ON EVALUATION OF RIEMANN ZETA FUNCTION $\zeta(s)$

QIU-MING LUO, BAI-NI GUO, AND FENG QI

Abstract. In this paper, by using Fourier series theory, several summing formulae for Riemann Zeta function $\zeta(s)$ and Dirichlet series are deduced.

1. Introduction

It is well-known that the Riemann Zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \Re(s)>1 \tag{1}$$

and Dirichlet series

$$D(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s},\quad \Re(s)>1 \tag{2}$$

are related to the gamma functions and have important applications in mathematics, especially in Analytic Number Theory.

In 1734, Euler gave some remarkably elementary proofs of the following Bernoulli series

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}. \tag{3}$$

The formula (3) has been studied by many mathematicians and many proofs have been provided, for example, see [2].

In 1748, Euler further gave the following general formula

$$\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}, \tag{4}$$

where B_{2k} denotes Bernoulli numbers for $k\in\mathbb{N}$.

2000 *Mathematics Subject Classification.* 11R40, 42A16.

Key words and phrases. Riemann Zeta function, Fourier series, recursion formula.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using $\mathcal{AMS}$-LaTeX.

The Bernoulli numbers B_k and Euler numbers E_k are defined in [22, 23] respectively by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k, \quad |t|<2\pi; \tag{5}$$

$$\frac{2e^t}{e^{2t}+1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k, \quad |t|\le\pi. \tag{6}$$

For other proofs concerning formula (4), please refer to the references in this paper, for example, [22] and [25].

In 1999, the paper [10] gave the following elementary expression for $\zeta(2k)$: Let $n\in\mathbb{N}$, then

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=A_k\pi^{2k}, \tag{7}$$

where

$$\begin{aligned} A_k&=\frac{1}{3!}A_{k-1}-\frac{1}{5!}A_{k-2}+\dots+(-1)^{k-2}\frac{1}{(2k-1)!}A_1+(-1)^{k-1}\frac{k}{(2k+1)!}\\ &=(-1)^{k-1}\frac{k}{(2k+1)!}+\sum_{i=1}^{k-1}\frac{(-1)^{k-i-1}}{(2k-2i+1)!}A_i. \end{aligned} \tag{8}$$

For several centuries, the problem of proving the irrationality of $\zeta(2k+1)$ has remained unsolved. In 1978, R. Apéry, a French mathematician, proved that the number $\zeta(3)$ is irrational. However, one cannot generalize his proof to other cases. Therefore, many mathematicians have much interest in the evaluation of $\zeta(s)$ and sums of related series. For some examples, see [11, 24, 26].

In [12], the lower and upper bounds for $\zeta(3)$ are given by using an integral expression $\zeta(3)=\frac{8}{7}\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3}=\frac{2}{7}\int_0^{\pi/2}\frac{x(\pi-x)}{\sin x}\,\mathrm{d}x$ in [9, p. 81] and refinements of the Jordan inequality $x-\frac{1}{6}x^3\le\sin x\le x-\frac{1}{6}x^3+\frac{1}{120}x^5$ in [13, 14].

The following formulae involving $\zeta(2k+1)$ were given by Ramanujan, see [24], as follows:

(1) If $k>1$ and $k\in\mathbb{N}$,

$$\alpha^k\left[\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\alpha}-1}\right]=(-\beta)^k\left[\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\beta}-1}\right], \tag{9}$$

(2) if $k > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} 0 = & \frac{1}{(4\alpha)^k}\left[\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}\right] \\ & -\frac{1}{(-4\beta)^k}\left[\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\beta}-1)}\right] \\ & +\sum_{j=0}^{\left[\frac{k+1}{2}\right]}{}' \frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2K-2J+2)!}\left[\alpha^{k-2j+1}+(-\beta)^{k-2j+1}\right], \end{aligned} \tag{10}$$

where B_j is the j-th Bernoulli number, $\alpha > 0$ and $\beta > 0$ satisfy $\alpha\beta = \pi^2$, and $\sum'$ means that, when k is an odd number $2m-1$, the last term of the left hand side in (10) is taken as $\frac{(-1)^m\pi^{2m}B_{2m}^2}{(m!)^2}$.

In 1928, Hardy in [6] proved (9). In 1970, E. Grosswald in [3] proved (10). In 1970, E. Grosswald in [4] gave another expression of $\zeta(2k+1)$.

