Blow-up solutions of logistic equations with absorption: uniqueness and asymptotics

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This paper is dedicated with deep respect to Professor Haim Brezis on his 60th birthday anniversary

Abstract. We study the uniqueness and expansion properties of the positive blow-up boundary solution of the logistic equation $\Delta u + au = b(x)f(u)$ in a smooth bounded domain Ω. The absorbtion term f is a positive function satisfying the Keller–Osserman condition and such that the mapping $f(u)/u$ is increasing on $(0, +\infty)$, b is nonnegative, while the values of the real parameter a are related to an appropriate semilinear eigenvalue problem. Our analysis is based on the Karamata regular variation theory.

Key words: logistic equation, boundary blow-up, uniqueness, Karamata theory, Keller–Osserman condition, population dynamics.
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1 Introduction

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ be a smooth bounded domain. Denote by $\mathcal{B}$ either the Dirichlet boundary operator $\mathcal{D}u := u$ or the Neumann/Robin boundary operator $\mathcal{R}u := u_\nu + \beta(x)u$, where ν is the unit outward normal to $\partial\Omega$ and $\beta \geq 0$ is in $C^{1,\mu}(\partial\Omega)$, $0 < \mu < 1$.

Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{1}$$

where $f \in C^1[0, \infty)$, $a \in \mathbf{R}$ is a parameter and $b \in C^{0,\mu}(\overline{\Omega})$ satisfies $b \geq 0$, $b \not\equiv 0$ in Ω.

Such equations are also known as the stationary version of the Fisher equation [18] and the Kolmogoroff–Petrovsky–Piscounoff equation [30] and they have been studied by Kazdan–Warner [28], Ouyang [39], del Pino [14] and Du–Huang [15]. We point out that if $f(u) = u^{(N+2)/(N-2)}$, then this equation originates from the Yamabe problem, which is a basic problem in Riemannian geometry (see, e.g., [34]).

The existence of positive solutions of (1) subject to the boundary condition

$$\mathcal{B}u = 0 \qquad \text{on } \partial\Omega \tag{2}$$

has been intensively studied in the case $f(u) = u^p$, $p > 1$ (see, e.g., [1], [2], [12], [14], [19] and [39]); this problem is basic population model (see [24]) and is also related to some prescribed curvature problems in Riemannian

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geometry (see [28] and [39]). Moreover, if $b > 0$ in $\overline{\Omega}$, then it is referred to as the logistic equation and it has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $(-\Delta)$ in Ω subject to the boundary condition (2).

Our general setting includes some simple prototype models from population dynamics. For instance, the problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

is (cf. [37]) the paradigmatic model in population dynamics. More precisely, the above problem has been proposed as a model for population density of a steady-state single species $u(x)$ when Ω is fully surrounded by inhospitable areas. Since the unknown u corresponds to the density of the population, we investigate only nonnegative solutions of this problem. There is a huge amount of recent papers dealing with the logistic equation when $b(x)$ is positive and bounded away from zero. Quite surprisingly, the general problem when the species u is free from crowding effects on some subdomain of Ω, i.e., when $b(x)$ vanishes on some subdomain of Ω, has not been tackled until very recently. Assume that $b(x)$ is positive in a proper smooth subdomain $\Omega' \subset\subset \Omega$ and vanishes outside it. In this context, the heterogeneous environment Ω represents the region inhabited by the species u, a measures its birth rate, while $b(x)$ denotes the capacity of Ω' to support the species u. From the results of Brezis–Oswald [8] (see also [19, 39]), a positive solution of (3) can only exist when $a \in (\lambda_1(\Omega), \lambda_1(\Omega'))$, being unique in that range. Here $\lambda_1(\Omega)$ (resp., $\lambda_1(\Omega')$) stands for the first Dirichlet eigenvalue of $(-\Delta)$ in Ω (resp., Ω'). Paper [21] ascertains the exact pointwise growth of the positive solutions as $a \nearrow \lambda_1(\Omega')$: the solutions grow to infinity uniformly on compact subsets of $\Omega \setminus \Omega'$ and they stabilize in Ω' to the minimal solution of the boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega' \\ u = \infty & \text{on } \partial\Omega' \end{cases} \tag{4}$$

being $a = \lambda_1(\Omega')$ and $b \equiv 0$ on $\partial\Omega'$ in this precise case.

In the understanding of (1), as well as for (3), an important role is played by the zero set of b, namely

$$\Omega_0 := \operatorname{int}\{x \in \Omega : \ b(x) = 0\}$$

where population is free from crowding and symbiosis effects.

We suppose throughout this paper that Ω_0 is smooth (possibly empty), $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.

Our framework includes the Holling–Tanner population model (see [25]) that corresponds to the case $b \equiv 1$ and $f(u) = u^2 + mu/(1+u)$, where m is a real constant. If $m = 0$ we regain the well-known diffusive logistic problem. If $m > 0$ then the term $-mu/(1+u)$ in the equation

$$\Delta u + au - u^2 - \frac{mu}{1+u} = 0$$

is one example of a predation term. In this case, u is considered to be a population of prey whose growth rate is decreased because of the existence of some predators.

By *large* (or *blow-up*) solution of (1), we mean any nonnegative solution u such that $u(x) \to \infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$. Problems related to large solutions have a long history and have been studied by many authors and in many contexts.

Singular value problems having large solutions have been initially studied for the special case $f(u) = e^u$ by Bieberbach [5] (if $N = 2$). Problems of this type arise in Riemannian geometry. More precisely, if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-g^2$ then $\Delta u = g^2 e^{2u}$. This study was continued by Rademacher [40] (if $N = 3$), in connection with some concrete questions arising in the theory of Riemann surfaces, automorphic functions and in the theory of the electric potential in a glowing hollow metal body.

The question of large solutions was later considered in N-dimensional domains and for other classes of nonlinearities (see [3], [4], [9]–[11], [13], [15], [22], [29], [31]–[33], [35], [36], [38]). For instance, Lazer and McKenna [32] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbf{R}^N$ satisfying a uniform external sphere condition and for nonlinearities of the type $b(x)e^u$, where b is continuous and strictly positive on $\overline{\Omega}$.

If $a \equiv 0$ and $b \equiv 1$, Keller and Osserman (see [29], [38]) supplied a necessary and sufficient condition for the existence of large solutions of (1), namely

$$(A_0) \quad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad \text{where} \quad F(t) = \int_0^t f(s)\,ds$$

provided that $f \in C^1[0,\infty)$ is positive and nondecreasing on $(0,\infty)$ with $f(0) = 0$.

In higher dimensions the notion of Gaussian curvature has to be replaced by the scalar curvature. It turns out that if a metric of the form $|ds|^2 = u(x)^{4/(N-2)}|dx|^2$ has constant scalar curvature $-g^2$, then u satisfies (1) for $f(u) = u^{(N+2)/(N-2)}$, $a = 0$ and $b(x) = [(N-2)g^2]/[4(N-1)]$. In a celebrated paper, Loewner and Nirenberg [35] described the precise asymptotic behavior at the boundary of large solutions to this equation and used this result in order to establish the uniqueness of the solution. Their main result is derived under the assumption that $\partial\Omega$ consists of the disjoint union of finitely compact C^∞ manifolds, each having codimension less than $N/2+1$. More precisely, the uniqueness of a large solution is a consequence of the fact that every large solution u satisfies

$$u(x) = \Gamma(d(x)) + o(\Gamma(d(x))) \quad \text{as } d(x) \to 0, \tag{5}$$

where Γ is defined by

$$\int_{\Gamma(t)}^\infty \frac{ds}{\sqrt{2F(s)}} = \left(\frac{(N-2)g^2}{4(N-1)}\right)^{1/2} t, \quad \text{for all } t > 0. \tag{6}$$

Kondrat'ev and Nikishkin [31] established the uniqueness of a large solution for the case $a = 0$, $b = 1$ and $f(u) = u^p$ $(p \geq 3)$, when $\partial\Omega$ is a C^2-manifold and Δ is replaced by a more general second order elliptic operator.

Recently, Dynkin [16] showed that there exist certain relations between hitting probabilities for some Markov processes called superdiffusions and maximal solutions of (1) with $a = 0$, $b = 1$ and $f(u) = u^p$ $(1 < p \leq 2)$. By means of a probabilistic representation, a uniqueness result in domains with nonsmooth boundary was established by le Gall [20] when $p = 2$. We point out that the case $p = 2$ arises in the study of the subsonic motion of a gas. In this connection the question of uniqueness is of special interest.

