NEW TAYLOR-LIKE EXPANSIONS FOR FUNCTIONS OF TWO VARIABLES AND ESTIMATES OF THEIR REMAINDERS

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Abstract. In this article, a generalisation of Sard's inequality for Appell polynomials is obtained. Estimates for the remainder are also provided.

1. Introduction

Let $x \in [a,b]$ and $y \in [c,d]$. If $f(x,y)$ is a function of two variables we shall adopt the following notation for partial derivatives of $f(x,y)$:

$$\begin{aligned} f^{(i,j)}(x,y) &\triangleq \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, \\ f^{(0,0)}(x,y) &\triangleq f(x,y), \\ f^{(i,j)}(\alpha,\beta) &\triangleq f^{(i,j)}(x,y)\big|_{(x,y)=(\alpha,\beta)} \end{aligned} \tag{1}$$

for $0 \le i,j \in \mathbb{N}$ and $(\alpha,\beta) \in [a,b] \times [c,d]$.

A. H. Stroud has pointed out in [6] that one of the most important tools in the numerical integration of double integrals is the following Taylor's formula [6, p. 138 and p. 157] due to A. Sard [5]:

Theorem A. *If $f(x,y)$ satisfies the condition that all the derivatives $f^{(i,j)}(x,y)$ for $i+j \le m$ are defined and continous on $[a,b] \times [c,d]$, then $f(x,y)$ has the expansion*

$$\begin{aligned} f(x,y) = &\sum_{i+j \le m} \frac{(x-a)^i}{i!} \frac{(y-c)^j}{j!} f^{(i,j)}(a,c) \\ &+ \sum_{j<q} \frac{(y-c)^j}{j!} \int_a^x \frac{(x-u)^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(u,c)\,du \\ &+ \sum_{i<p} \frac{(x-a)^i}{i!} \int_c^y \frac{(y-v)^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,v)\,dv \\ &+ \int_a^x \int_c^y \frac{(x-u)^{p-1}}{(p-1)!} \frac{(y-v)^{q-1}}{(q-1)!} f^{(p,q)}(u,v)\,dv\,du, \end{aligned} \tag{2}$$

where i,j are nonnegative integers; p,q are positive integers; and $m \triangleq p+q \ge 2$.

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Essentially, the representation (2) is used for obtaining the fundamental Kernel Theorems and Error Estimates in numerical integration of double integrals [6, p. 142, p. 145 and p. 158] and has both theoretical and practical importance in the domain as a whole.

Definition 1. A sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ is called *harmonic* [4] if it satisfies the recursive formula

$$P_i'(x) = P_{i-1}(x) \tag{3}$$

for $i \in \mathbb{N}$ and $P_0(x) = 1$.

A slightly different concept that specifies the connection between the variables is the following one.

Definition 2. We say that a sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ satisfies the *Appell condition* [2] if

$$\frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \tag{4}$$

and $P_0(t,x) = 1$ for all defined (t,x) and $n \in \mathbb{N}$.

It is wellknown that the Bernoulli polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in \mathbb{R}. \tag{5}$$

It can be shown that the polynomials $B_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the two formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t) = 1; \tag{6}$$

$$\text{and} \quad B_i(t+1) - B_i(t) = it^{i-1}. \tag{7}$$

The Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x + 1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in \mathbb{R}. \tag{8}$$

It can also be shown that the polynomials $E_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the two properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t) = 1; \tag{9}$$

$$\text{and} \quad E_i(t+1) + E_i(t) = 2t^i. \tag{10}$$

For further details about Bernoulli polynomials and Euler polynomials, please refer to [1, 23.1.5 and 23.1.6].

There are many examples of Appell polynomials. For instance, for i a nonnegative integer, $\theta \in \mathbb{R}$ and $\lambda \in [0,1]$,

$$P_{i,\lambda}(t) \triangleq P_{i,\lambda}(t;x;\theta) = \frac{[t - (\lambda\theta + (1-\lambda)x)]^i}{i!}, \tag{11}$$

$$P_{i,B}(t) \triangleq P_{i,B}(t;x;\theta) = \frac{(x-\theta)^i}{i!} B_i\left(\frac{t-\theta}{x-\theta}\right) \text{ ([4])}, \tag{12}$$

$$P_{i,E}(t) \triangleq P_{i,E}(t;x;\theta) = \frac{(x-\theta)^i}{i!} E_i\left(\frac{t-\theta}{x-\theta}\right) \text{ ([4])}. \tag{13}$$

In [4], the following generalized Taylor's formula was established.

