ON SOME ANALOGUES OF KY FAN-TYPE INEQUALITIES

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Abstract. We study the behavior of means under equal increments of their variables and we apply the results to Ky Fan-type inequalities and certain bounds for the differences of means. We also give a sharpening of Sierpiński's inequality and prove a Rado-type inequality.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} \omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n} \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume $0 \leq x_1 \leq x_2 \cdots \leq x_n, m = \min\{x_i\}, M = \max\{x_i\}$. We denote $\sigma_n = \sum_{i=1}^{n} \omega_i (x_i - A_n)^2$.

To any given $\mathbf{x}, t \geq 0$ we associate $\mathbf{x}' = (1 - x_1, 1 - x_2, \cdots, 1 - x_n), \mathbf{x}_t = (x_1 + t, \cdots, x_n + t)$. When there is no risk of confusion, We shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, $P_{n,r,t}$ for $P_{n,r}(\mathbf{x}_t)$ and $P^{'}_{n,r}$ for $P_{n,r}(\mathbf{x}')$ if $1 - x_i \geq 0$ for all i. We also define $A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1}$ and similarly for $A_{n,t}, G_{n,t}, H_{n,t},\ A'_n, G^{'}_n, H^{'}_n$.

To simplify expressions, we define

$$\Delta_{r,s,t,\alpha} = \frac{P^{\alpha}_{n,r,t} - P^{\alpha}_{n,s,t}}{P^{\alpha}_{n,r} - P^{\alpha}_{n,s}}, \Delta'_{r,s} = \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} \tag{1.1}$$

with $\Delta_{r,s,t,0} = (\ln \frac{P_{n,r,t}}{P_{n,s,t}})/(\ln \frac{P_{n,r}}{P_{n,s}})$. We also write $\Delta_{r,s,t}$ for $\Delta_{r,s,t,1}$. In order to include the case of equality for various inequalities in our discussions, for any given inequality, we define $0/0$ to be the number which makes the inequality an equality.

Recently, the author([8],[9]) proved the following result:

Theorem 1.1. *For $r > s, m > 0, t \geq 0$, the following inequalities are equivalent:*

$$\frac{r-s}{2m}\sigma_n \geq \quad P_{n,r} - P_{n,s} \quad \geq \frac{r-s}{2M}\sigma_n \tag{1.2}$$

$$\frac{M}{1-M} \geq \quad \Delta'_{r,s} \quad \geq \frac{m}{1-m} \tag{1.3}$$

$$\frac{M}{t+m} \geq \quad \Delta_{r,s,t} \quad \geq \frac{m}{t+M} \tag{1.4}$$

where in (1.3) *we require $M < 1$.*

D.Cartwright and M.Field[6] first proved the validity of (1.2) for $r = 1, s = 0$. For other extensions and refinements of (1.2), see [3], [7],[11] and [12]. (1.3) is commonly referred as the additive Ky Fan's inequality. We refer the reader to the survey article[2] and the references therein for an account of Ky Fan's inequality.

J.Aczél and Zs. Pâles[1] proved $\Delta_{1,s,t} \leq 1$ for any $s \neq 1$. We can interpret their result as an assertion of the monotonicity of $A_{n,t} - P_{n,s,t}$ as a function of t. The asymptotic behavior of $t(P_{n,r,t} - A_{n,t})$ was studied by J.Brenner and B. Carlson[5] and in this paper, we will study the

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monotonicities of $(t+M)(P_{n,r,t}-P_{n,s,t})$ and $(t+m)(P_{n,r,t}-P_{n,s,t})$ as functions of t for $r=1$ or $s=1$ and then apply the result to inequalities of the type (1.2).

The following inequality connecting three classical means(with $\omega_i=1/n$ here) is due to P.F.Wang and W.L.Wang[15](right-hand side inequality), H. Alzer, S. Ruscheweyh and L. Salinas[4](left-hand side inequality):

$$(\frac{H_n}{H'_n})^{n-1}\frac{A_n}{A'_n}\leq(\frac{G_n}{G'_n})^n\leq(\frac{A_n}{A'_n})^{n-1}\frac{H_n}{H'_n} \tag{1.5}$$

(1.5) was refined in [8] and in section 5 we will give a further refinement of the above inequality. We will also prove a Rado-type inequality in the last section.