An existence result for the problem $\Delta u = u^p$ with $1 < p < \infty$ was obtained in [36], where Matero constructed a boundary blow-up solution in a two-dimensional domain with fractal boundary called the von Koch snowflake domain. His approach is based on the comparison with boundary blow-up solutions in a cut-off open cone.

2 Preliminaries

In [1] it is developed an exhaustive study of positive solutions of (1), subject to $u = 0$ on $\partial\Omega$, under the assumption

(A_1) $\quad f \geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

Let H_∞ define the Dirichlet Laplacian on the set $\Omega_0 \subset \Omega$ as the unique self-adjoint operator associated to the quadratic form $\psi_1(u) = \int_\Omega |\nabla u|^2\,dx$ with form domain

$$H^1_D(\Omega_0) = \{u \in H^1_0(\Omega) : \; u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies an exterior cone condition, then $H^1_D(\Omega_0)$ coincides with $H^1_0(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. Set $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$.

Define $\mu_0 := \lim_{u\searrow 0} \frac{f(u)}{u}$ and $\mu_\infty := \lim_{u\to\infty} \frac{f(u)}{u}$. Denote by $\lambda_1(\mu_0)$ (resp., $\lambda_1(\mu_\infty)$) the first eigenvalue of $H_{\mu_0} = -\Delta + \mu_0 b$ (resp., $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H^1_0(\Omega)$.

Cf. [1, Theorem A (bis)], problem (1) subject to $u = 0$ on $\partial\Omega$ has a positive solution if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$; moreover, this solution is unique for a in the above range.

A corresponding result but for large solutions is provided by Theorem 1.1 in [10]. More precisely, if (A_0) and (A_1) are fulfilled, then (1) has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.

Note that, assuming (A_1), any large solution of (1) is *positive* and it can exist only if the Keller–Osserman condition (A_0) holds (see [10, Remark 3.1 and Corollary A.2]).

3 Aims and outcomes

Our purpose is to find general uniqueness results of large solutions of (1) and then to give an exact two-term asymptotic expansion of the blow-up solution near $\partial\Omega$.

These questions find an answer in [22], but only in the special case $f(u) = u^p$ $(p > 1)$, $b > 0$ in Ω and $b \equiv 0$ on $\partial\Omega$ such that

$$b(x) = c[d(x)]^{2\alpha} + o([d(x)]^{2\alpha}) \text{ as } d(x) \to 0, \text{ for some constants } c, \alpha > 0. \tag{7}$$

However, it was shown there that the degenerate case $b \equiv 0$ on $\partial\Omega$ is a *natural* restriction for b inherited from the logistic equation. Therefore, we are still interested in this case and replace (7) by a general condition, namely

(B) $\quad b(x) = k^2(d) + o(k^2(d))$ as $d(x) \to 0$, for some k in $\mathcal{K}$, defined later in §3.2.

But how can be extended the results of [22] to nonlinearities f other than the superlinear powers in order to cover situations when the potential b vanishes in Ω and its behavior near $\partial\Omega$ is different from the power case.

The special feature of this paper is that the theory of *regular variation* plays the key role in developing the answer.

3.1 Regular variation

The theory of regular variation was instituted in 1930 by Karamata [26, 27] and subsequently developed by him and many others. Although Karamata originally introduced his theory in order to use it in Tauberian theorems, regularly varying functions have been later applied in several branches of Analysis: Abelian theorems (asymptotic of series and integrals—Fouries ones in particular), analytic (entire) functions, analytic number theory, etc. The great potential of regular variation for probability theory and its applications was realised by Feller [17] and also stimulated by de Haan [23]. The first monograph on regularly varying functions was written by Seneta [41], while the theory and various applications of the subject are presented in the comprehensive treatise of Bingham, Goldie and Teugels [6].

We denote by RV_ρ the set of all functions that are regularly varying at infinity with index $\rho \in \mathbf{R}$. For brevity, we do not specify at infinity when the regular variation occurs there. We recall in Appendix the basic definitions and main properties of the class of regularly varying functions.

Definition 1 extends to *regular variation at the origin.* Precisely, we say that Z is regularly varying (on the right) at the origin with index ρ (and write, $Z \in RV_\rho(0+)$) if $Z(1/u) \in RV_{-\rho}$. Moreover, by $Z \in NRV_\rho(0+)$ we mean that $Z(1/u) \in NRV_{-\rho}$. The meaning of NRV_ρ is given by (72) in the Appendix.

For $\alpha \geq 0$, we define

$$\mathcal{P}_\alpha = \left\{ \begin{array}{ll} k: & k(\frac{1}{u}) = \frac{c_0}{u^\alpha} \exp\{\int_{c_1}^u \frac{E(t)}{t}\, dt\} \ (u \geq c_1), \text{ where } \alpha \geq E \in C[c_1, \infty), \\ & \lim_{u\to\infty} E(u) = 0 \text{ and } c_i > 0 \text{ are constants} \end{array} \right\}.$$

By Proposition 5, $\mathcal{P}_\alpha = NRV_\alpha(0+)$, for $\alpha > 0$ and $\mathcal{P}_0$ is the set of all normalised slowly varying functions at the origin that are nondecreasing on $(0, \nu)$, for some $\nu > 0$.

3.2 Our framework

In [10] we prove the existence of large solutions of (1) in the general setting (A_0) and (A_1). Note that the paper [22] gives the uniqueness of a large solution for the canonical varying function $f(u) = u^p \in NRV_p$ $(p > 1)$. Thus, it is natural to assume (A_1) and $f \in NRV_{\rho+1}$, for some $\rho > 0$. We do not need to require (A_0), since this is automatically fulfilled (see Remark 1 (ii)).

Recall that $f \in NRV_{\rho+1}$ ($\rho > 0$) if and only if $f(u)$ can be written as

$$f(u) = Cu^{\rho+1}\exp\left\{\int_B^u \frac{\varphi(t)}{t}\,dt\right\}, \quad \forall u \geq B, \tag{8}$$

for some constants B, $C > 0$, where $\varphi \in C[B,\infty)$ vanishes at infinity. But, for B large enough, $f(u)/u$ is increasing on $[B,\infty)$. So, the assumptions on f are achieved once we "paste" on $[0,B]$ a suitable smooth function to fulfil (A_1). For instance, we may simply define $f(u) = Cu^{\rho+1}\exp\left\{\int_0^u \frac{z(t)}{t}\,dt\right\}$, for all $u \geq 0$, where $0 \leq z \in C[0,\infty)$ satisfies $\lim_{t\searrow 0} z(t)/t \in [0,\infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p\ln(u+1)$, and $f(u) = u^p\arctan u$, $p > 1$, fall into this category.

Remark 1 *Let (A_1) be fulfilled and $f \in NRV_{\rho+1}$. Then*

(i) $\rho \geq 0$. Indeed, if $\rho < 0$ then Proposition 6 yields $\lim_{u\to\infty} f(u)/u = 0$, which contradicts (A_1).

(ii) *If $\rho \neq 0$, then (A_0) holds* (since $\lim_{u\to\infty} f(u)/u^r = \infty$, for all $r \in (1, 1+\rho)$). *The converse implication is not always true as we can see by taking $f(u) = u\ln^4(u+1)$. But, there are cases for which $\rho = 0$ and (A_0) fails so that (1) has* no *large solutions. This is illustrated by $f(u) = u\ln^j(u+1)$ with $j \in [0,2]$.*

Regarding b, assume that (B) holds with $\mathcal{K}$ defined as the set of all positive, nondecreasing $k \in C^1(0,\nu)$ that satisfy

$$\lim_{t\searrow 0}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i,\ i = \overline{0,1}.$$

Remark 2 *For every $k \in \mathcal{K}$, $\ell_0 = 0$ and $\ell_1 \in [0,1]$.* Indeed, $\ell_0 = 0$ since $\lim_{t\searrow 0}\ln\left(\int_0^t k(s)\,ds\right) = -\infty$. By l'Hospital's rule, we derive $\ell_1 \geq 0$. By the definition of ℓ_1 and monotonicity of k, we deduce $\ell_1 \leq 1$.

Consequently, $\mathcal{K} = \mathcal{K}_{(01]} \cup \mathcal{K}_0$, where $\mathcal{K}_{(01]} := \{k \in \mathcal{K} :\ \ell_1 \in (0,1]\}$ and $\mathcal{K}_0 := \{k \in \mathcal{K} :\ \ell_1 = 0\}$.