Theorem B. *Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then, for any $x \in I$, we have*

$$f(x) = f(a) + \sum_{k=1}^{m} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f; a, x), \tag{14}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{15}$$

The fundamental aim of this article is to obtain a generalisation of the Taylor-like formula (2) for Appell polynomials and to study its impact on the numerical integration of double integrals.

2. Two New Taylor-like Expansions

Following a similar argument to the proof of Theorem 2 in [4], we obtain the following result.

Theorem 1. *If $g : [a, b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a, b]$, then we have the generalised integration by parts formula for $x \in [a, b]$*

$$\begin{aligned}\int_a^b g(t)\, dt = &\sum_{k=1}^{n} (-1)^{k+1} \left[P_k(b, x) g^{(k-1)}(b) - P_k(a, x) g^{(k-1)}(a) \right] \\ &+ (-1)^n \int_a^b P_n(t, x) g^{(n)}(t)\, dt.\end{aligned} \tag{16}$$

Proof. By integration by parts we obtain, on using the Appell condition (4),

$$\begin{aligned}&(-1)^n \int_a^b P_n(t, x) g^{(n)}(t)\, dt \\ &= (-1)^n P_n(t, x) g^{(n-1)}(t) \Big|_a^b + (-1)^{n-1} \int_a^b P_{n-1}(t, x) g^{(n-1)}(t)\, dt \\ &= (-1)^n \left[P_n(b, x) g^{(n-1)}(b) - P_n(a, x) g^{(n-1)}(a) - \int_a^b P_{n-1}(t, x) g^{(n-1)}(t)\, dt \right].\end{aligned} \tag{17}$$

Clearly, the same procedure can be used for the term $\int_a^b P_{n-1}(t, x) g^{(n-1)}(t)\, dt$. Therefore, formula (16) follows from successive integration by parts. ∎

Theorem 2. *Let D be a domain in $\mathbb{R}^2$ and the point $(a, c) \in D$. Also, let $\{P_i(t, x)\}_{i=0}^{\infty}$ and $\{Q_j(s, y)\}_{j=0}^{\infty}$ be two Appell polynomials. If $f : D \to \mathbb{R}$ is such that $f^{(i,j)}(x, y)$ are continuous on D for all $0 \le i \le m$ and $0 \le j \le n$, then*

$$f(x, y) = f(a, c) + C(f, P_m, Q_n) + D(f, P_m, Q_n) + S(f, P_m, Q_n) + T(f, P_m, Q_n), \tag{18}$$

where

$$\begin{aligned}C(f, P_m, Q_n) = &\sum_{i=1}^{m} (-1)^{i+1} \left[P_i(x, x) f^{(i,0)}(x, c) - P_i(a, x) f^{(i,0)}(a, c) \right] \\ &+ \sum_{j=1}^{n} (-1)^{j+1} \left[Q_j(y, y) f^{(0,j)}(a, y) - Q_j(c, y) f^{(0,j)}(a, c) \right],\end{aligned} \tag{19}$$

$$
\begin{aligned}
&D(f,P_m,Q_n)\\
&\quad=\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(x,x)\big[Q_j(y,y)f^{(i,j)}(x,y)-Q_j(c,y)f^{(i,j)}(x,c)\big]\\
&\qquad-\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,x)\big[Q_j(y,y)f^{(i,j)}(a,y)-Q_j(c,y)f^{(i,j)}(a,c)\big], \quad (20)
\end{aligned}
$$

$$
\begin{aligned}
&S(f,P_m,Q_n)\\
&\quad=(-1)^m\int_a^x P_m(t,x)f^{(m+1,0)}(t,c)\,dt+(-1)^n\int_c^y Q_n(s,y)f^{(0,n+1)}(a,s)\,ds\\
&\quad+\sum_{i=1}^{m}(-1)^{n+i+1}\int_c^y Q_n(s,y)\big[P_i(x,x)f^{(i,n+1)}(x,s)-P_i(a,x)f^{(i,n+1)}(a,s)\big]\,ds\\
&\quad+\sum_{j=1}^{n}(-1)^{m+j+1}\int_a^x P_m(t,x)\big[Q_j(y,y)f^{(m+1,j)}(t,y)-Q_j(c,y)f^{(m+1,j)}(t,c)\big]\,dt \quad (21)
\end{aligned}
$$

and

$$
T(f,P_m,Q_n)=(-1)^{m+n}\int_a^x\int_c^y P_m(t,x)Q_n(s,y)f^{(m+1,n+1)}(t,s)\,ds\,dt. \quad (22)
$$