In 1983, N.-Y. Zhang in [24] not only proved Ramanujan formulae (9) and (10), but also gave an explicit expression of $\zeta(2k+1)$ as follows:

(1) If k is odd, then we have

$$\zeta(2k+1) = -2\psi_{-k}(\pi) - (2\pi)^{2k+1}\sum_{j=0}^{\left[\frac{k+1}{2}\right]}{}' \frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2k-2j+2)!}; \tag{11}$$

(2) if k is even,

$$\zeta(2k+1) = -2\psi_{-k}(\pi) + \frac{2\pi}{k}\psi'_{-k}(\pi)\frac{(2\pi)^{2k+1}}{k}\sum_{j=0}^{\frac{k}{2}}\frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2k-2j+2)!}, \tag{12}$$

where $\psi_{-k}(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}$, and $\psi'_{-k}(\alpha)$ is the derivative of $\psi_{-k}(\alpha)$ with respect to α.

There is much literature on calculating of $\zeta(s)$, for example, see [2, p. 435] and [22, pp. 144–145; p. 149; pp. 150–151].

As a matter of fact, many other recent investigations and important results on the subject of the Riemannian Zeta function $\zeta(s)$ can be found in the papers [15, 16, 17, 18, 20, 21] by H.M. Srivastava, and others. Furthermore, Chapter 4 entitled "Evaluations and Series Representations" of the book [19] contains a rather systematic presentation of much of these recent developments.

The aim of this paper is to obtain recursion formulae of sums for the Riemann Zeta function and Dirichlet series through expanding the power function x^n on

$[-\pi, \pi]$ by using the Dirichlet theorem in Fourier series theory. These recursion formulae are more beautiful than those from (4) to (12). To the best of our knowledge, these formulae are new.

2. Lemmas

Lemma 1 (Dirichlet Theorem [7, p. 281]). *Let $f(x)$ be a piecewise differentiable function on $[-\pi, \pi]$.*

(1) *If $f(x)$ is even on $[-\pi, \pi]$, then the Fourier series expansion of $f(x)$ on $[-\pi, \pi]$ is*

$$\frac{f(x+0)+f(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{13}$$

where

$$a_0 = \frac{\pi}{2}\int_0^{\pi} f(x)\,\mathrm{d}x, \quad a_n = \frac{\pi}{2}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x; \tag{14}$$

(2) *if $f(x)$ is odd on $[-\pi, \pi]$, then we have*

$$\frac{f(x+0)+f(x-0)}{2} = \sum_{n=1}^{\infty} b_n \sin nx, \tag{15}$$

where

$$b_n = \frac{\pi}{2}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x. \tag{16}$$

Lemma 2 ([5, pp. 272–273]). *Let $n \in \mathbb{N}$ and $s \in \mathbb{R}^+$, then*

$$\begin{aligned}\int x^s \cos nx\,\mathrm{d}x = \sin nx &\sum_{i=0}^{[\frac{s}{2}]}\binom{s}{2i}\frac{(-1)^i(2i)!x^{s-2i}}{n^{2i+1}}\\ &+ \cos nx \sum_{i=0}^{[\frac{s-1}{2}]}\binom{s}{2i+1}\frac{(-1)^i(2i+1)!x^{s-2i-1}}{n^{2i+2}},\end{aligned} \tag{17}$$

$$\begin{aligned}\int x^s \sin nx\,\mathrm{d}x = \cos nx &\sum_{i=0}^{[\frac{s}{2}]}\binom{s}{2i}\frac{(-1)^{i+1}(2i)!x^{s-2i}}{n^{2+1}}\\ &+ \sin nx \sum_{i=0}^{[\frac{s-1}{2}]}\binom{s}{2i+1}\frac{(-1)^i(2i+1)!x^{s-2i-1}}{n^{2i+2}}.\end{aligned} \tag{18}$$

Lemma 3. *For $s > 1$, let $\delta(s) \triangleq \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}$ and $\sigma(s) \triangleq \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$. Then*

$$\zeta(s) = \frac{2^s}{2^s-1}\delta(s), \tag{19}$$

$$\zeta(s) = \frac{2^{s-1}}{2^{s-1}-1}D(s), \tag{20}$$

$$\delta(s) = \sum_{n=1}^{\infty} \frac{1}{(4n-3)^s} + \sum_{n=1}^{\infty} \frac{1}{(4n-1)^s}, \tag{21}$$

$$\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(4n-3)^s} - \sum_{n=1}^{\infty} \frac{1}{(4n-1)^s}. \tag{22}$$