We now show how to built at once functions $k \in \mathcal{K}$ for each $\ell_1 \in [0,1]$.

Proposition 1 *Let $S \in NRV_m$, for some $m > 0$. Hence*

(i) *$k(t) = \exp\{-S(1/t)\} \in \mathcal{K}$ with $\ell_1 = 0$.*

(ii) *$k(t) = 1/S(1/t) \in \mathcal{K}$ with $\ell_1 = 1/(m+1) \in (0,1)$.*

(iii) *$k(t) = 1/\ln[S(1/t)] \in \mathcal{K}$ with $\ell_1 = 1$.*

Note that (7) is the particular case of (B) with $k(t) = \sqrt{c}\,t^\alpha \in \mathcal{K} \cap \mathcal{P}_\alpha$ ($\alpha > 0$). Hence, it is natural to ask us if there is any connection between $\mathcal{K}$ and $\mathcal{P}_\alpha$. The answer is that $\mathcal{K}$ is a huge class of functions, even larger than $\bigcup_{\alpha\geq 0}\mathcal{P}_\alpha$. More precisely, we prove that $\mathcal{K}_{(01]} \equiv \bigcup_{\alpha\geq 0}\mathcal{P}_\alpha$ and $\mathcal{K}_0 \equiv \mathcal{R}_0$, where

$$\mathcal{R}_0 = \left\{\begin{array}{l} k:\quad k(\frac{1}{u}) = \frac{d_0 u}{\Lambda(u)}\exp\{-\int_{d_1}^u \frac{ds}{s\Lambda(s)}\}\ (u \geq d_1),\ \text{where } 0 < \Lambda \in C^1[d_1,\infty), \\ \qquad \lim_{u\to\infty}\Lambda(u) = 0,\ \lim_{u\to\infty} u\Lambda'(u) = 0 \text{ and } d_i > 0 \text{ are constants} \end{array}\right\}.$$

Proposition 2 (Characterisation Theorem of $\mathcal{K}$). *We have*

$$\mathcal{K} = \bigcup_{\alpha\geq 0}\mathcal{P}_\alpha \bigcup \mathcal{R}_0.$$

Moreover, $k \in \mathcal{K}_{(01]}$ if and only if $k \in \mathcal{P}_\alpha$ for some $\alpha \geq 0$, where $\alpha = (1-\ell_1)/\ell_1$.

4 Main results

We first prove the uniqueness of a large solution for (1). This lies upon the crucial observation that all blow-up solutions have the same boundary behavior (9). More exactly, we find

Theorem 1 *Assume $f \in NRV_{\rho+1}$ with $\rho > 0$, (A_1) and (B) hold. Then, for any $a \in (-\infty, \lambda_{\infty,1})$, (1) admits a unique large solution u_a. Moreover, the asymptotic behavior is given by*

$$u_a(x) = \xi_0 h(d) + o(h(d)) \quad \text{as } d(x) \to 0, \tag{9}$$

where $\xi_0 = \left(\frac{2+\ell_1\rho}{2+\rho}\right)^{1/\rho}$ and h is defined by

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\, ds, \quad \forall t \in (0, \nu). \tag{10}$$

Through a very general and unitary approach, our Theorem 1 recovers the previous results of [35] and [22]. Indeed, for $k(t) = [(N-2)g^2/4(N-1)]^{1/2}$ in (B) and $f(u) = u^{(N+2)/(N-2)}$, (9) reduces to relation (5), prescribed by Loewner and Nirenberg [35] for their problem. Moreover, if $f(u) = u^p$ $(p > 1)$ and $k(t) = \sqrt{c}\, t^\alpha$ $(\alpha > 0)$, then we regain the uniqueness results of [22].

The aim of Theorems 2 and 3 is to find the two-term blow-up rate of u_a under the assumptions of Theorem 1, but with (B) subsequently replaced by

$(\tilde{B})$ $\quad b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta))$ as $d(x) \to 0$, where $\theta > 0$, $\tilde{c} \in \mathbf{R}$ are constants.

The two-term asymptotic expansion of u_a near $\partial\Omega$ depends on the chosen subclass for $k \in \mathcal{K}$ and the additional hypotheses on f (by means of φ given by (8)).

In order to avoid repetition, we define

$$\begin{aligned}
\mathcal{F}_{\rho\eta} &= \{f \in NRV_{\rho+1}\ (\rho > 0) : \text{ either } \varphi \in RV_\eta \text{ or } -\varphi \in RV_\eta\},\ \eta \in (-\rho-2, 0] \\
\mathcal{F}_{\rho 0,\tau} &= \{f \in \mathcal{F}_{\rho 0} : \lim_{u\to\infty} (\ln u)^\tau \varphi(u) = \ell^\star \in \mathbf{R}\},\ \tau \in (0, \infty) \\
\mathcal{P}_{\alpha,\tau} &= \{k \in \mathcal{P}_\alpha : \lim_{u\to\infty} (\ln u)^\tau E(u) = \ell_\sharp \in \mathbf{R}\},\ \alpha \in [0, \infty) \\
\mathcal{R}_{0,\zeta} &= \left\{k \in \mathcal{R}_0 : \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbf{R}\right\},\ \zeta \in (0, \infty).
\end{aligned}$$

Further in the paper, η, τ, α, and ζ are understood in the above range.

Theorem 2 *Assume $k \in \mathcal{R}_{0,\zeta}$, (A_1), $(\tilde{B})$ and any one of the cases*

(i) *$f(u) = Cu^{\rho+1}$ in a neighborhood of infinity (i.e., $\varphi \equiv 0$ in (8)).*

(ii) *$f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$.*

(iii) *$f \in \mathcal{F}_{\rho 0,\tau_1}$ with $\tau_1 = \varpi/\zeta$, where $\varpi = \min\{\theta, \zeta\}$.*

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, the two-term blow-up rate of u_a is

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{as } d(x) \searrow 0 \tag{11}$$

where

$$\chi = \begin{cases} -\dfrac{(1+\zeta)\ell_\star}{2\zeta} \operatorname{Heaviside}(\theta - \zeta) - \dfrac{\tilde{c}}{\rho} \operatorname{Heaviside}(\zeta - \theta) := \chi_1 & \text{for (i) and (ii)} \\ \chi_1 - \dfrac{\ell^\star}{\rho}\left(\dfrac{-\rho\ell_\star}{2}\right)^{\tau_1}\left(\dfrac{1}{\rho+2} + \ln\xi_0\right) & \text{for (iii).} \end{cases}$$

Theorem 3 *Assume $k \in \mathcal{P}_{\alpha,\tau}$, (A_1), $(\tilde{B})$ and any one of the cases*

(i) *$f \in \mathcal{F}_{\rho\eta}$ with $\eta\ell_\sharp \neq 0$.*

(ii) *$f \in \mathcal{F}_{\rho 0,\tau}$ with $(\alpha^2 + \ell_\sharp^2)[(\ell^\star)^2 + \ell_\sharp^2] \neq 0$.*

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, the two-term blow-up rate of u_a is

$$u_a(x) = \xi_0 h(d)[1 + \tilde{\chi}\,(-\ln d)^{-\tau} + o((-\ln d)^{-\tau})] \quad \textit{as } d(x) \searrow 0, \tag{12}$$

where

$$\tilde{\chi} = \begin{cases} \dfrac{\ell_\sharp}{(\alpha+1)(\rho+2\alpha+2)} := \chi_2 \quad \textit{for } \text{(i)} \\ \chi_2 - \dfrac{\ell^\star}{\rho}\left(\dfrac{\rho}{2(\alpha+1)}\right)^\tau \left[\dfrac{2\alpha}{(\rho+2)(\rho+2\alpha+2)} + \ln \xi_0\right] \quad \textit{for } \text{(ii)}. \end{cases}$$

We point out that the asymptotic general results stated in the above theorems do not concern the difference or the quotient of $u(x)$ and $\psi(d(x))$, as established in [4, 5, 33, 40] for $a = 0$ and $b = 1$, where ψ is a large solution of

$$\psi''(r) = f(\psi(r)) \qquad \text{on } (0,\infty)\,.$$

For instance, Bieberbach [5] and Rademacher [40] proved that $|u(x) - \psi(d(x))|$ is bounded in a neighborhood of the boundary. Their result was improved by Bandle and Essén [3] who established the more precise estimate $\lim_{d(x)\to 0}(u(x) - \psi(d(x))) = 0$.