Proof. Let $P_m(t,x)$ be an Appell polynomial. Applying formula (14) to the function $f(x,y)$ with respect to variable x yields

$$
\begin{aligned}
f(x,y)&=f(a,y)+\sum_{i=1}^{m}(-1)^{i+1}\big[P_i(x,x)f^{(i,0)}(x,y)-P_i(a,x)f^{(i,0)}(a,y)\big]\\
&\quad+(-1)^m\int_a^x P_m(t,x)f^{(m+1,0)}(t,y)\,dt.
\end{aligned} \quad (23)
$$

Similarly, for the functions $f^{(i,0)}(x,y)$, $f^{(i,0)}(a,y)$, $f^{(m+1,0)}(t,y)$ and $f(a,y)$, we have

$$
\begin{aligned}
f^{(i,0)}(x,y)&=f^{(i,0)}(x,c)+(-1)^n\int_c^y Q_n(s,y)f^{(i,n+1)}(x,s)\,ds\\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(i,j)}(x,y)-Q_j(c,y)f^{(i,j)}(x,c)\big],
\end{aligned} \quad (24)
$$

$$
\begin{aligned}
f^{(i,0)}(a,y)&=f^{(i,0)}(a,c)+(-1)^n\int_c^y Q_n(s,y)f^{(i,n+1)}(a,s)\,ds\\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(i,j)}(a,y)-Q_j(c,y)f^{(i,j)}(a,c)\big],
\end{aligned} \quad (25)
$$

$$
\begin{aligned}
f^{(m+1,0)}(t,y)&=f^{(m+1,0)}(t,c)+(-1)^n\int_c^y Q_n(s,y)f^{(m+1,n+1)}(a,s)\,ds\\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(m+1,j)}(t,y)-Q_j(c,y)f^{(m+1,j)}(t,c)\big],
\end{aligned} \quad (26)
$$

$$
\begin{aligned}
f(a,y)=f(a,c)&+(-1)^n\int_c^y Q_n(s,y)f^{(0,n+1)}(a,s)\,ds\\
&+\sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(0,j)}(a,y)-Q_j(c,y)f^{(0,j)}(a,c)\big].
\end{aligned}\tag{27}
$$

Substituting formulae (24)–(27) into (23) produces

$$
\begin{aligned}
f(x,y)&=f(a,c)+\sum_{i=1}^{m}(-1)^{i+1}\big[P_i(x,x)f^{(i,0)}(x,c)-P_i(a,x)f^{(i,0)}(a,c)\big]\\
&+\sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(0,j)}(a,y)-Q_j(c,y)f^{(0,j)}(a,c)\big]\\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(x,x)\big[Q_j(y,y)f^{(i,j)}(x,y)-Q_j(c,y)f^{(i,j)}(x,c)\big]\\
&-\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,x)\big[Q_j(y,y)f^{(i,j)}(a,y)-Q_j(c,y)f^{(i,j)}(a,c)\big]\\
&+(-1)^m\int_a^x P_m(t,x)f^{(m+1,0)}(t,c)\,dt+(-1)^n\int_c^y Q_n(s,y)f^{(0,n+1)}(a,s)\,ds\\
&+\sum_{i=1}^{m}(-1)^{n+i+1}\int_c^y Q_n(s,y)\big[P_i(x,x)f^{(i,n+1)}(x,s)-P_i(a,x)f^{(i,n+1)}(a,s)\big]\,ds\\
&+\sum_{j=1}^{n}(-1)^{m+j+1}\int_a^x P_m(t,x)\big[Q_j(y,y)f^{(m+1,j)}(t,y)-Q_j(c,y)f^{(m+1,j)}(t,c)\big]\,dt\\
&+(-1)^{m+n}\int_a^x\int_c^y P_m(t,x)Q_n(s,y)f^{(m+1,n+1)}(t,s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}\tag{28}
$$

