AN APPROXIMATION FOR THE FINITE-FOURIER TRANSFORM OF TWO INDEPENDENT VARIABLES

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Abstract. In this paper we develop some approximations of the two-dimensional Finite-Fourier transform in terms of the complex exponential mean. A cubature formula is developed as a numerical application and explored via a numerical experiment.

1. Introduction

The Fourier transform has long been a principle analytical tool in such diverse fields as linear systems, optics, random process modeling, probability theory, quantum physics, and boundary-value problems [3]. In particular, it has been very successfully applied to the restoration of astronomical data [2]. The Fourier transform, a pervasive and versatile tool, is used in many fields of science as a mathematical or physical tool to alter a problem into one that can be more easily solved. Some scientists understand Fourier theory as a physical phenomenon, not simply as a mathematical tool. In some branches of science, the Fourier transform of one function may yield another physical function [1]. Utilizing some integral identities and inequalities developed in [4, 5, 6], we point out some approximations of the two-dimensional Finite-Fourier transform in terms of the complex exponential mean $E(z,w)$ and estimate the error of approximation for different classes of continuous mappings defined on finite intervals.

In this paper $f:[a,b]\times[c,d]\to\mathbb{R}$ will be a continuous mapping defined on the finite interval $[a,b]\times[c,d]$ and $\mathcal{F}(f)$ its Finite-Fourier transform. That is

$$\begin{aligned}&\mathcal{F}(f)(u,v;a,b,c,d)\\&\quad=\int_a^b\int_c^d f(x,y)\,e^{-2\pi i(ux+vy)}dydx,\end{aligned} \tag{1}$$

$(u,v)\in[a,b]\times[c,d]$. For a function of one variable we use the notation

$$\mathcal{F}(g)(u;a,b)=\int_a^b g(x)\,e^{-2\pi iux}dx.$$

2. Some Integral Inequalities

In this section we employ an identity obtained in [4] and develop inequalities for the estimation of the two dimensional Fourier transform. The following inequality holds.

Theorem 1. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[a,b] \times [c,d]$ *and assume that* $f''_{x,y} := \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$, *then we have the inequality*

$$\left| \mathcal{F}(f)(u,v;a,b,c,d) - \mathcal{I}_1 - \mathcal{I}_2 + \mathcal{I}_3 \right| \leq \begin{cases} \dfrac{(b-a)^2 (d-c)^2}{9} \left\| f''_{x,y} \right\|_\infty, & \text{if } f''_{x,y} \in L_\infty([a,b]\times[c,d]); \\ \left[\dfrac{2\left[(b-a)(d-c)\right]^{\frac{q+1}{2}}}{(q+1)(q+2)} \right]^{\frac{2}{q}} \left\| f''_{x,y} \right\|_p, & \text{if } f''_{x,y} \in L_p([a,b]\times[c,d]), \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ (b-a)(d-c) \left\| f''_{x,y} \right\|_1, & \text{if } f''_{x,y} \in L_1([a,b]\times[c,d]) \end{cases} \quad (2)$$

for all $(u,v) \in [a,b] \times [c,d]$, *where*

$$\mathcal{I}_1 := \mathcal{I}_1(u,v;a,b,c,d) = E(u) \int_a^b \mathcal{F}(f(s,\cdot))(v;c,d)\, ds,$$

$$\mathcal{I}_2 := \mathcal{I}_2(u,v;a,b,c,d) = E(v) \int_c^d \mathcal{F}(f(\cdot,t))(u;a,b)\, dt,$$

$$\mathcal{I}_3 := \mathcal{I}_3(u,v;a,b,c,d) = E(u)\, E(v) \int_a^b \int_c^d f(s,t)\, dt ds$$

with

$E(u) := E(-2\pi i u b, -2\pi i u a)$, *and* $E(v) := E(-2\pi i v d, -2\pi i v c)$, *given that* E *is the exponential mean of complex numbers, that is*

$$E(z,w) := \begin{cases} \dfrac{e^z - e^w}{z - w} & \text{if } z \neq w \\ e^w & \text{if } z = w \end{cases} \text{for } z, w \in \mathbb{C}.$$