The next result specifies the subset $\mathcal{K}_{0,\zeta}$ of $\mathcal{K}_0$ (resp., the subset $\mathcal{K}_{(01],\tau}$ of $\mathcal{K}_{(01]}$) which is equivalent to $\mathcal{R}_{0,\zeta}$ (resp., $\bigcup_{\alpha\geq 0}\mathcal{P}_{\alpha,\tau}$). This reveals, on the one hand, the properties of k in $\mathcal{R}_{0,\zeta}$ (resp., $\mathcal{P}_{\alpha,\tau}$) captured through $\Lambda(u)$ (resp., $E(u)$). On the other hand, the selection of functions k in $\mathcal{K}_{0,\zeta}$ (resp., $\mathcal{K}_{(01],\tau}$) can be carried out by deciding whether or not the form of k is like that described by $\mathcal{R}_{0,\zeta}$ (resp., $\mathcal{P}_{\alpha,\tau}$). Set

$$\begin{aligned} \mathcal{K}_{0,\zeta} &= \left\{ k \in \mathcal{K}_0 : \lim_{t\searrow 0} \frac{1}{t^\zeta}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' := L_\star \in \mathbf{R} \right\}, \\ \mathcal{K}_{(01],\tau} &= \left\{ k \in \mathcal{K}_{(01]} : \lim_{t\searrow 0} (-\ln t)^\tau \left[\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' - \ell_1\right] := L_\sharp \in \mathbf{R} \right\}. \end{aligned}$$

Proposition 3 *The following hold:*
(i) $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$ *and the relation between $L_\star$ and $\ell_\star$ is*

$$\zeta L_\star + (1+\zeta)\ell_\star = 0.$$

(ii) $\bigcup_{\alpha\geq 0}\mathcal{P}_{\alpha,\tau} \equiv \mathcal{K}_{(01],\tau}$. *Moreover, $k \in \mathcal{P}_{\alpha,\tau}$ ($\alpha \geq 0$) if and only if $k \in \mathcal{K}_{(01],\tau}$, where $\alpha = (1-\ell_1)/\ell_1$ and the relation between $L_\sharp$ and $\ell_\sharp$ is*

$$(1+\alpha)^2 L_\sharp - \ell_\sharp = 0.$$

For the proofs of Propositions 1–3 we refer to §6.

5 Properties of f and h

Firstly, we show that $f \in NRV_{\rho+1}$ can be stated in different ways, provided that (A_1) holds. Secondly, under the assumptions of Theorem 1, we explore a whole range of properties for the function h, defined implicitely by (10) (see Lemma 2). Function h can be seen as the *link* between f and k. In view of (9), the information we can get about h is critical in describing the unique large solution u_a of (1). Thirdly, we study some asymptotical properties of $f \in \mathcal{F}_{\rho\eta}$ (see Lemma 3).

Lemma 1 *Assume (A_1). The following assertions are equivalent:*
(i) $f \in NRV_{\rho+1}$.
(ii) $f' \in RV_\rho$.
(iii) $\lim_{u\to\infty} uf'(u)/f(u) := \vartheta < \infty$.
(iv) $\lim_{u\to\infty} (F/f)'(u) := \gamma > 0$.
Moreover, ρ, ϑ and γ are connected by $\gamma = 1/(\rho+2) = 1/(\vartheta+1)$.

Proof. For the equivalence between (i) and (iii) we refer to Appendix.

Since $\rho \geq 0$ (by Remark 1 (i)), Proposition 7 (i) shows that (iii) is implied by (ii) and $\vartheta = \rho+1$. The converse implication follows by Proposition 8 (i).

We now prove that (iii) is equivalent to (iv). If (iii) holds, then by L'Hospital's rule $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = 1+\vartheta$. Hence $\frac{\vartheta}{1+\vartheta} = \lim_{u\to\infty}[1-(F/f)'(u)]$ i.e., $\gamma = \frac{1}{\vartheta+1}$.

Conversely, (iv) yields $(F/f)' \geq \gamma/2$ on $[s_1,\infty)$, for some $s_1 > 0$. Thus,

$$(F/f)(u) \geq \gamma(u-s_1)/2 + (F/f)(s_1)\,, \qquad \forall u \geq s_1\,.$$

Passing to the limit as $u \to \infty$, we find $\lim_{u\to\infty} \frac{F(u)}{f(u)} = \infty$. Thus, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = \frac{1}{\gamma}$. Since $1-\gamma :=$ $\lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)}$, we obtain $\lim_{u\to\infty} \frac{uf'(u)}{f(u)} = \frac{1-\gamma}{\gamma}$. So, (iii) holds with $\vartheta = (1-\gamma)/\gamma$. ∎

Corollary 1 *If $f \in NRV_{\rho+1}$ with $\rho > 0$ and (A_1) holds, then*

$$\lim_{u\to\infty} \Xi(u) := \lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u)\int_u^\infty [F(s)]^{-1/2}\,ds} = \frac{\rho}{2(\rho+2)}.$$

Proof. By L'Hospital's rule, $\lim_{u\to\infty} \frac{F(u)}{f^2(u)} = 0$. Using Lemma 1 and L'Hospital's rule, we deduce

$$\lim_{u\to\infty} \Xi(u) = \lim_{u\to\infty} \left(-\frac{1}{2} + \frac{f'(u)F(u)}{f^2(u)}\right) = \frac{1}{2} - \gamma = \frac{\rho}{2(\rho+2)}.$$

Lemma 2 *Assume $f \in NRV_{\rho+1}$ with $\rho > 0$, (A_1) and (B) hold. Then function h, defined by (10), has the following properties:*

(i) $h \in C^2(0,\nu)$, $\lim_{t\searrow 0} h(t) = \infty$ *and* $\lim_{t\searrow 0} h'(t) = -\infty$.

(ii) $\lim_{t\searrow 0} h''(t)/[k^2(t)f(h(t)\xi)] = (2+\rho\ell_1)/[\xi^{\rho+1}(2+\rho)]$, $\forall \xi > 0$.

(iii) $\lim_{t\searrow 0} h(t)/h''(t) = \lim_{t\searrow 0} h'(t)/h''(t) = \lim_{t\searrow 0} h(t)/h'(t) = 0$.

(iv) $\lim_{t\searrow 0} h(t)h''(t)/[h'(t)]^2 = (2+\rho\ell_1)/2$.

(v) $\lim_{t\searrow 0}[\ln k(t)]/[\ln h(t)] = \rho(\ell_1-1)/2$.

(vi) $\lim_{t\searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2+\rho\ell_1)$.

(vii) $\lim_{t\searrow 0} h(t)/[t^2h''(t)] = \rho^2\ell_1^2/[2(2+\rho\ell_1)]$.

(viii) $\lim_{t\searrow 0} h(t)/[th'(t)] = \lim_{t\searrow 0}[\ln t]/[\ln h(t)] = -\rho\ell_1/2$.

(ix) *Assuming $\ell_1 = 0$, we find $\lim_{t\searrow 0} t^j h(t) = \infty$, for all $j > 0$.*

(x) *Furthermore, if $k \in \mathcal{R}_{0,\zeta}$ then*

$$\lim_{t\searrow 0} \frac{1}{t^\zeta \ln h(t)} = -\frac{\rho\ell_\star}{2} \quad \text{and} \quad \lim_{t\searrow 0} \frac{h'(t)}{t^{\zeta+1}h''(t)} = \frac{\rho\ell_\star}{2\zeta}\,. \tag{13}$$

Proof. (i) and (ii). Using (10), we see that $h \in C^2(0,\nu)$ and $\lim_{t\searrow 0} h(t) = \infty$. Next, for any $t \in (0,\nu)$, we have $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t)f(h(t))\left\{1 + 2\Xi(h(t))\left[\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' - 1\right]\right\}. \tag{14}$$

By Corollary 1 and (14), we conclude (ii) for $\xi = 1$. But $f \in RV_{\rho+1}$ so that we easily derive (ii), for each $\xi > 0$. Consequently, $h''(t) > 0$ on $(0,\delta)$ for some $\delta > 0$. This and $\lim_{t\searrow 0} h(t) = \infty$ imply $\lim_{t\searrow 0} h'(t) = -\infty$.