Proof. It is easy to see that, for $s > 1$, $\zeta(s)$, $D(s)$, $\delta(s)$, and $\sigma(s)$ converge absolutely. Since

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \delta(s) + \frac{1}{2^s}\zeta(s), \tag{23}$$

the formula (19) follows from rewriting (23). Further,

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \delta(s) - \frac{1}{2^s}\zeta(s), \tag{24}$$

combining (19) with (24) yields (20). □

Lemma 4 ([22, p. 151])**.** *For $k \in \mathbb{N}$, we have*

$$\sigma(2k+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = \frac{\pi^{2k+1}}{2^{2k+2}(2k)!} E_k. \tag{25}$$

3. Main results and proofs

We will use the usual convention that an empty sum is taken to be zero. For example, if $k = 0$ and $k = 1$, we take $\sum_{i=1}^{k-1} = 0$ in this paper.

Theorem 1. *For $k \in \mathbb{N}$, we have*

$$\zeta(2k) = \frac{(-1)^{k-1} k \pi^{2k}}{(2k+1)!} + \sum_{i=1}^{k-1} \frac{(-1)^{k+i+1} \pi^{2k-2i}}{(2k-2i+1)!} \zeta(2i), \tag{26}$$

$$\zeta(2k+1) = \frac{2^{2k+1}}{2^{2k+1}-1} \left[2 \sum_{n=1}^{\infty} \frac{1}{(4n-1)^{2k+1}} + \sigma(2k+1) \right]. \tag{27}$$

Proof. Let $f(x) = x^s$, $s \in \mathbb{N}$, then $f(x)$ is differentiable on $[-\pi, \pi]$.

If s is even, then $f(x)$ is an even function on $[-\pi, \pi]$. From (14) and (17), we obtain

$$\begin{aligned} a_0 &= \frac{2\pi^s}{s+1}, \\ a_n &= \frac{2}{\pi} \sum_{i=0}^{\left[\frac{s-1}{2}\right]} \binom{s}{2i+1} \frac{(-1)^{n+i} (2i+1)! \pi^{s-2i-1}}{n^{2i+2}}. \end{aligned} \tag{28}$$

Substituting (28) into (13) leads to a Fourier series expansion of $f(x)=x^s$ below

$$\sum_{i=0}^{\left[\frac{s-1}{2}\right]}\binom{s}{2i+1}(-1)^i(2i+1)!\pi^{s-2i-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2i+2}}\cos nx=\frac{\pi}{2}\left(x^s-\frac{\pi^s}{s+1}\right). \tag{29}$$

If s is odd, then $f(x)$ is an odd function on $[-\pi,\pi]$. Using (16) and (18) yields

$$b_n=\frac{2}{\pi}\sum_{i=0}^{\left[\frac{s}{2}\right]}\binom{s}{2i}\frac{(-1)^{n+i+1}(2i)!\pi^{s-2i}}{n^{2i+1}}. \tag{30}$$

Substituting (30) into (15) give us the following

$$\sum_{i=0}^{\left[\frac{s}{2}\right]}\binom{s}{2i}(-1)^{i+1}(2i)!\pi^{s-2i}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2i+1}}\sin nx=\frac{\pi}{2}x^s. \tag{31}$$

Taking $x=\pi$ and $s=2k$ in (29) produces

$$\sum_{i=1}^{k}\binom{2k}{2i-1}(-1)^{i-1}(2i-1)!\pi^{2k-2i-1}\sum_{n=1}^{\infty}\frac{1}{n^{2i}}=\frac{k\pi^{2k+1}}{2k+1}. \tag{32}$$

Formula (26) follows from (32).

Set $s=2k+1$ in (19), (21), and (22), then we have

$$\zeta(2k+1)=\frac{2^{2k+1}}{2^{2k+1}-1}\delta(2k+1), \tag{33}$$

$$\delta(2k+1)=\sum_{n=0}^{\infty}\frac{1}{(2n-1)^{2k+1}}=\sum_{n=1}^{\infty}\frac{1}{(4n-3)^{2k+1}}+\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}, \tag{34}$$

$$\sigma(2k+1)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}}=\sum_{n=1}^{\infty}\frac{1}{(4n-3)^{2k+1}}-\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}. \tag{35}$$

Formula (27) follows from combining (33), (34), and (35). □

Remark 1. Using (26), we can obtain values of $\zeta(2k)$, for examples,

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \zeta(8)=\frac{\pi^8}{9450},\quad \zeta(10)=\frac{\pi^{10}}{93555}.$$

Using (27), we also can approximate values of $\zeta(2k+1)$.