The proof of Theorem 2 is complete. ∎

Remark 1. If we take

$$
P_i(t,x)=P_{m,\lambda}(t,x;a),\quad Q_j(s,y)=Q_{j,\mu}(s,y;c)\tag{29}
$$

for $0\le i\le m$, $0\le j\le n$ and $\lambda,\mu\in[0,1]$ in Theorem 2, then the expressions simplify to give, on using (11),

$$
\begin{aligned}
C(f,P_m,Q_n)=\sum_{i=1}^{m}\frac{(x-a)^i}{i!}&\big[(1-\lambda)^i f^{(i,0)}(a,c)+\lambda^i f^{(i,0)}(x,c)\big]\\
&+\sum_{j=1}^{n}\frac{(y-c)^j}{j!}\big[(1-\mu)^j f^{(0,j)}(a,c)+\mu^j f^{(0,j)}(a,y)\big],
\end{aligned}\tag{30}
$$

$$
D(f,P_m,Q_n)=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\lambda^i(x-a)^i(y-c)^j}{i!\cdot j!}\big[\mu^j f^{(i,j)}(x,y)+(1-\mu)^j f^{(i,j)}(x,c)\big]
$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(1-\lambda)^i(x-a)^i(y-c)^j}{i!\cdot j!}\left[\mu^j f^{(i,j)}(a,y)+(1-\mu)^j f^{(i,j)}(a,c)\right], \quad (31)$$

$$\begin{aligned}S(f,P_m,Q_n)&=(-1)^m\int_a^x\frac{[t-(\lambda a+(1-\lambda)x)]^m}{m!}f^{(m+1,0)}(t,c)\,\mathrm{d}t\\&\quad+(-1)^n\int_c^y\frac{[s-(\mu c+(1-\mu)y)]^n}{n!}f^{(0,n+1)}(a,s)\,\mathrm{d}s\\&+\sum_{i=1}^{m}\int_c^y\frac{[\mu c+(1-\mu)y-s]^n(x-a)^i}{n!\cdot i!}\left[(\lambda-1)^i f^{(i,n+1)}(a,s)-\lambda^i f^{(i,n+1)}(x,s)\right]ds\\&+\sum_{j=1}^{n}\int_a^x\frac{[\lambda a+(1-\lambda)x-t]^m(y-c)^j}{m!\cdot j!}\left[(\mu-1)^j f^{(m+1,j)}(t,c)-\mu^j f^{(m+1,j)}(t,y)\right]dt, \quad (32)\end{aligned}$$

and

$$T(f,P_m,Q_n)=\int_a^x\int_c^y\frac{[(\lambda a+(1-\lambda)x)-t]^m[(\mu c+(1-\mu)y)-s]^n}{m!\cdot n!}f^{(m+1,n+1)}(t,s)\,ds\,dt. \quad (33)$$

Notice that, taking $\lambda=0$ and $\mu=0$ in (29), then we can deduce Theorem A from Theorem 2. Other choices of Appell type polynomials will provide generalizations of Theorem A.

The following approximation of double integrals in terms of Appell polynomials holds.

Theorem 3. *Let $\{P_i(t,x)\}_{i=0}^{\infty}$ and $\{Q_j(s,y)\}_{j=0}^{\infty}$ be two Appell polynomials and $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ such that $f^{(i,j)}(x,y)$ are continuous on $[a,b]\times[c,d]$ for all $0\le i\le m$ and $0\le j\le n$. We then have*

$$\int_a^b\int_c^d f(t,s)\,ds\,dt=A(f,P_m,Q_n)+B(f,P_m,Q_n)+R(f,P_m,Q_n), \quad (34)$$

where

$$\begin{aligned}&A(f,P_m,Q_n)=\\&\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,b)\left[Q_j(d,d)f^{(i-1,j-1)}(a,d)-Q_j(c,d)f^{(i-1,j-1)}(a,c)\right]\\&-\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(b,b)\left[Q_j(d,d)f^{(i-1,j-1)}(b,d)-Q_j(c,d)f^{(i-1,j-1)}(b,c)\right],\end{aligned} \quad (35)$$

$$\begin{aligned}B(f,P_m,Q_n)&=\sum_{j=1}^{n}(-1)^jQ_j(c,d)\int_a^b f^{(0,j-1)}(t,c)\,dt\\&-\sum_{j=1}^{n}(-1)^jQ_j(d,d)\int_a^b f^{(0,j-1)}(t,d)\,dt\\&+\sum_{i=1}^{m}(-1)^iP_i(a,b)\int_c^d f^{(i-1,0)}(a,s)\,ds\\&-\sum_{i=1}^{m}(-1)^iP_i(b,b)\int_c^d f^{(i-1,0)}(b,s)\,ds\end{aligned}\tag{36}$$

and

$$R(f,P_m,Q_n)=(-1)^{m+n}\int_a^b\int_c^d P_m(t,b)Q_n(s,d)f^{(m,n)}(t,s)\,ds\,dt.\tag{37}$$