2. A Few Lemmas

Lemma 2.1. *Let $J(x)$ be the smallest closed interval that contains all of x_i and $f(x),g(x)\in C^2(J(x))$ be two twice differentiable functions, then*

$$\frac{\sum_{i=1}^n\omega_if(x_i)-f(\sum_{i=1}^n\omega_ix_i)}{\sum_{i=1}^n\omega_ig(x_i)-g(\sum_{i=1}^n\omega_ix_i)}=\frac{f''(\xi)}{g''(\xi)} \tag{2.1}$$

for some $\xi\in J(x)$, provided that the denominator of the left-hand side is nonzero.

Lemma 2.1 and the following consequence of it are due to A.M.Mercer[10]:

Lemma 2.2. *For $w>u,w\neq v,u\neq v,x_1>0$*

$$|\frac{u(u-v)}{w(w-v)}|\frac{1}{x_1^{w-u}}\geq|\frac{(P^u_{n,u}-P^u_{n,v})}{(P^w_{n,w}-P^w_{n,v})}|\geq|\frac{u(u-v)}{w(w-v)}|\frac{1}{x_n^{w-u}} \tag{2.2}$$

with equality holding if and only if $x_1=\cdots=x_n$.

Apply Lemma 2.1 to $f(x)=(t+x)^r,g(x)=x^r,r\neq0$ and $f(x)=\ln(t+x),g(x)=\ln x$ when $r=0$, we obtain

Corollary 2.1. *For $x_1>0$*

$$\min\{(\frac{t+x_n}{x_n})^{r-2},(\frac{t+x_1}{x_1})^{r-2}\}\leq\Delta_{r,1,t,r}\leq\max\{(\frac{t+x_n}{x_n})^{r-2},(\frac{t+x_1}{x_1})^{r-2}\} \tag{2.3}$$

We now give a generalization of the result of Aczél and Pâles:

Lemma 2.3. *Let $r>s,t\geq0,\alpha\leq1$.*

(i). For $s\neq1,\Delta_{1,s,t,\alpha}\leq1$.
(ii). If $\Delta_{r,s,t}\leq\frac{x_n}{t+x_n}$, then $\Delta_{r,s,t,\alpha}\leq(\frac{x_n}{t+x_n})^{2-\alpha}$.
(iii). If $\Delta_{r,s,t}\geq\frac{x_1}{t+x_1}$, then $\Delta_{r,s,t,\alpha}\geq(\frac{x_1}{t+x_1})^{2-\alpha}$.

Proof. We will prove (i) for $s<1,\alpha\neq0$, (ii) for $0<\alpha<1$ and the other proofs are similar. For (i), let $f(t)=A^\alpha_{n,t}-P^\alpha_{n,s,t}$, then

$$f'(t)=\alpha(A^{\alpha-1}_{n,t}-P^{\alpha-1}_{n,s,t}(\frac{P_{n,s,t}}{P_{n,s-1,t}})^{1-s})\begin{cases}\leq\alpha(A^{\alpha-1}_{n,t}-P^{\alpha-1}_{n,s,t}) & \leq0,0<\alpha\leq1\\ \geq\alpha P^{\alpha-1}_{n,s,t}(1-(\frac{P_{n,s,t}}{P_{n,s-1,t}})^{1-s}) & \geq0,\alpha<0\end{cases}$$

The conclusion then follows from the monotonicity of $f(t)$.

For (ii), let $f(t)=(t+x_n)^{2-\alpha}(P^\alpha_{n,r,t}-P^\alpha_{n,s,t})$, then it suffices to show $f'(0)\leq0$ or equivalently

$$(2-\alpha)(P^\alpha_{n,r}-P^\alpha_{n,s})\leq\alpha x_n(P^{\alpha-1}_{n,s}(\frac{P_{n,s}}{P_{n,s-1}})^{1-s}-P^{\alpha-1}_{n,r}(\frac{P_{n,r}}{P_{n,r-1}})^{1-r})$$