Furthermore we define the usual Lebesgue norms

$$\left\| f''_{x,y} \right\|_\infty = \sup_{(s,t)\in[a,b]\times[c,d]} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| < \infty, \text{ and}$$

$$\left\| f''_{x,y} \right\|_p = \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right|^p dt ds \right)^{\frac{1}{p}}, 1 \leq p < \infty.$$

Proof. Using the identity obtained by Barnett and Dragomir in [4],

$$f(x,y) = \frac{\int_a^b f(s,y)\,ds}{b-a} + \frac{\int_c^d f(x,t)\,dt}{d-c} - \frac{\int_a^b \int_c^d f(s,t)\,dtds}{(b-a)(d-c)} + \frac{\int_a^b \int_c^d P(x,s)\,Q(y,t)\,f''_{x,y}(s,t)\,dtds}{(b-a)(d-c)} \tag{3}$$

provided that f is continuous on $[a,b]\times[c,d]$ and

$$P(x,s) = \begin{cases} s-a, & a \le s \le x \\ s-b, & x < s \le b \end{cases} \quad \text{and} \quad Q(y,t) = \begin{cases} t-c, & c \le t \le y \\ t-d, & y < t \le d. \end{cases}$$

If we replace $f(x,y)$ in (1) by its representation from (3), we get

$$\begin{aligned}
&\mathcal{F}(f)(u,v;a,b,c,d) \\
&= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a} \int_a^b f(s,y)\,ds \right) dydx \\
&+ \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c} \int_c^d f(x,t)\,dt \right) dydx \\
&- \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,dtds \right) dydx \\
&+ R(f,u,v;a,b,c,d),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&R(f,u,v;a,b,c,d) \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(e^{-2\pi i(ux+vy)} \right) \\
&\times \left[\int_a^b \int_c^d P(x,s)\,Q(y,t)\,f''_{x,y}(s,t)\,dtds \right] dydx.
\end{aligned} \tag{5}$$

Let

$$\mathcal{I}_1 = \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a} \int_a^b f(s,y)\,ds \right) dydx, \text{ then}$$

$$\begin{aligned}
\mathcal{I}_1 &= \int_a^b \frac{e^{-2\pi iux}}{b-a} dx \left(\int_c^d e^{-2\pi ivy} \left(\int_a^b f(s,y)\,ds \right) dy \right) \\
&= \frac{e^{-2\pi iub} - e^{-2\pi iua}}{-2\pi iu(b-a)} \int_a^b \left(\int_c^d e^{-2\pi ivy} f(s,y)\,dy \right) ds \\
&= E(u) \int_a^b \mathcal{F}(f(s,\cdot))(v;c,d)\,ds.
\end{aligned}$$

In a similar fashion we obtain

$$\begin{aligned}\mathfrak{I}_2 &= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c} \int_c^d f\left(x,t\right) dt \right) dydx \\ &= E\left(v\right) \int_c^d \mathfrak{F}\left(f\left(\cdot,t\right)\right)\left(u;a,b\right) dt\end{aligned}$$

and

$$\begin{aligned}\mathfrak{I}_3 &= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \cdot \int_a^b \int_c^d f\left(s,t\right) dtds \right) dydx \\ &= \frac{1}{\left(b-a\right)\left(d-c\right)} \int_a^b \int_c^d f\left(s,t\right) dtds \times \int_a^b \int_c^d e^{-2\pi iux} \cdot e^{-2\pi ivy} dydx \\ &= E\left(u\right) E\left(v\right) \int_a^b \int_c^d f\left(s,t\right) dtds. \qquad (6)\end{aligned}$$