(iii). Using (ii), Corollary 1 and Remark 2, we obtain

$$\lim_{t\searrow 0} \frac{h'(t)}{h''(t)} = \frac{-2(2+\rho)}{2+\rho\ell_1} \lim_{t\searrow 0} \frac{\int_0^t k(s)\,ds}{k(t)}\,\Xi(h(t)) = \frac{-\rho\ell_0}{2+\rho\ell_1} = 0. \tag{15}$$

Thus, from $\lim_{t\searrow 0} h'(t) = -\infty$ and L'Hospital's rule, we infer that $\lim_{t\searrow 0} \frac{h(t)}{h'(t)} = 0$. This and (15) lead to $\lim_{t\searrow 0} \frac{h(t)}{h''(t)} = 0$ which concludes (iii).

(iv). By L'Hospital's rule, $\lim_{u\to\infty}\frac{uf(u)}{F(u)}=\frac{1}{\gamma}=\rho+2$. Using (ii), we have

$$\lim_{t\searrow 0}\frac{h(t)h''(t)}{[h'(t)]^2}=\lim_{t\searrow 0}\frac{h''(t)}{k^2(t)f(h(t))}\,\frac{h(t)f(h(t))}{2F(h(t))}=\frac{2+\rho\ell_1}{2}. \tag{16}$$

(v). By Corollary 1, we obtain

$$\lim_{t\searrow 0}\frac{k'(t)}{k(t)}\,\frac{h(t)}{h'(t)}=\lim_{t\searrow 0}\frac{h(t)f(h(t))}{F(h(t))}\,\frac{-k'(t)\left(\int_0^t k(s)\,ds\right)}{k^2(t)}\,\Xi(h(t))=\frac{\rho(\ell_1-1)}{2}. \tag{17}$$

By L'Hospital's rule, it follows that $\lim_{t\searrow 0}\frac{\ln k(t)}{\ln h(t)}=\frac{\rho(\ell_1-1)}{2}$.

(vi). Using (ii) and Corollary 1, we find

$$\lim_{t\searrow 0}\frac{h'(t)}{th''(t)}=\frac{-2(2+\rho)}{2+\rho\ell_1}\lim_{t\searrow 0}\frac{\int_0^t k(s)\,ds}{tk(t)}\,\Xi(h(t))=\frac{-\rho\ell_1}{2+\rho\ell_1}.$$

(vii). By (iv) and (vi), we have

$$\lim_{t\searrow 0}\frac{h(t)}{t^2h''(t)}=\lim_{t\searrow 0}\frac{h(t)h''(t)}{[h'(t)]^2}\left[\frac{h'(t)}{th''(t)}\right]^2=\frac{\rho^2\ell_1^2}{2(2+\rho\ell_1)}.$$

(viii). If $\ell_1\neq 0$, then by (vi) and (vii), we have

$$\lim_{t\searrow 0}\frac{h(t)}{th'(t)}=\lim_{t\searrow 0}\frac{h(t)}{t^2h''(t)}\,\frac{th''(t)}{h'(t)}=\frac{-\rho\ell_1}{2}.$$

If $\ell_1=0$, then

$$\lim_{t\searrow 0}\frac{k(t)}{tk'(t)}=\lim_{t\searrow 0}\frac{k^2(t)}{k'(t)\left(\int_0^t k(s)\,ds\right)}\,\frac{\int_0^t k(s)\,ds}{tk(t)}=0 \tag{18}$$

which, together with (17), yields

$$\lim_{t\searrow 0}\frac{h(t)}{th'(t)}=\lim_{t\searrow 0}\frac{k'(t)h(t)}{k(t)h'(t)}\,\frac{k(t)}{tk'(t)}=0.$$

Therefore, by L'Hospital's rule, we conclude that

$$\lim_{t\searrow 0}\frac{\ln t}{\ln h(t)}=\lim_{t\searrow 0}\frac{h(t)}{th'(t)}=\frac{-\rho\ell_1}{2}.$$

(ix). By (viii), $\lim_{t\searrow 0}\frac{\ln t}{\ln h(t)}=0$ provided that $\ell_1=0$. Thus,

$$\lim_{t\searrow 0}\ln[t^j h(t)]=\lim_{t\searrow 0}\left[1+j\frac{\ln t}{\ln h(t)}\right]\ln h(t)=\infty,\qquad \forall j>0$$

which proves (ix).

(x). Assume that $k\in\mathcal{R}_{0,\zeta}$, for some $\zeta>0$. Then, by Proposition 3, $k\in\mathcal{K}_{0,\zeta}$ and $\lim_{t\searrow 0}\frac{\int_0^t k(s)\,ds}{t^{\zeta+1}k(t)}=L_\star/(\zeta+1)=-\ell_\star/\zeta$. It follows that

$$\frac{-\ell_\star}{\zeta}=\lim_{t\searrow 0}\frac{\int_0^t k(s)\,ds}{t^{\zeta+1}k(t)}\,\frac{k^2(t)}{k'(t)\left(\int_0^t k(s)\,ds\right)}=\lim_{t\searrow 0}\frac{k(t)}{t^{\zeta+1}k'(t)}. \tag{19}$$

By L'Hospital's rule, we obtain

$$\lim_{t\searrow 0}\frac{1}{t^\zeta\ln k(t)}=\lim_{t\searrow 0}\frac{-\zeta k(t)}{t^{\zeta+1}k'(t)}=\ell_\star. \tag{20}$$

Using assertion (v) and (20), we find

$$\lim_{t\searrow 0}\frac{1}{t^{\zeta}\ln h(t)}=\lim_{t\searrow 0}\frac{1}{t^{\zeta}\ln k(t)}\,\frac{\ln k(t)}{\ln h(t)}=\frac{-\rho\ell_{\star}}{2}.$$

By Corollary 1 and (15), we deduce

$$\lim_{t\searrow 0}\frac{h'(t)}{t^{\zeta+1}h''(t)}=-\frac{\rho}{2}\lim_{t\searrow 0}\frac{\int_0^t k(s)\,ds}{t^{\zeta+1}k(t)}=\frac{\rho\ell_{\star}}{2\zeta}.$$

This completes the proof of the lemma. ■

For any $u>0$, define

$$T_{1,\tau}(u)=\left[\frac{\rho}{2(\rho+2)}-\Xi(u)\right](\ln u)^{\tau}\quad\text{and}\quad T_{2,\tau}(u)=\left[\frac{f(\xi_0 u)}{\xi_0 f(u)}-\xi_0^{\rho}\right](\ln u)^{\tau}. \tag{21}$$

Remark 3 *If $\varphi\equiv 0$ in (8), then $T_{1,\tau}(u)=T_{2,\tau}(u)=0$, for any $u\geq B$.*

Lemma 3 *Assume (A_1) and $f\in\mathcal{F}_{\rho\eta}$. The following hold:*
(i) *If $f\in\mathcal{F}_{\rho 0,\tau}$, then*

$$\lim_{u\to\infty}T_{1,\tau}(u)=\frac{-\ell^{\star}}{(\rho+2)^2}\quad\text{and}\quad\lim_{u\to\infty}T_{2,\tau}(u)=\xi_0^{\rho}\ell^{\star}\ln\xi_0.$$

(ii) *If $f\in\mathcal{F}_{\rho\eta}$ with $\eta\neq 0$, then $\lim_{u\to\infty}T_{1,\tau}(u)=\lim_{u\to\infty}T_{2,\tau}(u)=0$.*

Proof. Using $\lim_{u\to\infty}\frac{uf(u)}{F(u)}=\rho+2$, Corollary 1 and L'Hospital's rule, we find

$$\begin{aligned}\lim_{u\to\infty}T_{1,\tau}(u)&=\frac{\rho}{2}\lim_{u\to\infty}\frac{\frac{\rho}{2(\rho+2)}\int_u^{\infty}[F(s)]^{-1/2}\,ds-\frac{\sqrt{F(u)}}{f(u)}}{u[F(u)]^{-1/2}(\ln u)^{-\tau}}\\&=\lim_{u\to\infty}\left[\frac{\rho+1}{\rho+2}-\frac{F(u)f'(u)}{f^2(u)}\right](\ln u)^{\tau}:=\lim_{u\to\infty}Q_{1,\tau}(u).\end{aligned}$$