Proof. Using the generalized integration by parts formula consecutively yields

$$\begin{aligned}&\int_a^b\int_c^d P_m(t,b)Q_n(s,d)f^{(m,n)}(t,s)\,ds\,dt\\&=\int_a^b P_m(t,b)\left[\int_c^d Q_n(s,d)f^{(m,n)}(t,s)\,ds\right]dt\\&=(-1)^m\int_a^b P_m(t,b)\Bigg\{\int_c^d f^{(m,0)}(t,s)\,ds\\&\quad+\sum_{j=1}^{n}(-1)^j\left[Q_j(d,d)f^{(m,j-1)}(t,d)-Q_j(c,d)f^{(m,j-1)}(t,c)\right]\Bigg\}\,dt\\&=(-1)^m\int_a^b\int_c^d P_m(t,b)f^{(m,0)}(t,s)\,ds\,dt\\&\quad+\sum_{j=1}^{n}(-1)^{m+j}Q_j(d,d)\int_a^b P_m(t,b)f^{(m,j-1)}(t,d)\,dt\\&\quad-\sum_{j=1}^{n}(-1)^{m+j}Q_j(c,d)\int_a^b P_m(t,b)f^{(m,j-1)}(t,c)\,dt\\&=(-1)^m\int_c^d(-1)^n\Bigg\{\int_a^b f(t,s)\,dt\\&\quad+\sum_{i=1}^{m}(-1)^i\Big[P_i(b,b)f^{(i-1,0)}(b,s)-P_i(a,b)f^{(i-1,0)}(a,s)\Big]\Bigg\}\,ds\\&\quad+\sum_{j=1}^{n}(-1)^{n+j}Q_j(d,d)\Bigg\{(-1)^m\Bigg[\int_a^b f^{(0,j-1)}(t,d)\,dt\\&\quad+\sum_{i=1}^{m}(-1)^j\Big(P_i(b,b)f^{(i-1,j-1)}(b,d)-P_i(a,b)f^{(i-1,j-1)}(a,d)\Big)\Bigg]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{n} (-1)^{n+j} Q_j(c,d) \bigg\{ (-1)^m \bigg[\int_a^b f^{(0,j-1)}(t,c)\,dt \\
&+ \sum_{i=1}^{m} (-1)^i \bigg(P_i(b,b) f^{(i-1,j-1)}(b,c) - P_i(a,b) f^{(i-1,j-1)}(a,c) \bigg) \bigg] \bigg\} \\
= {}& (-1)^{m+n} \int_a^b \int_c^d f(t,s)\,ds\,dt \\
&+ \sum_{i=1}^{m} (-1)^{m+n+i} \int_c^d \big[P_i(b,b) f^{(i-1,0)}(b,s) - P_i(a,b) f^{(i-1,0)}(a,s) \big]\,ds \\
&+ \sum_{j=1}^{n} (-1)^{m+n+j} Q_j(d,d) \int_a^b f^{(0,j-1)}(t,d)\,dt \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(b,b) Q_j(d,d) f^{(i-1,j-1)}(b,d) \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(a,b) Q_j(d,d) f^{(i-1,j-1)}(a,d) \\
&- \sum_{j=1}^{n} (-1)^{m+n+j} Q_j(c,d) \int_a^b f^{(0,j-1)}(t,c)\,dt \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(a,b) Q_j(c,d) f^{(i-1,j-1)}(a,c) \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(b,b) Q_j(c,d) f^{(i-1,j-1)}(b,c) \\
= {}& (-1)^{m+n} \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(b,b) \big[Q_j(d,d) f^{(i-1,j-1)}(b,d) \\
&\qquad\qquad\qquad\qquad - Q_j(c,d) f^{(i-1,j-1)}(b,c) \big] \\
&+ (-1)^{m+n} \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(a,b) \big[Q_j(c,d) f^{(i-1,j-1)}(a,c) \\
&\qquad\qquad\qquad\qquad - Q_j(d,d) f^{(i-1,j-1)}(a,d) \big] \\
&+ (-1)^{m+n} \sum_{i=1}^{m} (-1)^i P_i(b,b) \int_c^d f^{(i-1,0)}(b,s)\,ds \\
&- (-1)^{m+n} \sum_{i=1}^{m} (-1)^j P_i(a,b) \int_c^d f^{(i-1,0)}(a,s)\,ds \\
&+ (-1)^{m+n} \sum_{j=1}^{n} (-1)^j Q_j(d,d) \int_a^b f^{(0,j-1)}(t,d)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- (-1)^{m+n}\sum_{j=1}^{n}(-1)^i Q_j(c,d)\int_a^b f^{(0,j-1)}(t,c)\,dt\\
&+ (-1)^{m+n}\int_a^b\int_c^d f(t,s)\,ds\,dt.
\end{aligned}
$$