Theorem 2. *For $k\in\mathbb{N}$, we have*

$$\zeta(2k)=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}, \tag{36}$$

$$\zeta(2k+1)=\frac{\pi^{2k+1}}{(2^{2k+2}-2)(2k)!}E_k+\frac{2^{2k+2}}{2^{2k+1}-1}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}. \tag{37}$$

where B_{2k} and E_k denote Bernoulli numbers and Euler numbers, respectively.

Proof. This follows from substituting (25) into (27). □

Theorem 3. *For $k \in \mathbb{N}$, we have*

$$D(2k) = \frac{(-1)^{k-1}\pi^{2k}}{2(2k+1)!} + \sum_{i=1}^{k-1} \frac{(-1)^{k+i+1}\pi^{2k-2i}}{(2k-2i+1)!} D(2i), \tag{38}$$

$$\sigma(2k+1) = \frac{(-1)^{k}\pi^{2k+1}}{2^{2k+2}(2k+1)!} + \sum_{i=0}^{k-1} \frac{(-1)^{k+i+1}\pi^{2k-2i}}{(2k-2i+1)!} \sigma(2i+1). \tag{39}$$

Proof. Since

$$\sin\frac{n\pi}{2} = \begin{cases} 0, & \text{for } n = 2\ell, \\ (-1)^{\ell-1}, & \text{for } n = 2\ell-1, \end{cases} \tag{40}$$

In (29), taking $x = 0$ and $s = 2k$ gives us

$$\sum_{i=0}^{k-1} \binom{2k}{2i+1} (-1)^i (2i+1)! \pi^{2k-2i-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2i+2}} = \frac{\pi^{2k+1}}{2(2k+1)}. \tag{41}$$

From (41), formula (38) follows.

In (31), taking $x = \frac{\pi}{2}$ and $s = 2k+1$ $(k = 0, 1, 2\dots)$ and using (40) yields

$$\sigma(2k+1) = \frac{(-1)^{k+1}}{(2k+1)!\pi} \left[\sum_{i=0}^{k-1} \binom{2k+1}{2i} (-1)^i (2i)! \pi^{2k-2i+1} \sigma(2i+1) - \left(\frac{\pi}{2}\right)^{2k+2} \right]. \tag{42}$$

From (42), we obtain (39). □

Remark 2. From (38), we can calculate values of Dirichlet series $D(2k)$, for example,

$$D(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

$$D(4) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7\pi^4}{720},$$

$$D(6) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6} = \frac{31\pi^6}{30240}.$$

From (39), we can obtain values of $\sigma(2k+1)$, for example,

$$\sigma(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4},$$

$$\sigma(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32},$$

$$\sigma(5) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5\pi^5}{1536}.$$

Theorem 4. *For $k \in \mathbb{N}$, we have*

$$D(2k) = \frac{(-1)^{k-1}(2^{2k-1}-1)\pi^{2k}}{(2k)!}B_{2k}, \tag{43}$$

$$\delta(2k) = \frac{(-1)^{k-1}(2^{2k}-1)\pi^{2k}}{2(2k)!}B_{2k}. \tag{44}$$

where B_{2k} denotes a Bernoulli number.

Proof. In (19) and (20), taking $s = 2k$ and $k \in \mathbb{N}$ and using (36) leads to (43) and (44). □

Remark 3. From (44), we obtain

$$\delta(2) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

$$\delta(4) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96},$$

$$\delta(6) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

Acknowledgments. The authors are indebted to the anonymous referees and the Editor, Professor H. M. Srivastava, for their fairly detailed comments and many valuable additions to the list of references.

References

[1] Alfred van der Poorten, *A proof missed by Euler*, Shùxué Yìlín (Translations of Mathematics) (1980), no. 2, 47–63. (Chinese)

[2] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978.

[3] E. Grosswald, *Comments on Some formulae of Ramnujan*, Acta. Arith. **21** (1972), 25–34.

[4] E. Grosswald, *Die Werte der Riemannschen Zeta Function an ungeraden Argumenstellen*, Göttingen Nachrichten (1970), 9–13.