We also have

$$\frac{P^{1-\alpha}_{n,s}}{\alpha}(P^\alpha_{n,r}-P^\alpha_{n,s})\leq P_{n,r}-P_{n,s}\leq x_n((\frac{P_{n,s}}{P_{n,s-1}})^{1-s}-(\frac{P_{n,r}}{P_{n,r-1}})^{1-r}) \tag{2.4}$$

where the first inequality above follows from the mean value theorem and the second inequality follows from $\Delta_{r,s,t} \leq \frac{x_n}{t+x_n}$. Similarly, by using the mean value theorem, we get

$$\frac{P_{n,r}^{\alpha} - P_{n,s}^{\alpha}}{P_{n,s}^{\alpha-1} - P_{n,r}^{\alpha-1}} \leq \frac{\alpha}{1-\alpha} P_{n,r} \leq \frac{\alpha}{1-\alpha} x_n \left(\frac{P_{n,r}}{P_{n,r-1}}\right)^{1-r} \tag{2.5}$$

where the last inequality follows from $P_{n,r}^{r} = \sum_{i=1}^{n} \omega_i x_i^r \leq \sum_{i=1}^{n} \omega_i x_n x_i^{r-1} = x_n P_{n,r-1}^{r-1}$. Now (ii) follows by rewriting (2.4), (2.5) as

$$P_{n,r}^{\alpha} - P_{n,s}^{\alpha} \leq \alpha P_{n,s}^{\alpha-1} x_n \left(\left(\frac{P_{n,s}}{P_{n,s-1}}\right)^{1-s} - \left(\frac{P_{n,r}}{P_{n,r-1}}\right)^{1-r}\right) \tag{2.6}$$

$$(1-\alpha)(P_{n,r}^{\alpha} - P_{n,s}^{\alpha}) \leq \alpha x_n (P_{n,s}^{\alpha-1} - P_{n,r}^{\alpha-1}) \left(\frac{P_{n,r}}{P_{n,r-1}}\right)^{1-r} \tag{2.7}$$

and adding (2.6) and (2.7). □

The following two lemmas will be needed in section 5.

Lemma 2.4. *Let x, b, u, v, t be real numbers with $0 < x \leq b, u \geq 1, v \geq 1, t \geq 0$, then $f(u, v, x, b) \leq f(u, v, x+t, b+t)$ where*

$$f(u, v, x, b) = b^2 \left(\frac{u+v-1}{ux+vb} + \frac{1}{x^2(u/x + v/b)} - \frac{1}{x}\right)$$

with equality holding if and only if $x = b$ or $u = v = 1$ or $t = 0$.

Proof. Let $x < b, t > 0$ and $u > 1, v > 1$. Write $D(u, v, x, b, t) = f(u, v, x, b) - f(u, v, x+t, b+t)$,then

$$\begin{aligned} D(u, v, x, b, t) &= v(b-x)\Big[-\frac{(u-1)b/x + (v-1)}{(v+ux/b)(u+vx/b)} + \\ &\quad + \frac{(u-1)(b+t)/(x+t) + (v-1)}{(v+u(x+t)/(b+t))(u+v(x+t)/(b+t))}\Big] \\ &< \frac{v(b-x)}{(v+ux/b)(u+vx/b)}[(u-1)(b+t)/(x+t) + (v-1) - ((u-1)b/x + (v-1))] \\ &= -\frac{v(u-1)(b-x)^2 t}{(v+ux/b)(u+vx/b)x(x+t)} < 0 \end{aligned}$$

since $(x+t)/(b+t) \geq x/b$. Thus we conclude that $D(u, v, x, b, t) \leq 0$ for $0 < x \leq b, u \geq 1, v \geq 1$. □

We remark here from the proof of the Lemma 2.4, one finds $f(u, v, s, b) \leq 0$ and we have $D \leq 0$ as long as the condition $u + v \geq 2, u \geq 1, v \geq 0$ is satisfied, we don't really need $v \geq 1$.

Lemma 2.5. *Let x, a, b, u, v, s, t be real numbers with $t \geq 0, 0 < x \leq a \leq b, u \geq 1, v \geq 1, u+v \geq 3$ and $0 \leq s \leq v$, then $g(u, s, v, x, a, b) \leq g(u, s, v, x+t, a+t, b+t)$ where*

$$g(u, s, v, x, a, b) = b^2 \left[\frac{u+v-1}{ux+sa+(v-s)b} + \frac{1}{x^2(u/x + s/a + (v-s)/b)} - \frac{1}{x}\right]$$

with equality holding if and only if one of the following cases is true: 1. $x = a = b$; 2. $s = 0, x = b$; 3. $t = 0$.