The proof of Theorem 3 is complete. ∎

Remark 2. As usual, let B_i, $i\in\mathbb{N}$, denote Bernoulli numbers.
From properties (6) and (7), (9) and (10) of the Bernoulli and Euler polynomials respectively, we can easily obtain that, for $i\geq 1$,

$$
B_{i+1}(0)=B_{i+1}(1)=B_{i+1},\quad B_1(0)=-B_1(1)=-\frac{1}{2}, \tag{38}
$$

and, for $j\in\mathbb{N}$,

$$
E_j(0)=-E_j(1)=-\frac{2}{j+1}(2^{j+1}-1)B_{j+1}. \tag{39}
$$

It is also a well known fact that $B_{2i+1}=0$ for all $i\in\mathbb{N}$.

As an example, taking $P_i(t,x)=P_{i,B}(t,x;a)$ and $Q_j(s,y)=P_{j,E}(s,y;c)$ from (12) and (13) for $0\leq i\leq m$ and $0\leq j\leq n$ in Theorem 3 and using (38) and (39) yields

$$
\begin{aligned}
A(f,P_m,Q_n) &= \sum_{i=1}^{m}\sum_{j=2}^{n}\frac{(a-b)^i(c-d)^j}{i!\cdot j!}\cdot\frac{2(2^{j+1}-1)}{j+1}B_iB_{j+1}\\
&\times\big[f^{(i-1,j-1)}(a,d)+f^{(i-1,j-1)}(a,c)-f^{(i-1,j-1)}(b,d)-f^{(i-1,j-1)}(b,c)\big]\\
&+(b-a)\sum_{i=1}^{m}\frac{(2^{i+1}-1)(c-d)^j}{(i+1)!}B_{j+1}\\
&\times\big[f^{(i-1,0)}(a,d)+f^{(i-1,0)}(a,c)+f^{(i-1,0)}(b,d)+f^{(i-1,0)}(b,c)\big],
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&B(f,P_m,Q_n)=\\
&\quad 2\sum_{j=1}^{n}\frac{(1-2^{j+1})(c-d)^j}{(j+1)!}B_{j+1}\int_a^b\big[f^{(0,j-1)}(t,c)+f^{(0,j-1)}(t,d)\big]\,dt\\
&\quad+\sum_{j=2}^{n}\frac{(a-b)^j}{j!}B_j\int_c^d\big[f^{(i-1,0)}(a,s)-f^{(i-1,0)}(b,s)\big]\,ds\\
&\quad+\frac{b-a}{2}\int_c^d\big[f(a,s)+f(b,s)\big]\,ds,
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
&R(f,P_m,Q_n)=\\
&\qquad\frac{(a-b)^m(c-d)^n}{m!\cdot n!}\int_a^b\int_c^d B_m\Big(\frac{t-a}{b-a}\Big)E_n\Big(\frac{s-c}{d-c}\Big)f^{(m,n)}(t,s)\,ds\,dt.
\end{aligned} \tag{42}
$$

3. Estimates of the Remainders

In this section, we will give some estimates for the remainders of expansions in Theorem 2 and Theorem 3.

We firstly need to introduce some notation.
For a function $\ell : [a,b] \times [c,d] \to \mathbb{R}$, then for any $x, y \in [a,b]$, $u,\ v \in [c,d]$ we define

$$\|\ell\|_{[x,y]\times[u,v],\infty} := ess\sup\left\{ |\ell(t,s)| \right\}, \qquad t \in [x,y] \ \text{ or } \ [y,x] \ \text{ and } s \in [u,v] \ \text{ or } \ [v,u]$$

and

$$\|\ell\|_{[x,y]\times[u,v],p} := \left| \int_x^y \int_u^v |h(t,s)|^p \, dsdt \right|^{\frac{1}{p}}, \quad p \geq 1.$$

The following result establishing bounds for the remainder in the Taylor-like formula (18) holds.