Using the properties of modulus on (4), we have

$$\begin{aligned}&\left|\mathfrak{F}\left(f\right)\left(u,v;a,b,c,d\right) - \mathfrak{I}_1 - \mathfrak{I}_2 + \mathfrak{I}_3\right| \\ &= \left| \int_a^b \int_c^d (\int_a^b \int_c^d \frac{e^{-2\pi i(ux+vy)}}{\left(b-a\right)\left(d-c\right)} \cdot P\left(x,s\right) Q\left(y,t\right) \times f''_{x,y}(s,t) dtds) dydx \right| \\ &\leq \int_a^b \int_c^d \int_a^b \int_c^d \left| \frac{e^{-2\pi i(ux+vy)}}{\left(b-a\right)\left(d-c\right)} \right| \left|P\left(x,s\right)\right| \left|Q\left(y,t\right)\right| \times \left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (7) \\ &= \int_a^b \int_c^d \int_a^b \int_c^d \frac{\left|P\left(x,s\right)\right| \left|Q\left(y,t\right)\right|}{\left(b-a\right)\left(d-c\right)} times \left|f''_{x,y}(s,t)\right| dtdsdydx. \qquad (8)\end{aligned}$$

Now, we observe that

$$\begin{aligned}&\int_a^b \int_c^d \int_a^b \int_c^d \left|P\left(x,s\right)\right| \left|Q\left(y,t\right)\right| \times \left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (9) \\ &\leq \left\|f''_{x,y}\right\|_\infty \left[\int_a^b \left(\int_a^b \left|P\left(x,s\right)\right| ds \right) dx \int_c^d \left(\int_c^d \left|Q\left(y,t\right)\right| dt \right) dy \right] \\ &= \left\|f''_{x,y}\right\|_\infty \left[\int_a^b \left\{ \left. \frac{\left(s-a\right)^2}{2} \right|_a^x + \left. \frac{\left(b-s\right)^2}{2} \right|_x^b \right\} dx \right. \\ &\qquad \left. \times \int_c^d \left\{ \left. \frac{\left(t-c\right)^2}{2} \right|_c^y + \frac{\left(d-t\right)^2}{2} \right\} dy \right] \\ &= \left\|f''_{x,y}\right\|_\infty \left[\left(\int_a^b \frac{\left(x-a\right)^2}{2} dx + \int_a^b \frac{\left(b-x\right)^2}{2} dx \right) \right. \\ &\qquad \left. \times \left(\int_c^d \frac{\left(y-c\right)^2}{2} dy + \int_c^d \frac{\left(d-y\right)^2}{2} dy \right) \right] \\ &= \left\|f''_{x,y}\right\|_\infty \left[\frac{\left(b-a\right)^3}{3} \cdot \frac{\left(d-c\right)^3}{3} \right].\end{aligned}$$

Substituting in (8) with (9), we obtain the first inequality in (2).

Applying Hölder's integral inequality for double integrals, we get

$$\begin{aligned}
&\int_a^b\int_c^d\int_a^b\int_c^d |P(x,s)\,Q(y,t)|\left|f''_{x,y}(s,t)\right| dtdsdydx\\
&\quad\le \left(\int_a^b\int_c^d\int_a^b\int_c^d |P(x,s)\,Q(y,t)|^q\, dtdsdydx\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_a^b\int_c^d\int_a^b\int_c^d \left|f''_{x,y}(s,t)\right|^p dtdsdydx\right)^{\frac{1}{p}} \qquad (10)\\
&\quad=\left\|f''_{x,y}\right\|_p((b-a)(d-c))^{\frac{1}{p}}\times\left(\int_a^b\left(\int_a^b|P(x,s)|^q\,ds\right)dx\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_c^d\left(\int_c^d|Q(y,t)|^q\,dt\right)dy\right)^{\frac{1}{q}}\\
&\quad=\left\|f''_{x,y}\right\|_p((b-a)(d-c))^{\frac{1}{p}}\times\left(\int_a^b\left(\frac{(x-a)^{q+1}}{q+1}+\frac{(b-x)^{q+1}}{q+1}\right)dx\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_c^d\left(\frac{(y-c)^{q+1}}{q+1}+\frac{(d-y)^{q+1}}{q+1}\right)dy\right)^{\frac{1}{q}}\\
&\quad=\left\|f''_{x,y}\right\|_p\left[\frac{2^{\frac{2}{q}}(b-a)^{1+\frac{2}{q}}(d-c)^{1+\frac{2}{q}}}{((q+1)(q+2))^{\frac{2}{q}}}\right]. \qquad (11)
\end{aligned}$$