Proof. We may assume $t > 0$ and let $M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b\}$. Furthermore, we define $H(s, a) = g(u, s, v, x, a, b) - g(u, s, v, x+t, a+t, b+t)$, where $(s, a) \in M$. It suffices to show $H(s, a) \leq 0$. Let $m = (s_0, a_0)$ be the point in which the absolute minimum of H is reached. If m

is an interior point of M, then we obtain

$$\begin{aligned} 0 &= \frac{1}{s}\frac{\partial H}{\partial a} - \frac{1}{a-b}\frac{\partial H}{\partial s}|_{(s,a)=(s_0,a_0)} = \frac{(b-a)b/x}{xa^2(u+sx/a+(v-s)x/b)^2} - \\ &- \frac{(b-a)(b+t)/(x+t)}{(x+t)(a+t)^2((u+s(x+t)/(a+t)+(v-s)(x+t)/(b+t))^2} > 0 \end{aligned}$$

where the inequality follows from $b/x > (b+t)/(x+t), (x+t)/(a+t) > x/a$. Hence, m is a boundary point of M, so that we get $m \in \{(s_0, x), (s_0, b), (0, a_0), (v, a_0)\}$. Using Lemma 2.4 we obtain $H(s_0, b) = H(0, a_0) = D(u, v, x, b, t) \le 0$ and

$$H(s_0, x) = D(u + s_0, v - s_0, x, b, t) \le 0$$

The above inequality follows from the remark after the proof of the Lemma 2.4, since here $v - s_0 \ge 0$ but may not exceed 1. Finally,

$$H(v, a_0) = b^2/a_0^2 f(u, v, x, a_0) - (b+t)^2/(a_0+t)^2 f(u, v, x+t, a_0+t) \le 0$$

The above inequality holds since $f(u, v, x, a_0) \le f(u, v, x+t, a_0+t) \le 0$ by the remark after the proof of the Lemma 2.4 and $b/a_0 \ge (b+t)/(a_0+t)$. Thus if $(s, a) \in M$, then $H(s, a) \le 0$. The conditions for equality can be easily checked by using Lemma 2.4 and noticing the condition $u + v \ge 3$. □

3. The Main Theorem

Theorem 3.1. *For $t \ge 0, x_1 > 0$, $-1 \le s \ne 1 \le 2$*

$$\frac{x_1}{t+x_1} \le \Delta_{1,s,t} \le \frac{x_n}{t+x_n} \tag{3.1}$$

with equality holding if and only if $t = 0$ or $x_1 = \cdots = x_n$.

Proof. The case $s = 0$ has been treated in [9] so we will assume $s \ne 0$ and prove the left-hand side inequality of (3.1) and the other proofs are similar. For $0 < s < 1$, let

$$D_n(\mathbf{x}, t) = x_n(A_n - P_{n,s}) - (t + x_n)(A_{n,t} - P_{n,s,t})$$

We want to show $D_n \ge 0$ here. We can assume $x_1 < x_2 < \cdots < x_n$ and prove by induction, the case $n = 1$ is clear so we will start with $n > 1$ variables assuming the inequality holds for $n - 1$ variables. Then

$$\begin{aligned} \frac{\partial D_n}{\partial x_n} &= (A_n - P_{n,s}) - (A_{n,t} - P_{n,s,t}) + \omega_n[(A_n - P_{n,s}^{1-s}x_n^s) - (A_{n,t} - P_{n,s,t}^{1-s}(t+x_n)^s)] \\ &\ge \omega_n[(A_n - P_{n,s}) - (A_{n,t} - P_{n,s,t}) + (A_n - P_{n,s}^{1-s}x_n^s) - (A_{n,t} - P_{n,s,t}^{1-s}(t+x_n)^s)] \\ &= \omega_n[P_{n,s,t}^{1-s}(t+x_n)^s + P_{n,s,t} - 2t - P_{n,s} - P_{n,s}^{1-s}x_n^s] \end{aligned}$$

where the inequality follows from $\Delta_{1,s,t} \le 1$. Now consider

$$g(t) = P_{n,s,t}^{1-s}(t + x_n)^s + P_{n,s,t} - 2t$$

and we have

$$\begin{aligned} g'(t) &= (1-s)(\frac{t+x_n}{P_{n,s,t}})^s(\frac{P_{n,s,t}}{P_{n,s-1,t}})^{1-s} + s(\frac{P_{n,s,t}}{t+x_n})^{1-s} + (\frac{P_{n,s,t}}{P_{n,s-1,t}})^{1-s} - 2 \\ &\ge (1-s)y^s + sy^{s-1} - 1 := h(y) \end{aligned}$$