[5] Group of Compilation, *Shùxué Shǒucè* (*Handbook of Mathematics*), Higher Education Press, Beijing, China, 1990. (Chinese)

[6] G. H. Hardy, *A formula of Ramanujan*, J. London Math. Soc. **3** (1928), 238–240.

[7] L.-G. Hua, *Gāoděng Shùxué Yǐnlùn* (*Introduction to Advanced Mathematics*), Vol. 1, Fascicule 2, Science Press, Beijing, China, 1979. pp. 281–283. (Chinese)

[8] R.-R. Jiang, *Generalization and Applications of Fundamentional Residul Theorem*, Shùxué de Shíjiàn yù Rènshī (Math. Practice Theory) **27** (1997), no. 3, 354–360. (Chinese)

[9] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd edition, Hunan Education Press, Changsha City, Hunan Province, China, 1993. (Chinese)

[10] J.-F. Lin, *Elementary solvabilities of* $\sum_{n=1}^{+\infty}\frac{1}{n^{2m}}$, Shùxué de Shíjiàn yù Rènshī (Math. Practice Theory), **29** (1999), no. 3, 14–18. (Chinese)

[11] G.-D. Liu, *A class of evaluating formulae including Riemann Zeta*, Kēxué Tōngbào (Chinese Sci. Bull.) **44** (1999), no. 2, 146–148. (Chinese)

[12] Q.-M. Luo, Z.-L. Wei, and F. Qi, *Lower and upper bounds of* $\zeta(3)$, Adv. Stud. Contemp. Math. **6** (2003), no. 1, 47–51. RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 7, 565–569. Available online at `http://rgmia.vu.edu.au/v4n4.html`.

[13] F. Qi, *Extensions and sharpenings of Jordan's and Kober's inequality*, Gōngkē Shùxué (Journal of Mathematics for Technology) **12** (1996), no. 4, 98–102. (Chinese)

[14] F. Qi and Q.-D. Hao, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (1998), no. 3, 116–120.

[15] H. M. Srivastava, *Certain families of rapidly convergent series representations for* $\zeta(2n+1)$, Math. Sci. Res. Hot-Line **1** (1997), no. 6, 1–6.

[16] H. M. Srivastava, *Further series representations for* $\zeta(2n+1)$, Appl. Math. Comput. **97** (1998), 1–15.

[17] H. M. Srivastava, *Some rapidly converging series for* $\zeta(2n+1)$, Proc. Amer. Math. Soc. **127** (1999), 385–396.

[18] H. M. Srivastava, *Some simple algorithms for the evaluations and representations of the Riemann Zeta function at positive integer arguments*, J. Math. Anal. Appl. **246** (2000), 331–351.

[19] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.

[20] H. M. Srivastava, M. L. Glasser, and V. S. Adamchik, *Some definite integrals associated with the Riemann Zeta function*, Z. Anal. Anwendungen **19** (2000), 831–846.

[21] H. M. Srivastava and H. Tsumura, *A certain class of rapidly convergent series representations for* $\zeta(2n+1)$, J. Comput. Appl. Math. 118(2000), 323–335.

[22] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn* (*Introduction to Special Function*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

[23] L.-Zh. Xu (L. C. Hsu) and X.-H. Wang, *Shùxué Fēnxī Zhōng de Fāngfǎ hé Lìtǐ Xuǎnjiǎng* (*Methods of Mathematical Analysis and Selected Examples*), Revised Edition, Higher Education Press, Beijing, China, 1983. pp. 281–283. (Chinese)

[24] N.-Y. Zhang, *Values of Riemannjan formula and Riemann Zeta function at positive odd numbers*, Shùxué Jìnzhǎn (Adv. Math. (China)) **12** (1983), no. 1, 61–71. (Chinese)

[25] Q.-T. Zhuang and N.-Y. Zhang, *Fùbiàn Hánshù* (*Complex Functions*), Beijing University Press, Beijing, China, 1984. p. 10 and p. 249. (Chinese)

[26] W.-P. Zhang, *On identities of Riemann Zeta function*, Kēxué Tōngbào (Chinese Sci. Bull.) **36** (1991), no. 4, 250–253. (Chinese)

Department of Broadcast-Television-Teaching, Jiaozuo University, Jiaozuo City, Henan 454003, CHINA

E-mail address: luoqm@jzu.edu.cn

Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: guobaini@jzit.edu.cn

Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: qifeng@jzit.edu.cn