Utilizing (8) with (11), we get the second inequality of (2).
Finally, we obtain that

$$\begin{aligned}
&\int_a^b\int_c^d\int_a^b\int_c^d |P(x,s)\,Q(y,t)|\times\left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (12)\\
&\le \sup_{(x,s)\in[a,b]^2}|P(x,s)|\sup_{(y,t)\in[c,d]^2}|Q(y,t)|\times\int_a^b\int_c^d\int_a^b\int_c^d\left|f''_{x,y}\right|dtdsdydx\\
&=(b-a)(d-c)\int_a^b\int_c^d\int_a^b\int_c^d\left|f''_{x,y}\right|dtdsdydx\\
&=\left\|f''_{x,y}\right\|_1(b-a)^2(d-c)^2.
\end{aligned}$$

Substituting in (8) with (12), gives the final inequality in (2), where we have used the fact that

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

Thus the theorem is completely proved. ∎

3. A Numerical Cubature Formula

To illustrate the use of a cubature formula, we form a composite rule from the in equality (2).
Let us consider the arbitrary divisions $I_n : a = x_0 < x_1 < \cdots < x_n = b$ on $[a, b]$ and $J_m : c = y_0 < y_1 < \cdots < y_n = d$ on $[c, d]$, define the sum

$$\mathfrak{F}(f, I_n, J_m, u, v) = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_1(\mathcal{SD}) + \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_2(\mathcal{SD}) - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_3(\mathcal{SD}) \tag{13}$$

where

$$(\mathcal{SD}) := (u, v; x_k, x_{k+1}, y_l, y_{l+1});$$

$h_k := x_{k+1} - x_k$ $(k = 0, 1, 2, \cdots, n-1)$ and $v_l := y_{l+1} - y_l$ $(l = 0, 1, \cdots, m-1)$

Under the above assumptions the following theorem can be obtained.

Theorem 2. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous mapping on $[a, b] \times [c, d]$, then we have the cubature formula*

$$\mathcal{F}(f)(u, v; a, b, c, d) = \mathfrak{F}(f, I_n, J_m, u, v) + R(f, I_n, J_m, u, v), \tag{14}$$

where $\mathfrak{F}(f, I_n, J_m, \cdot, \cdot)$ approximates the Fourier Transform $\mathcal{F}(f)$ at every point $(u, v) \in [a, b] \times [c, d]$, and the remainder term $R(f, I_n, J_m, \cdot, \cdot)$ satisfies the bounds

$$|R(f, I_n, J_m, u, v)| \leq \begin{cases} \frac{1}{9}\left(\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} h_k^2 v_l^2\right)\|f''_{x,y}\|_\infty \\ \left[\frac{2}{(q+1)(q+2)}\right]^{\frac{2}{q}} \left(\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} h_k v_l\right)^{\frac{q+1}{q}} \|f''_{x,y}\|_p \\ \kappa(h)\tau(\nu)\|f''_{x,y}\|_1 \end{cases} \tag{15}$$

where
$\kappa(h) := max\ \{h_k|\ k = 0, \cdots, n-1\}$, *and* $\tau(\nu) := max\ \{\nu_l|\ l = 0, \cdots, m-1\}$.

Proof. Applying Theorem 1 over every subinterval $[x_k, x_{k+1}]$ and $[y_l, y_{l+1}]$, we can state that

$$\left|\mathcal{F}(f)(\mathcal{SD}) - \mathcal{I}_1(\mathcal{SD}) - \mathcal{I}_2(\mathcal{SD}) + \mathcal{I}_3(\mathcal{SD})\right| \leq \begin{cases} \frac{1}{9} h_k^2 v_l^2 \sup_{(s,t)\in[x_k,x_{k+1}]\times[y_l,y_{l+1}]} \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right| \\ \left[\frac{2\left[h_k v_l\right]^{\frac{q+1}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}} \mathcal{DIS} \\ h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right|^p dtds \end{cases}$$

where

$$\mathfrak{DIS} := \left(\int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right|^p dtds \right)^{\frac{1}{p}},$$