where $y = \frac{t+x_n}{P_{n,s,t}} \ge 1$ and the inequality follows from $(\frac{P_{n,s,t}}{P_{n,s-1,t}})^{1-s} \ge 1$. Note $h'(y) = 0$ has only one root $y = 1$, which implies $h(y) \ge \min\{h(1), \lim_{y\to\infty} h(y)\} = 0$. Thus $g'(t) \ge 0$, hence $g(t) \ge g(0) = P_{n,s} + P_{n,s}^{1-s}x_n^s$ and it follows $\frac{\partial D_n}{\partial x_n} \ge 0$ and by letting x_n tend to x_{n-1}, we have $D_n \ge D_{n-1}$(with weights $\omega_1, \cdots, \omega_{n-2}, \omega_{n-1} + \omega_n$) and thus the right-hand side inequality of (3.1) holds by induction. It is easy to see the equality holds if and only if $t = 0$ or $x_1 = \cdots = x_n$.

For $-1 \leq s < 0$, we have

$$\frac{1}{\omega_1}\frac{\partial D_n}{\partial x_1} = -t - x_n(\frac{P_{n,s}}{x_1})^{1-s} + (t+x_n)(\frac{P_{n,s,t}}{t+x_1})^{1-s} := -t - f(x_1)$$

Consider

$$f'(x_1) = -(1-s)\sum_{j=2}^{n}\omega_j[(\frac{P_{n,s}}{x_1})^{1-2s}\cdot\frac{x_nx_j^s}{x_1^{s+1}} - (\frac{P_{n,s,t}}{t+x_1})^{1-2s}\frac{(t+x_n)(t+x_j)^s}{(t+x_1)^{s+1}}] \leq 0$$

The last inequality holds, since when $-1 \leq s < 0, j = 2, \cdots, n$, we have

$$(\frac{P_{n,s}}{x_1})^{1-2s} \geq (\frac{P_{n,s,t}}{t+x_1})^{1-2s}, \frac{x_j}{x_1} \geq \frac{t+x_j}{t+x_1}, \frac{x_n}{t+x_n}\cdot(\frac{x_j}{t+x_j})^s \geq (\frac{x_j}{t+x_j})^{1+s} \geq (\frac{x_1}{t+x_1})^{1+s}$$

Thus by a similar argument as above, we deduce $f(x_1) \geq -t$ and $\frac{\partial D_n}{\partial x_1} \leq 0$, which implies $D_n \geq 0$ with equality holding if and only if $t = 0$ or $x_1 = \cdots = x_n$.

For $1 < s \leq 2$, it suffices to show $\frac{\partial D_n}{\partial t} \leq 0$, which is equivalent to

$$\frac{P_{n,s}^{s-1}}{x_n} \leq \frac{(P_{n,s}^{s-1} - P_{n,s-1}^{s-1})}{(P_{n,s} - A_n)}$$

The above inequality follows from $\frac{P_{n,s}^{s-1}}{x_n} \leq x_n^{s-2}$ and Lemma 2.2 with $u = s-1, v = s, w = 1$. □

4. Some Consequences of Theorem 3.1

Corollary 4.1. (1.2) *holds for* $r = 1, -1 \leq s < 1$ *and* $1 < r \leq 2, s = 1$.

Proof. This follows from Theorems 3.1 and 1.1. □

The above result was first proved by the author in [8], in fact it was shown there those are the only cases (1.2) can hold for $r = 1$ or $s = 1$. Thus by Theorem 1.1, we have

Corollary 4.2. (3.1) *holds for all* $t \geq 0$ *if and only if* $-1 \leq s \neq 1 \leq 2$.

Corollary 4.3. *For* $-1 \leq s < 1$

$$\frac{x_1}{P_{n,s-1}^{1-s}} \leq \frac{(A_n - P_{n,s})}{(P_{n,s}^{1-s} - P_{n,s-1}^{1-s})} \leq \frac{x_n}{P_{n,s-1}^{1-s}} \tag{4.1}$$

Proof. Theorem 3.1 implies $f(t) = (t+x_n)(A_{n,t} - P_{n,s,t})$ is a decreasing function of t and $f'(0) \leq 0$ implies the right-hand side inequality of (4.1) and the proof of the left-hand side inequality of (4.1) is similar. □

By a change of variables $x_i \to 1/x_i$ and let $x_1 = m > 0$, the right-hand side inequality of (4.1) when $s = -1$ gives

$$A_n - H_n \leq \frac{H_n}{x_1A_n}\sigma_n \tag{4.2}$$

a refinement of the left-hand side inequality of (1.2) for $r = 1, s = -1$. We note here one can use the method in [9] to give a direct proof of (4.2) and show the equality holds if and only if $x_1 = \cdots = x_n$. We will leave the details to the reader.