A simple calculation shows that, for any $u>0$,

$$\begin{aligned}Q_{1,\tau}(u)&=\frac{1}{\rho+2}\left[\rho+1-\frac{uf'(u)}{f(u)}\right](\ln u)^{\tau}+\frac{uf'(u)}{f(u)}\left[\frac{1}{\rho+2}-\frac{F(u)}{uf(u)}\right](\ln u)^{\tau}\\&=:\frac{1}{\rho+2}\,Q_{2,\tau}(u)+\frac{uf'(u)}{f(u)}\,Q_{3,\tau}(u).\end{aligned}$$

Since (8) holds with $\varphi\in RV_{\eta}$ or $-\varphi\in RV_{\eta}$, we can assume $B>0$ such that $\varphi\neq 0$ on $[B,\infty)$. For any $u>B$, we have $Q_{2,\tau}(u)=-\varphi(u)(\ln u)^{\tau}$ and

$$Q_{3,\tau}(u)=\left(-\int_0^B f(s)\,ds+\frac{CB^{\rho+2}}{\rho+2}\right)\frac{(\ln u)^{\tau}}{uf(u)}+\frac{\int_B^u f(s)\varphi(s)\,ds}{(\rho+2)uf(u)\varphi(u)}\varphi(u)(\ln u)^{\tau}.$$

Since either $f\varphi\in RV_{\rho+\eta+1}$ or $-f\varphi\in RV_{\rho+\eta+1}$, Proposition 7 (i) leads to

$$\lim_{u\to\infty}\frac{uf(u)\varphi(u)}{\int_B^u f(x)\varphi(x)\,dx}=\rho+\eta+2.$$

If (i) holds, then $\lim_{u\to\infty}Q_{2,\tau}(u)=-\ell^{\star}$ and $\lim_{u\to\infty}Q_{3,\tau}(u)=\ell^{\star}/(\rho+2)^2$. Thus,

$$\lim_{u\to\infty}T_{1,\tau}(u)=\lim_{u\to\infty}Q_{1,\tau}(u)=-\ell^{\star}/(\rho+2)^2.$$

If (ii) holds, then $\lim_{u\to\infty}(\ln u)^\tau \varphi(u) = 0$ (see Proposition 6). It follows that

$$\lim_{u\to\infty} Q_{2,\tau}(u) = \lim_{u\to\infty} Q_{3,\tau}(u) = 0$$

which yields $\lim_{u\to\infty} T_{1,\tau}(u) = 0$. Note that the proof is finished if $\xi_0 = 1$, since $T_{2,\tau}(u) = 0$ for each $u > 0$.

Arguing by contradiction, let us suppose that $\xi_0 \neq 1$. Then, by (8),

$$T_{2,\tau}(u) = \xi_0^\rho \left[\exp\left\{ \int_u^{\xi_0 u} \frac{\varphi(t)}{t}\, dt \right\} - 1 \right] (\ln u)^\tau, \quad \forall u > B/\xi_0.$$

But, $\lim_{u\to\infty} \varphi(us)/s = 0$, uniformly with respect to $s \in [\xi_0, 1]$. So

$$\lim_{u\to\infty} \int_u^{\xi_0 u} \frac{\varphi(t)}{t}\, dt = \lim_{u\to\infty} \int_1^{\xi_0} \frac{\varphi(su)}{s}\, ds = 0$$

which leads to

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty} \left(\int_u^{\xi_0 u} \frac{\varphi(t)}{t}\, dt \right) (\ln u)^\tau.$$

If (i) occurs then, by Proposition 4,

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty} (\ln u)^\tau \varphi(u) \int_1^{\xi_0} \frac{\varphi(tu)}{\varphi(u)} \frac{dt}{t} = \xi_0^\rho \ell^\star \ln \xi_0.$$

If (ii) occurs, then by L'Hospital's rule and Proposition 6, we infer that

$$\lim_{u\to\infty} T_{2,\tau}(u) = \frac{-\xi_0^\rho}{\tau} \lim_{u\to\infty} \left[\varphi(\xi_0 u) - \varphi(u)\right] (\ln u)^{\tau+1} = 0.$$

The proof of our lemma is now complete. ∎

6 Characterisation of $\mathcal{K}$ and its subclasses

We present here the proofs of Propositions 1–3.

Proof of Proposition 1. The assumption $S \in NRV_m$ yields $\lim_{u\to\infty} \frac{uS'(u)}{S(u)} = m > 0$. Thus, in any of the cases (i), (ii) or (iii), $\lim_{t\searrow 0} k(t) = 0$ and k is an increasing C^1-function on $(0, \nu)$, for $\nu > 0$ small enough.

(i) It is clear that $\lim_{t\searrow 0} \frac{tk'(t)}{k(t)\ln k(t)} = \lim_{t\searrow 0} \frac{-S'(1/t)}{tS(1/t)} = -m$. By l'Hospital's rule, $\ell_0 = \lim_{t\searrow 0} \frac{k(t)}{k'(t)} = 0$ and $\lim_{t\searrow 0} \frac{\left(\int_0^t k(s)\, ds\right)\ln k(t)}{tk(t)} = -\frac{1}{m}$. Consequently, $1 - \ell_1 := \lim_{t\searrow 0} \frac{\left(\int_0^t k(s)\, ds\right)k'(t)}{k^2(t)} = 1$.

(ii) We see that $\lim_{t\searrow 0} \frac{tk'(t)}{k(t)} = \lim_{t\searrow 0} \frac{S'(1/t)}{tS(1/t)} = m$. By l'Hospital's rule, $\ell_0 = 0$ and $\lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{tk(t)} = \frac{1}{m+1}$. So, $\ell_1 = 1 - \lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{tk(t)} \frac{tk'(t)}{k(t)} = \frac{1}{m+1}$.

(iii) We have $\lim_{t\searrow 0} \frac{tk'(t)}{k^2(t)} = \lim_{t\searrow 0} \frac{S'(1/t)}{tS(1/t)} = m$. By l'Hospital's rule, we find $\lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{tk(t)} = 1$. Thus, $\ell_0 = 0$ and $\ell_1 = 1 - \lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{t} \frac{tk'(t)}{k^2(t)} = 1$. ∎

Proof of Proposition 2. Let us denote $P(u) = k(1/u)$.

Let $k \in \mathcal{K}$ be arbitrary. Assume that $\ell_1 \neq 0$. A simple calculation shows that

$$(0,\infty) \ni \frac{1}{\ell_1} = \lim_{t\searrow 0} \frac{tk(t)}{\int_0^t k(s)\, ds} = \lim_{u\to\infty} \frac{\frac{P(u)}{u}}{\int_u^\infty \frac{P(s)}{s^2}\, ds} \tag{22}$$

$$\lim_{t\searrow 0} \frac{tk'(t)}{k(t)} = \lim_{t\searrow 0} \frac{\left(\int_0^t k(s)\, ds\right)k'(t)}{k^2(t)} \frac{tk(t)}{\int_0^t k(s)\, ds} = \frac{1-\ell_1}{\ell_1}. \tag{23}$$

From Proposition 8 (ii) and (22), we find $P \in RV_{1-1/\ell_1}$. Furthermore, by (23)

$$\lim_{u\to\infty} \frac{uP'(u)}{P(u)} = \lim_{u\to\infty} \frac{-k'(1/u)}{uk(1/u)} = \frac{\ell_1 - 1}{\ell_1}$$

so that $P \in NRV_{1-1/\ell_1}$. Hence, $k \in NRV_{1/\ell_1 - 1}(0+)$. This implies $\mathcal{K}_{(01]} \subseteq \bigcup_{\alpha \geq 0} \mathcal{P}_\alpha$.

Conversely, assume that $k \in \mathcal{P}_\alpha$ for an arbitrary $\alpha \geq 0$. Hence, $P \in NRV_{-\alpha}$.

On the one hand, we have

$$\alpha = \lim_{u\to\infty} \frac{-uP'(u)}{P(u)} = \lim_{u\to\infty} \frac{k'(1/u)}{uk(1/u)} = \lim_{t\searrow 0} \frac{tk'(t)}{k(t)}. \tag{24}$$

If $k \in \mathcal{P}_\alpha$, $\alpha > 0$, then k is increasing on some neighborhood (on the right) of zero.