Summing over k from 0 to $n-1$ and l from 0 to $m-1$, and using the triangle inequality, we obtain

$$\begin{aligned}
&|R(f, I_n, J_m, u, v)| \\
&\quad = |\mathcal{F}(f)(u, v; a, b, c, d) - \mathfrak{F}(f, I_n, J_m, u, v)| \\
&\quad \leq \begin{cases}
\dfrac{1}{9} \displaystyle\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sup_{(s,t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| h_k^2 v_l^2 \\[2ex]
\left[\dfrac{2 \left(\displaystyle\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} [h_k v_l]^{q+1} \right)^{\frac{1}{2}}}{(q+1)(q+2)} \right]^{\frac{2}{q}} \mathfrak{DIS} \\[2ex]
\displaystyle\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} h_k v_l \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right|^p dtds.
\end{cases}
\end{aligned}$$

where

$$\sup_{(s,t) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| \leq \sup_{(s,t) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right| = \|f''_{x,y}\|_\infty$$

thus the first inequality in (15) is obtained. Using Hölder's discrete inequality, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} [h_k \nu_l]^{\frac{q+1}{q}} \left(\int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right|^p dtds \right)^{\frac{1}{p}} \\
&\quad \leq \left[\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left([h_k \nu_l]^{\frac{q+1}{q}} \right)^q \right]^{\frac{1}{q}} \\
&\qquad \times \left[\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[\left(\int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} \left| \frac{\partial^2 f(s,t)}{\partial x \partial y} \right|^p dtds \right)^{\frac{1}{p}} \right]^p \right]^{\frac{1}{p}} \\
&\quad = \left(\sum_{k=0}^{n-1} \sum_{l=0}^{m-1} (h_k \nu_l)^{q+1} \right)^{\frac{1}{q}} \|f''_{x,y}\|_p
\end{aligned}$$

which proves the second inequality in (15).
For the last inequality, we observe that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} h_k \nu_l \int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}} \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right| dtds \\
&\leq \kappa(h)\tau(v) \sum_{l=0}^{m-1} h_k v_l \int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}} \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right| dtds \\
&= \kappa(h)\tau(v) \int_a^b \int_c^d \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right| dtds \\
&= \kappa(h)\tau(\nu) \|f''_{x,y}\|_1
\end{aligned}$$

and the theorem is completely proved. ∎

In practical applications, it is convenient to consider the equidistant partitioning of the region $[a,b] \times [c,d]$. Thus let

$$\begin{aligned}
&I_n : x_k = a + k \cdot \frac{b-a}{n}, \quad k = 0,1,\cdots,n \quad \text{and} \\
&J_m : y_l = c + l \cdot \frac{d-c}{m}, \quad l = 0,1,\cdots,m,
\end{aligned}$$

and we defined the sum

$$\begin{aligned}
&\mathfrak{F}_{\mathfrak{n},\mathfrak{m}}(f, I_n, J_m, u, v) \\
&\quad = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_1(\mathcal{ES}) + \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_2(\mathcal{ES}) - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \mathcal{I}_3(\mathcal{ES}) \qquad (16)
\end{aligned}$$

where $(\mathcal{ES}) := (u, v; a + k \cdot \frac{b-a}{n}, a + (k+1) \cdot \frac{b-a}{n}, c + l \cdot \frac{d-c}{m}, c + (l+1) \cdot \frac{d-c}{m})$.
The following corollary of Theorem 2 holds:

Corollary 1. *Let f be as defined in Theorem 2. Then we have*

$$\mathcal{F}(f)(u, v; a, b, c, d) = \mathfrak{F}_{\mathfrak{n},\mathfrak{m}}(f, I_n, J_m, u, v) + R_{n,m}(f, I_n, J_m, u, v), \qquad (17)$$

where $\mathfrak{F}_{\mathfrak{n},\mathfrak{m}}(f, I_n, J_m, ., .)$ *approximates the Fourier Transform* $\mathcal{F}(f)$ *at every point* $(u,v) \in [a,b] \times [c,d]$*, and the remainder term* $R_{n,m}(f, I_n, J_m, ., .)$ *satisfies the bounds*

$$\begin{aligned}
&|R_{n,m}(f, I_n, J_m, u, v)| \\
&\leq \begin{cases}
\dfrac{(b-a)^2(d-c)^2}{9nm} \|f''_{x,y}\|_\infty; \\[2ex]
\left[\dfrac{2[(b-a)(d-c)]^{\frac{1+q}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}} \dfrac{\|f''_{x,y}\|_p}{nm}; \\[2ex]
\dfrac{(b-a)(d-c)}{nm} \|f''_{x,y}\|_1.
\end{cases} \qquad (18)
\end{aligned}$$