5. A sharpening of Sierpiński's inequality

Theorem 5.1. *For* $0 < x_1 \leq \cdots \leq x_n, t \geq 0,\ q = \min\{\omega_i\}$

$$(\frac{x_n}{x_n+t})^2 \geq \frac{(1-q)\ln A_{n,t} + q\ln H_{n,t} - \ln G_{n,t}}{(1-q)\ln A_n + q\ln H_n - \ln G_n} \geq (\frac{x_1}{x_1+t})^2 \tag{5.1}$$

$$(\frac{x_n}{x_n+t})^2 \geq \frac{\ln G_{n,t} - q\ln A_{n,t} - (1-q)\ln H_{n,t}}{\ln G_n - q\ln A_n - (1-q)\ln H_n} \geq (\frac{x_1}{x_1+t})^2 \tag{5.2}$$

with equality holding if and only if $t = 0$ *or* $q = 1/2$ *or* $x_1 = \cdots = x_n$.

Proof. The proof uses the ideas in [4]. We will prove the left-hand side inequality of (5.1) and the proofs for other inequalities are similar. We may assume $t > 0$ being fixed and $q > 0, 0 < x = x_1, x_n = b$ with $x_1 < x_n$, we define

$$\begin{aligned} f_n(\mathbf{x}_n, q) &= x_n^2[(1-q)\ln A_n + q\ln H_n - \ln G_n] - \\ &\quad -(x_n+t)^2[(1-q)\ln A_{n,t} + q\ln H_{n,t} - \ln G_{n,t}] \end{aligned}$$

where we regard $A_n, G_n, H_n, A_{n,t}, G_{n,t}, H_{n,t}$ as functions of $\mathbf{x}_n = (x_1, \cdots, x_n)$. Then

$$g_n(x_2, \cdots, x_{n-1}) := \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} = x_n^2[\frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1}] - (x_n+t)^2[\frac{1-q}{A_{n,t}} + \frac{qH_{n,t}}{(x_1+t)^2} - \frac{1}{x_1+t}]$$

We want to show $g_n \leq 0$. Let $D = \{(x_2, \cdots, x_{n-1}) \in R^{n-2} | 0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b\}$. Let $\mathbf{a} = (a_2, \cdots, a_{n-1}) \in D$ be the point in which the absolute minimum of g_n is reached. Next, we show that

$$\mathbf{a} = (x, \cdots, x, a, \cdots, a, b, \cdots, b) \text{ with } x < a < b \tag{5.3}$$

where the numbers x, a, and b appear u, v, and w times, respectively, with $u, v, w \geq 0, u+v+w = n-2$.

Suppose not, this implies two components of $\mathbf{a}$ have different values and are interior points of D. We denote these values by a_k and a_l. Partial differentiation shows a_l, a_l are the roots of

$$h(x) = \frac{B}{x^2} - \frac{B'}{(x+t)^2} + C = 0 \tag{5.4}$$

where

$$B = q\frac{H_n^2 x_n^2}{x_1^2}, B' = q\frac{H_{n,t}^2(x_n+t)^2}{(x_1+t)^2}, C = \frac{(1-q)(x_n+t)^2}{A_{n,t}^2} - \frac{(1-q)x_n^2}{A_n^2}$$

It's easy to show $h'(x)$ only has one positive root, which implies $h(x)$ can have at most two distinct positive roots, but $\lim_{x\to 0} h(x) = \infty, \lim_{x\to\infty} h(x) = C < 0$ implies $h(x)$ can have at most one positive root. Thus (5.4) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus (5.3) is valid and it suffices to show $g_n \leq 0$ for the cases $n = 2, 3$.

When $n = 2$, by setting $x_1 = x, x_2 = b, \omega_1/q = u, \omega_2/q = v$, $g_2 \leq 0$ follows from Lemma 2.4.