On the other hand, by Proposition 7 (ii) we find

$$\alpha + 1 = \lim_{u\to\infty} \frac{\frac{P(u)}{u}}{\int_u^\infty \frac{P(x)}{x^2}\, dx} = \lim_{t\searrow 0} \frac{tk(t)}{\int_0^t k(s)\, ds} \tag{25}$$

which yields $\ell_0 = \lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{k(t)} = 0$. Combining (24) and (25), we deduce

$$\ell_1 = \lim_{t\searrow 0} \left(\frac{\int_0^t k(s)\, ds}{k(t)} \right)' = 1 - \lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{tk(t)} \frac{tk'(t)}{k(t)} = \frac{1}{\alpha+1}.$$

We easily conclude that $\mathcal{P}_\alpha \subseteq \mathcal{K}_{(01]}$.

We now prove that $\mathcal{K}_0 \equiv \mathcal{R}_0$. Let $k \in \mathcal{K}_0$. For any $u \in (1/\nu, \infty)$, we define

$$\Lambda(u) = \frac{\int_u^\infty \frac{P(s)}{s^2}\, ds}{\frac{P(u)}{u}} = \frac{\int_0^{1/u} k(s)\, ds}{\frac{1}{u} k(\frac{1}{u})}. \tag{26}$$

From the definition of ℓ_1 and L'Hospital's rule, we infer that

$$\lim_{t\searrow 0} \frac{\left(\int_0^t k(s)\, ds\right) k'(t)}{k^2(t)} = 1 \quad \text{and} \quad \lim_{t\searrow 0} \frac{\int_0^t k(s)\, ds}{tk(t)} = 0. \tag{27}$$

It follows that $\lim_{u\to\infty} \Lambda(u) = 0$. A simple calculation gives

$$\frac{1}{x\Lambda(x)} = \frac{\frac{P(x)}{x^2}}{\int_x^\infty \frac{P(s)}{s^2}\, ds} = -\left[\ln \left(\int_x^\infty \frac{P(s)}{s^2}\, ds \right) \right]', \quad \forall x > d_1 := 1/\nu.$$

Integrating with respect to x over $[d_1, u]$ ($u > d_1$), we obtain

$$\begin{aligned} \int_{d_1}^u \frac{dx}{x\Lambda(x)} &= -\ln \left(\int_u^\infty \frac{P(s)}{s^2}\, ds \right) + \ln \left(\int_{d_1}^\infty \frac{P(s)}{s^2}\, ds \right) \\ &= -\ln \left(\frac{P(u)\Lambda(u)}{u} \right) + \ln \left(\int_{d_1}^\infty \frac{P(s)}{s^2}\, ds \right), \quad \forall u > d_1. \end{aligned}$$

This yields

$$k(1/u) = \frac{d_0 u}{\Lambda(u)} \exp\left\{ -\int_{d_1}^u \frac{ds}{s\Lambda(s)} \right\} \quad (u > d_1), \quad \text{where } d_0 = \text{Const.} > 0. \tag{28}$$

By a direct computation, we find

$$\frac{k'(1/u)}{uk(1/u)} = -1 + \frac{1}{\Lambda(u)} + \frac{u\Lambda'(u)}{\Lambda(u)}. \tag{29}$$

Combining (26), (27) and (29), we arrive at

$$1=\lim_{u\to\infty}\frac{\left(\int_0^{1/u}k(s)\,ds\right)k'(1/u)}{k^2(1/u)}=1+\lim_{u\to\infty}(u\Lambda'(u)-\Lambda(u))$$

which produces $\lim_{u\to\infty}u\Lambda'(u)=0$. Consequently, $\mathcal{K}_0\subseteq\mathcal{R}_0$.

Conversely, suppose $k\in\mathcal{R}_0$ is arbitrary. Notice that $\lim_{u\to\infty}\Lambda(u)=0$ yields $\lim_{u\to\infty}\int_{d_1}^u\frac{ds}{s\Lambda(s)}=\infty$. Since k has the form (28), we recover (29) and, moreover,

$$\frac{P(u)}{u^2}=\frac{k(1/u)}{u^2}=-d_0\left(\exp\left\{-\int_{d_1}^u\frac{ds}{s\Lambda(s)}\right\}\right)',\quad\forall u>d_1.$$

Hence, by integration we obtain

$$\int_u^\infty\frac{P(x)}{x^2}\,dx=d_0\exp\left\{-\int_{d_1}^u\frac{ds}{s\Lambda(s)}\right\}=\frac{\Lambda(u)P(u)}{u},\quad\forall u>d_1$$

which gives the expression of $\Lambda(u)$, namely (26). Since $\lim_{u\to\infty}\Lambda(u)=0$, we can read (26) as $\lim_{t\searrow0}\frac{\int_0^t k(s)\,ds}{tk(t)}=0$. So, in particular, $\lim_{t\searrow0}\frac{\int_0^t k(s)\,ds}{k(t)}=0$.

Using the given properties of Λ and the regained relations (26), (29) we deduce

$$\lim_{u\to\infty}\frac{\left(\int_0^{1/u}k(s)\,ds\right)k'(1/u)}{k^2(1/u)}=1+\lim_{u\to\infty}(u\Lambda'(u)-\Lambda(u))=1.$$

It follows that $\lim_{t\searrow0}\frac{\left(\int_0^t k(s)\,ds\right)k'(t)}{k^2(t)}=1$ which shows that $\ell_1=0$ and $k'>0$ on some interval $(0,\nu)$. Therefore, $\mathcal{R}_0\subseteq\mathcal{K}_0$. This concludes the proof. ■

Proof of Proposition 3. Note that Proposition 2 tells us that $\mathcal{R}_0\equiv\mathcal{K}_0$ and $\bigcup_{\alpha\geq0}\mathcal{P}_\alpha\equiv\mathcal{K}_{(01]}$.

(i) By the proof of Proposition 2, the expression of Λ in terms of k is given by (26) and

$$1-\frac{\left(\int_0^{1/u}k(s)\,ds\right)k'(1/u)}{k^2(1/u)}=\Lambda(u)-u\Lambda'(u).\tag{30}$$

We first prove that $\mathcal{R}_{0,\zeta}\subseteq\mathcal{K}_{0,\zeta}$. Let $k\in\mathcal{R}_{0,\zeta}$ be arbitrary. By L'Hospital's rule, $\lim_{u\to\infty}u^\zeta\Lambda(u)=-\ell_\star/\zeta$. By using (30), we find

$$\lim_{u\to\infty}u^\zeta\left[1-\frac{\left(\int_0^{1/u}k(s)\,ds\right)k'(1/u)}{k^2(1/u)}\right]=-\frac{(1+\zeta)\ell_\star}{\zeta}.$$

Therefore, $k\in\mathcal{K}_{0,\zeta}$ and $L_\star=-(1+\zeta)\ell_\star/\zeta$.

Second, we choose arbitrarily $k\in\mathcal{K}_{0,\zeta}$ in order to prove that $k\in\mathcal{R}_{0,\zeta}$. Since $\ell_0=0$ (see Remark 2), by L'Hospital's rule, we derive $\lim_{t\searrow0}\frac{\int_0^t k(s)\,ds}{k(t)t^{\zeta+1}}=\frac{L_\star}{\zeta+1}$. This, combined with (26), yields $\lim_{u\to\infty}u^\zeta\Lambda(u)=L_\star/(\zeta+1)$.

By the definition of $L_\star$ and (30), we conclude that

$$L_\star=\lim_{u\to\infty}u^\zeta(\Lambda(u)-u\Lambda'(u))=\frac{L_\star}{\zeta+1}-\lim_{u\to\infty}u^{\zeta+1}\Lambda'(u).$$

Consequently, $\lim_{u\to\infty}u^{\zeta+1}\Lambda'(u)=-\zeta L_\star/(\zeta+1)$, i.e., $k\in\mathcal{R}_{0,\zeta}$.