Theorem 4. *Assume that $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ and f satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies the estimate*

$$|T(f,P_m,Q_n)| \leq \begin{cases} \|P_m(\cdot,x)\|_{[a,x],\infty} \|Q_n(\cdot,y)\|_{[c,y],\infty} \left\| f^{(m+1,n+1)} \right\|_{[a,x]\times[c,y],1}, \\ \\ \|P_m(\cdot,x)\|_{[a,x],p} \|Q_n(\cdot,y)\|_{[c,y],p} \left\| f^{(m+1,n+1)} \right\|_{[a,x]\times[c,y],p}, \\ \qquad\qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \|P_m(\cdot,x)\|_{[a,x],1} \|Q_n(\cdot,y)\|_{[c,y],1} \left\| f^{(m+1,n+1)} \right\|_{[a,x]\times[c,y],\infty}. \end{cases} \tag{43}$$

The proof follows in a straightforward fashion on using Hölder's inequality applied for the integral representation of the remainder $T(f,P_m,Q_n)$ provided by equation (22). We omit the details.

The integral remainder in the cubature formula (34) may be estimated in the following manner.

Theorem 5. *Assume that $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ and f satisfy the assumptions in Theorem 3. Then one has the cubature formula (34) and, the remainder $R(f,P_m,Q_n)$ satisfies the estimate:*

$$|R(f,P_m,Q_n)| \leq \begin{cases} \|P_m(\cdot,b)\|_{[a,b],\infty} \|Q_n(\cdot,d)\|_{[c,d],\infty} \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],1}, \\ \\ \|P_m(\cdot,b)\|_{[a,b],p} \|Q_n(\cdot,d)\|_{[c,d],p} \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],p}, \\ \qquad\qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \|P_m(\cdot,b)\|_{[a,b],1} \|Q_n(\cdot,d)\|_{[c,d],1} \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],\infty}. \end{cases} \tag{44}$$

Remark 1. If we consider the particular instances of Appell polynomials provided by (11), (12) and (13), then a number of particular formulae may be obtained. Their remainder may be estimated by the use of Theorems 4 and 5, providing a $2-$dimensional version of the results in [4].

For instance, if we consider from (11),

$$P_{m,\lambda}(t,x;a) = \frac{[t-(\lambda\ a + (1-\lambda)\ x)]^m}{m!} \tag{45}$$

$$Q_{n,\mu}(s,y;c) = \frac{[s-(\mu\ c + (1-\mu)\ y)]^n}{n!} \tag{46}$$

then we obtain the following result.

Theorem 6. *Let $\{P_{m,\lambda}(t,x;a)\}_{m=0}^{\infty}$, $\{Q_{n,\mu}(s,y)\}_{n=0}^{\infty}$ and f satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies for $a \le x$, $c \le y$, the estimate*

$$|T(f,P_{m,\lambda},Q_{n,\mu})| \le \begin{cases} \frac{(x-a)^m(y-c)^n}{m!n!}\lambda_\infty\ \mu_\infty \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],1}, \\ \frac{1}{m!n!}\left[\frac{(x-a)^{mq+1}(y-c)^{nq+1}}{(mq+1)(nq+1)}\right]^{\frac{1}{q}}\lambda_p\ \mu_p \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],q}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(x-a)^{m+1}(y-c)^{n+1}}{(m+1)!(n+1)!}\lambda_1\mu_1 \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],\infty}. \end{cases} \tag{47}$$

where

$$\lambda_1 = \left[\lambda^{m+1}+(1-\lambda)^{m+1}\right],\quad \lambda_p = \left[\lambda^{mq+1}+(1-\lambda)^{mq+1}\right]^{\frac{1}{p}} \quad \text{and} \quad \lambda_\infty = \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^m.$$

and similar for μ_1, μ_p and μ_∞

Proof. Utilizing equations (45) and (46) and using Hölder's inequality for double integrals and the properties of the modulus on equation (22), then we have that

$$\begin{aligned} &\left|\int_a^x\int_c^y T(f,P_{m,\lambda},Q_{n,\mu})\right| \\ &\qquad = \left|\int_a^x\int_c^y P_{m,\lambda}(t,x;a)\ Q_{n,\mu}(s,y;c) f^{(m+1,n+1)}\ ds\ dt\right| \\ &\qquad \le \int_a^x\int_c^y |P_{m,\lambda}(t,x;a)\ Q_{n,\mu}(s,y;c)|\left|f^{(m+1,n+1)}\right|\ ds\ dt \\ &\qquad \le \begin{cases} \sup_{(t,s)\in[a,x]\times[c,y]} |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|\ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],1}. \\ \left(\int_a^x\int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|^q\,dt\,ds\right)^{\frac{1}{q}}\ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],p}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \int_a^x\int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|\,dt\,ds\ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],\infty}. \end{cases} \end{aligned} \tag{48}$$

Now, the result in equation (48) can be further simplified by the application of equations (45) and (46), given that,

$$\alpha = (1-\lambda)\ x + \lambda\ a \quad \text{and} \quad \beta = (1-\mu)\ y + \mu\ c.$$