When $n = 3$, by setting $x_1 = x, x_2 = a, x_3 = b, \omega_1/q = u, \omega_2/q = s, \omega_3/q = v - s$, $g_3 \leq 0$ follows from Lemma 2.5.

Thus we have shown that $g_n = \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} \leq 0$ with equality holding if and only if $n = 1$ or $n = 2, q = 1/2$. By letting x_1 tend to x_2, we have

$$f_n(\mathbf{x}_n, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q) \geq f_{n-1}(\mathbf{x}_{n-1}, q')$$

where $\mathbf{x}_{n-1} = (x_2, \cdots, x_n)$ with weights $\omega_1 + \omega_2, \cdots, \omega_{n-1}, \omega_n$ and $q' = min\{\omega_1 + \omega_2, \cdots, \omega_n\}$. Here we have used $\Delta_{1,-1,t,0} \leq (\frac{x_n}{t+x_n})^2$,which is a consequence of Theorem 3.1 and Lemma 2.3.

It then follows by induction that $f_n \geq f_{n-1} \geq \cdots \geq f_2 = 0$ when $q = 1/2$ in f_2 or else $f_n \geq f_{n-1} \geq \cdots \geq f_1 = 0$ and this completes the proof. □

By letting $t \to \infty$ in (5.1), (5.2), we recover the following result of the author[8], which can be regarded as sharpenings of Sierpiński's inequality[13] for the weighted cases:

Corollary 5.1. *For $0 < x_1 \leq \cdots \leq x_n$, $q = \min\{\omega_i\}$*

$$\frac{1-2q}{2x_1^2}\sigma_n \geq (1-q)\ln A_n + q\ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{5.5}$$

$$\frac{1-2q}{2x_1^2}\sigma_n \geq \ln G_n - q\ln A_n - (1-q)\ln H_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{5.6}$$

with equality holding if and only if $q = 1/2$ or $x_1 = \cdots = x_n$.

6. A Rado-Type Inequality

By letting $\omega_i = q_i/Q_n, Q_n = \sum_{i=1}^n q_i, q_i > 0$(note for different n, ω_i's take different values), C.L.Wang[14] proved the following Rado-type inequality:

Theorem 6.1. *If $x_i \in (0, 1/2], i = 1, \cdots, n$, then*

$$Q_n(A_nG'_n - A'_nG_n) \geq Q_{n-1}(A_{n-1}G'_{n-1} - A'_{n-1}G_{n-1}) \tag{6.1}$$

We end the paper by giving an analogue of Wang's theorem:

Theorem 6.2. *For $t > 0, q_i > 0, i = 1, \cdots, n$*

$$Q_n(A_nG_{n,t} - A_{n,t}G_n) \geq Q_{n-1}(A_{n-1}G_{n-1,t} - A_{n-1,t}G_{n-1})\left(\frac{A_{n-1,t} - A_{n-1}}{G_{n-1,t} - G_{n-1}}\right)^{\frac{q_n}{Q_n}} \tag{6.2}$$

Proof. Let $f(x_n) = Q_n(A_nG_{n,t} - A_{n,t}G_n)$, by setting

$$f'(x_n) = q_n(x_n + A_{n,t})\left(\frac{G_{n,t}}{t+x_n} - \frac{G_n}{x_n}\right) = 0$$

we get $x_n = tG_{n-1}/(G_{n-1,t} - G_{n-1})$. Moreover, at this point

$$f''(x_n) = \frac{q_nQ_{n-1}}{Q_n} \cdot \frac{G_n}{x_n}\left(\frac{A_{n,t}}{x_n} - \frac{A_n}{t+x_n}\right) > 0$$

and it is easy to see that $f(x_n)$ takes its absolute minimum at the point, which implies

$$f(x_n) \geq f\left(\frac{tG_{n-1}}{G_{n-1,t} - G_{n-1}}\right) = Q_{n-1}(A_{n-1}G_{n-1,t} - A_{n-1,t}G_{n-1})\left(\frac{A_{n-1,t} - A_{n-1}}{G_{n-1,t} - G_{n-1}}\right)^{\frac{q_n}{Q_n}}$$

for any $x_n \geq 0$, with equality holding if and only if $x_n = tG_{n-1}/(G_{n-1,t} - G_{n-1})$. □

We note here by letting $t \to \infty$ in (6.2), we get back Rado's inequality:

$$Q_n(A_n - G_n) \geq Q_{n-1}(A_{n-1} - G_{n-1})$$

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