4. Numerical Experiment

To illustrate the use of the cubature formula, we will employ (13) to approximate the finite Fourier transform of

$$f(x, y) = e^{3x-2y}(x - y), \qquad 0 \leq x, y \leq 1. \tag{19}$$

Since $\mathcal{F}(f)$ can be computed analytically we can gauge the performance of the cubature rule as well as compare it to the theoretical error bound (18).
The results are shown in Table 1 where n^2 is the number of uniform partitions of the domain $[0, 1] \times [0, 1]$. It is clearly evident that the cubature rule performs extremely well and achieves single precision accuracy when $n = 16$. Halving the interval size will increase the accuracy by approximately one and a half orders, and a simple analysis shows that the rate of convergence is at least $O((nm)^{-2})$. The contrasts with the theoretical error which is $O(1/(nm))$. Extending the Peano kernel, that is using a higher order identity to that of (3), may provide a higher order theoretical error result.

In Figure 1, we show a three dimensional plot of the finite Fourier transformed obtained using (13).

n	Num. Error	Ratio	Th. Error
1	0.32E+00	3.11	0.13E+02
2	0.13E-01	25.28	0.33E+01
4	0.48E-04	267.37	0.82E+00
8	0.16E-05	30.63	0.20E+00
16	0.23E-07	67.49	0.51E-01
32	0.34E-09	68.02	0.13E-01
64	0.77E-11	44.09	0.32E-02

Table 1: Numerical error (column 2) and theoretical error (column 4) in approximating the finite Fourier transform of (19) using equation (13).

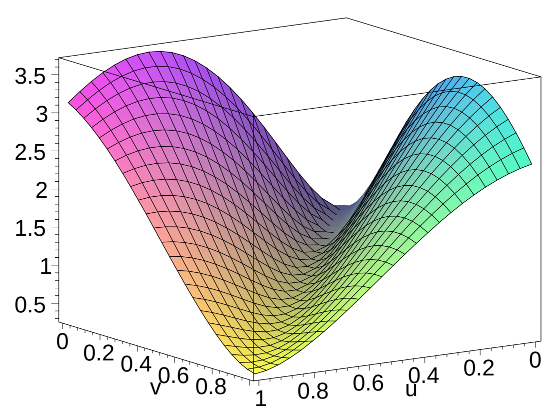

Figure 1: Finite Fourier transform of $f(x, y) = e^{3x-2y}(x - y)$, $0 \leq x, y \leq 1$ evaluated using the rule (13).

5. Conclusion

The current work has modelled a means for estimating the partition required in order to be guaranteed a certain acurracy for the two-dimensional Finite-Fourier transform in term of the complex exponential mean.

References

[1] Bracewell, R. N, *The Fourier Transform and Its Applications*, New York: McGraw-Hill Book Company, (1965).

[2] Brault, J. W, and White, O. R, The analysis and restoration of astronomical data via the fast Fourier transform, *Astron. & Astrophys.*, **13**(1971), 169-189.

[3] Brigham, E. O, *The Fast Fourier Transform and Its Applications*, Englewood Cliffs, NJ: Prentice-Hall, Inc., (1988).

[4] Barnett, N.S, and Dragomir, S.S, An Ostrowski type inequality for double integrals and application for cubature formulae, *Soochow J. Math.,* **27**(1) (2001), 1-10.

[5] Barnett, N.S, and Dragomir, S.S, A Trapezoid type inequality of double integrals, *NonLinear Analysis* , **47**(4) (2001), 2321-2332.

[6] Pachpatte, B.G, On a new Ostrowski type inequality in two independent variables, *Tamkang J. Math.,* **32**(1) (2001), 45-49.