It then follows

$$\begin{aligned}
&\sup_{(t,s)\in[a,x]\times[c,y]} |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)| \\
&= \sup_{t\in[a,c]} |P_{m,\lambda}(t,x;a)| \sup_{s\in[c,y]} |Q_{n,\mu}(s,y;c)| \\
&= \max\left\{\frac{(\alpha-a)^m}{m!}, \frac{(x-\alpha)^m}{m!}\right\} \times \max\left\{\frac{(\beta-c)^n}{n!}, \frac{(y-\beta)^n}{n!}\right\} \\
&= \frac{(x-a)^m (y-c)^n}{m!n!} [\max\{(1-\lambda),\lambda\}]^m \times [\max\{(1-\mu),\mu\}]^n \\
&= \frac{(x-a)^m (y-c)^n}{m!n!} \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^m \times \left[\frac{1}{2}+\left|\mu-\frac{1}{2}\right|\right]^n
\end{aligned}$$

giving the first inequality in (47) where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Further, we have

$$\begin{aligned}
&\left(\int_a^x \int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
&= \left(\int_a^x |P_{m,\lambda}(t,x;a)|^q \, dt\right)^{\frac{1}{q}} \left(\int_c^y |Q_{n,\mu}(s,y;c)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
&= \frac{1}{m!n!}\left[\int_a^\alpha (\alpha-t)^{mq}\, dt + \int_\alpha^x (t-\alpha)^{mq}\, dt\right]^{\frac{1}{q}} \\
&\quad \times \left[\int_c^\beta (\beta-s)^{nq}\, ds + \int_\beta^y (s-\beta)^{nq}\, ds\right]^{\frac{1}{q}} \\
&= \frac{1}{m!n!}\left[\frac{(x-a)^{mq+1}(y-c)^{nq+1}}{(mq+1)(nq+1)}\right]^{\frac{1}{q}} \lambda_p\, \mu_p
\end{aligned}$$

producing the second inequality in (47).
Finally,

$$\begin{aligned}
&\int_a^x \int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|\, dt\, ds \\
&= \int_a^x \left|\frac{(t-\alpha)^m}{m!}\right| dt \int_c^y \left|\frac{(s-\beta)^n}{n!}\right| ds \\
&= \left[\int_a^\alpha \frac{(\alpha-t)^m}{m!} dt + \int_\alpha^x \frac{(t-\alpha)^m}{m!} dt\right] \times \left[\int_c^\beta \frac{(\beta-s)^n}{n!} ds + \int_\beta^y \frac{(s-\beta)^n}{n!} ds\right] \\
&= \frac{(x-a)^{m+1}(y-c)^{n+1}}{(m+1)!\,(n+1)!}\left[(1-\lambda)^{m+1}+\lambda^{m+1}\right] \times \left[(1-\mu)^{n+1}+\mu^{n+1}\right]
\end{aligned}$$

gives the last inequality in (47). Thus the theorem is completely proved. ∎

Remark 2. By taking $\lambda = \mu = 0$ or 1, we recapture the result obtained by G. Hanna et al. in [3].

In a similar fashion, we can stat the remainder $R(f, P_{m,\lambda}, Q_{n,\mu})$ estimate in the cubature formula (34) as in the following

Theorem 7. *Let $\{P_{m,\lambda}(t,x;a)\}_{m=0}^{\infty}$, $\{Q_{n,\mu}(s,y)\}_{n=0}^{\infty}$ and f satisfy the assumptions of Theorem 3, then the remainder $R(f, P_{m,\lambda}, Q_{n,\mu})$ estimate in the cubature formula (34) satisfies the following*

$$|R(f, P_{m,\lambda}, Q_{n,\mu})| \leq \begin{cases} \frac{(b-a)^m (d-c)^n}{m!n!} \lambda_\infty \ \mu_\infty \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],1}, \\ \frac{1}{m!n!} \left[\frac{(b-a)^{mq+1} (d-c)^{nq+1}}{(mq+1)(nq+1)} \right]^{\frac{1}{q}} \lambda_p \ \mu_p \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],q}, \\ \qquad \text{where } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^{m+1} (d-c)^{n+1}}{(m+1)!(n+1)!} \lambda_1 \ \mu_1 \left\| f^{(m,n)} \right\|_{[a,b]\times[c,d],\infty}. \end{cases} \tag{49}$$

The proof is similar to the one in Theorem 6 applied on the interval $[a,b] \times [c,d]$, and we omit the details.

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