)$ and $1 \le j \le 2n$.

By formula (22), using the L'Hospital rule, from Lemma 2, integration by parts gives us

$$
\begin{aligned}
&\int_0^\infty \frac{\sin^{2m}x}{x^{2n}}\cos(bx)\,\mathrm{d}x\\
&= \frac{1}{2^{2m-1}}\int_0^\infty \frac{\sum_{i=0}^{m-1}(-1)^{m+i}C_{2m}^{i}\cos[2(m-i)x]\cos(bx)+\frac{1}{2}C_{2m}^{m}\cos(bx)}{x^{2n}}\,\mathrm{d}x\\
&= \frac{1}{2^{2m}}\int_0^\infty \frac{\sum_{i=0}^{m-1}(-1)^{m+i}C_{2m}^{i}\Big[\cos[(2m-2i+b)x]+\cos[(2m-2i-b)x]\Big]+C_{2m}^{m}\cos(bx)}{x^{2n}}\,\mathrm{d}x\\
&= -\frac{1}{2^{2m}}\cdot\frac{1}{2n-1}\int_0^\infty \Bigg[\sum_{i=0}^{m-1}(-1)^{m+i}C_{2m}^{i}\Big[\cos[(2m-2i+b)x]+\cos[(2m-2i-b)x]\Big]\\
&+ C_{2m}^{m}\cos(bx)\Bigg]\,\mathrm{d}\Big(\frac{1}{x^{2n-1}}\Big)\\
&= -\frac{1}{(2n-1)\cdot 2^{2m}}\Bigg\{\Bigg[\sum_{i=0}^{m-1}(-1)^{m+i}C_{2m}^{i}\Big[\cos[(2m-2i+b)x]\\
&+ \cos[(2m-2i-b)x]\Big]+C_{2m}^{m}\cos(bx)\Bigg]\frac{1}{x^{2n-1}}\Bigg|_0^\infty\\
&+ \int_0^\infty \Big\{\sum_{i=0}^{m-1}(-1)^{m+i}C_{2m}^{i}\Big[(2m-2i+b)\sin[(2m-2i+b)x]\\
&+ (2m-2i-b)\sin[(2m-2i-b)x]\Big]+C_{2m}^{m}b\sin(bx)\Big\}\frac{1}{x^{2n-1}}\,\mathrm{d}x\Bigg\}
\end{aligned}
$$

(by integration by part)

$$
\begin{aligned}
&= -\frac{1}{(2n-1)\cdot 2^{2m}} \int_0^\infty \bigg\{ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \Big[(2m-2i+b)\sin[(2m-2i+b)x] \\
&\quad + (2m-2i-b)\sin[(2m-2i-b)x] \Big] + C_{2m}^m b\sin(bx) \bigg\} \frac{1}{x^{2n-1}}\,\mathrm{d}x
\end{aligned}
$$

(by Lemma 2)

$$
\begin{aligned}
&= \frac{(2n-j-2)!}{2^{2m}(2n-1)!} \Bigg\{ \Bigg[\sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \bigg[(2m-2i+b)^j \sin\Big[(2m-2i+b)x + \frac{(j-1)\pi}{2}\Big] \\
&\quad + (2m-2i-b)^j \sin\Big[(2m-2i-b)x + \frac{(j-1)\pi}{2}\Big]\bigg] + C_{2m}^m b^j \sin\Big[bx + \frac{(j-1)\pi}{2}\Big]\Bigg] \frac{1}{x^{2n-j-1}}\bigg|_0^\infty \\
&\quad - \int_0^\infty \bigg\{ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \bigg[(2m-2i+b)^{j+1} \sin\Big[(2m-2i+b)x + \frac{j\pi}{2}\Big] \\
&+ (2m-2i-b)^{j+1} \sin\Big[(2m-2i-b)x + \frac{j\pi}{2}\Big]\bigg] + C_{2m}^m b^{j+1} \sin\Big[bx + \frac{j\pi}{2}\Big]\bigg\} \frac{1}{x^{2n-j-1}}\,\mathrm{d}x \Bigg\}
\end{aligned}
$$

(by integration by part)

$$
\begin{aligned}
&= \frac{(-1)^n}{2^{2m}(2n-1)!} \int_0^\infty \Bigg\{ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \bigg[(2m-2i+b)^{2n-1} \sin[(2m-2i+b)x] \\
&+ (2m-2i-b)^{2n-1} \sin[(2m-2i-b)x]\bigg] + C_{2m}^m b^{2n-1} \sin(bx) \Bigg\} \frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

(by mathematical induction on $j \le 2n-2$)

$$
\begin{aligned}
&= \frac{(-1)^n}{2^{2m}(2n-1)!} \Bigg\{ \sum_{i=0}^{m-1} (-1)^{m+i} C_{2m}^i \bigg[(2m-2i+b)^{2n-1} \int_0^\infty \frac{\sin[(2m-2i+b)x]}{x}\,\mathrm{d}x \\
&\quad + (2m-2i-b)^{2n-1} \int_0^\infty \frac{\sin[(2m-2i-b)x]}{x}\,\mathrm{d}x \bigg] + C_{2m}^m b^{2n-1} \int_0^\infty \frac{\sin(bx)}{x}\,\mathrm{d}x \Bigg\} \\
&= \Bigg\{ (-1)^{m+n} \sum_{i=0}^{m-1} (-1)^i C_{2m}^i \Big[(2m-2i+b)^{2n-1} + (2m-2i-b)^{2n-1} \operatorname{sgn}(2m-2i-b) \Big] \\
&\quad + (-1)^n C_{2m}^m b^{2n-1} \Bigg\} \frac{1}{2^{2m}(2n-1)!} \cdot \frac{\pi}{2}.
\end{aligned}
$$

(by formula (20))

$$
= \frac{(-1)^{m+n} \sum_{i=0}^{m-1} (-1)^i C_{2m}^i v(m,n,i,b) + (-1)^n C_{2m}^m b^{2n-1}}{2^{2m}(2n-1)!} \cdot \frac{\pi}{2},
$$

where $v(m,n,i,b) = (2m-2i+b)^{2n-1} + (2m-2i-b)^{2n-1} \operatorname{sgn}(2m-2i-b)$.

The proof of Theorem 1 is thus complete. $\square$

Proof of Theorem 2. From standard argument, for $\alpha \neq 0$, by transformation $\alpha x = t$, we have

$$\int_0^\infty \frac{\sin^r(\alpha x)}{x^s} \cos(bx) \,\mathrm{d}x = \alpha^{s-1} \operatorname{sgn} \alpha \int_0^\infty \frac{\sin^r t}{t^s} \cos\left(\frac{b}{\alpha}\right) \mathrm{d}t. \tag{28}$$

From Theorem 1, Theorem 2 follows. □

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