(ii) Suppose that $k\in\mathcal{P}_{\alpha,\tau}$, for some $\alpha\geq0$. A simple calculation leads to

$$\lim_{t\searrow0}(-\ln t)^\tau\left[\frac{1-\ell_1}{\ell_1}-\frac{tk'(t)}{k(t)}\right]=\lim_{u\to\infty}(\ln u)^\tau E(u)=\ell_\sharp\tag{31}$$

since α and ℓ_1 are connected by $\alpha = 1/\ell_1 - 1$. By L'Hospital's rule, we find

$$
\begin{aligned}
\lim_{t\searrow 0} (-\ln t)^\tau \left[\ell_1 - \frac{\int_0^t k(s)\,ds}{tk(t)} \right] &= \lim_{t\searrow 0} \frac{(\ell_1 - 1)k(t) + \ell_1 tk'(t)}{k(t)(-\ln t)^{-\tau} \left[1 + \frac{tk'(t)}{k(t)} - \frac{\tau}{\ln t} \right]} \\
= \quad -\ell_1^2 \lim_{t\searrow 0} (-\ln t)^\tau \left[\frac{1-\ell_1}{\ell_1} - \frac{tk'(t)}{k(t)} \right] &= \frac{-\ell_\sharp}{(\alpha+1)^2}.
\end{aligned}
\tag{32}
$$

Clearly, for each $t \in (0, \nu)$, we have

$$
\begin{aligned}
(-\ln t)^\tau \left[\left(\frac{\int_0^t k(s)\,ds}{k(t)} \right)' - \ell_1 \right] &= (-\ln t)^\tau \left[1 - \ell_1 - \frac{\left(\int_0^t k(s)\,ds\right) k'(t)}{k^2(t)} \right] \\
= \quad \frac{tk'(t)}{k(t)} (-\ln t)^\tau \left[\ell_1 - \frac{\int_0^t k(s)\,ds}{tk(t)} \right] &+ \ell_1 (-\ln t)^\tau \left[\frac{1-\ell_1}{\ell_1} - \frac{tk'(t)}{k(t)} \right].
\end{aligned}
\tag{33}
$$

By (31)–(33), we derive that $k \in \mathcal{K}_{(01],\tau}$ and $L_\sharp = \ell_\sharp/(1+\alpha)^2$.

Conversely, let $k \in \mathcal{K}_{(01],\tau}$ be arbitrary. Then $k \in \mathcal{P}_\alpha$ with $\alpha = 1/\ell_1 - 1$. Moreover, by L'Hospital's rule, we find

$$
\lim_{t\searrow 0} (-\ln t)^\tau \left(\frac{\int_0^t k(s)\,ds}{tk(t)} - \ell_1 \right) = \lim_{t\searrow 0} \frac{\left(\frac{\int_0^t k(s)\,ds}{k(t)} \right)' - \ell_1}{(-\ln t)^{-\tau}\left(1 - \frac{\tau}{\ln t}\right)} = L_\sharp. \tag{34}
$$

By virtue of (33) and (34), we deduce that

$$
L_\sharp = -\alpha L_\sharp + \frac{1}{\alpha+1} \lim_{u\to\infty} (\ln u)^\tau E(u).
$$

Consequently, $\lim_{u\to\infty} (\ln u)^\tau E(u) = (1+\alpha)^2 L_\sharp$. Hence, $k \in \mathcal{P}_{\alpha,\tau}$ with $\ell_\sharp = (1+\alpha)^2 L_\sharp$. This completes the proof. ∎

Lemma 4 *If $k \in \mathcal{R}_0$ or $k \in \mathcal{P}_{\alpha,\tau}$ with $\alpha^2 + \ell_\sharp^2 \neq 0$, then*

$$
\lim_{t\searrow 0} \frac{k'(t)}{k(t)t^{\theta-1}} = \infty, \quad \textit{for every } \theta > 0. \tag{35}
$$

Proof. If $k \in \mathcal{R}_0$ then, by Proposition 2, $k \in \mathcal{K}_0$. Hence, (27) holds. Consequently, $\lim_{t\searrow 0} \frac{tk'(t)}{k(t)} = \infty$, which leads to (35).

Assume that $k \in \mathcal{P}_{\alpha,\tau}$ with $\alpha^2 + \ell_\sharp^2 \neq 0$. Two cases may occur:

(i) $\alpha > 0$. We have $\lim_{t\searrow 0} \frac{tk'(t)}{k(t)} = \alpha$, which yields (35).

(ii) $\alpha = 0$ when $\alpha^2 + \ell_\sharp^2 \neq 0$ reads $\ell_\sharp \neq 0$. Since $k \in \mathcal{P}_{0,\tau}$, we deduce

$$
\lim_{t\searrow 0} \frac{k'(t)}{k(t)t^{\theta-1}} = \lim_{u\to\infty} -E(u)u^\theta = -\ell_\sharp \lim_{u\to\infty} \frac{u^\theta}{(\ln u)^\tau} = \infty.
$$

∎

7 Proof of Theorem 1

Fix $a \in (-\infty, \lambda_{\infty,1})$. Then, by [10, Theorem 1.1], equation (1) has at least a large solution. We show that, in order to establish the uniqueness, it is enough to prove that (9) holds for *any* large solution of (1). Indeed, if

u_1 and u_2 are two arbitrary large solutions of (1) then (9) yields $\lim_{d(x)\searrow 0} \frac{u_1(x)}{u_2(x)} = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \leq \delta. \tag{36}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where

$$C_\delta := \{x \in \Omega : \ d(x) > \delta\}.$$

Obviously, u_1 is a positive solution of the boundary value problem

$$\begin{cases} \Delta\varphi + a\varphi = b(x)f(\varphi) \quad \text{in } C_\delta, \\ \varphi = u_1 \quad \text{on } \partial C_\delta. \end{cases} \tag{37}$$

By (A_1) and (36), we see that $\varphi^- = (1-\varepsilon)u_2$ (resp., $\varphi^+ = (1+\varepsilon)u_2$) is a positive subsolution (resp., supersolution) of (37). By the sub and supersolutions method, (37) has a positive solution φ_1 satisfying $\varphi^- \leq \varphi_1 \leq \varphi^+$ in C_δ. Since $b > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}_0$ and $a \in (-\infty, \lambda_{\infty,1})$ by [10, Lemma 3.2] we derive that (37) has a *unique* positive solution, i.e., $u_1 \equiv \varphi_1$ in C_δ. This yields

$$(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x) \quad \text{in } C_\delta,$$

so that (36) holds in Ω. Passing to the limit as $\varepsilon \searrow 0$, we conclude that $u_1 \equiv u_2$.

In what follows, u_a denotes an arbitrary large solution of (1). Fix $\varepsilon \in (0, 1/2)$. Since (B) holds, we take $\delta > 0$ such that

(i) $d(x)$ is a C^2 function on the set $\{x \in \mathbf{R}^N : \ d(x) < \delta\}$.
(ii) k is nondecreasing on $(0,\delta)$.
(iii) $(1-\varepsilon)k^2(d(x)) < b(x) < (1+\varepsilon)k^2(d(x)), \quad \forall x \in \Omega$ with $0 < d(x) < \delta$.
(iv) $h'(t) < 0$ and $h''(t) > 0 \quad \forall t \in (0,\delta)$ (see (i) and (ii) of Lemma 2).

Define $\xi^\pm = \left[\frac{2+\ell_1\rho}{(1\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and $u^\pm(x) = \xi^\pm h(d(x))$, for any x with $d(x) \in (0,\delta)$.

The proof of (9) will be divided into three steps:

Step 1. There exists $\delta_1 \in (0,\delta)$ small such that

$$\begin{cases} \Delta u^+ + au^+ - (1-\varepsilon)k^2(d)f(u^+) \leq 0, \quad \forall x \text{ with } d(x) \in (0,\delta_1) \\ \Delta u^- + au^- - (1+\varepsilon)k^2(d)f(u^-) \geq 0, \quad \forall x \text{ with } d(x) \in (0,\delta_1). \end{cases} \tag{38}$$

Indeed, for every $x \in \Omega$ with $0 < d(x) < \delta$, we have

$$\begin{aligned} \Delta u^\pm \ &+ \ au^\pm - (1\mp\varepsilon)k^2(d)f(u^\pm) \\ &= \ \xi^\pm h''(d)\left(1 + a\frac{h(d)}{h''(d)} + \Delta d\,\frac{h'(d)}{h''(d)} - (1\mp\varepsilon)\frac{k^2(d)f(u^\pm)}{\xi^\pm h''(d)}\right). \end{aligned} \tag{39}$$

The definition of $u^\pm$, together with Lemma 2 (ii), yields

$$\lim_{d\searrow 0} \frac{k^2(d)f(u^\pm)}{\xi^\pm h''(d)} = \frac{2+\rho}{2+\rho\ell_1}\,(\xi^\pm)^\rho = \frac{1}{1\mp 2\varepsilon}. \tag{40}$$

By (39), (40), and Lemma 2 (iii) we easily deduce that (38) holds.

Step 2. There exists M^+, $\delta^+ > 0$ such that

$$u_a(x) \leq u^+(x) + M^+, \quad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

For $x \in \Omega$ with $d(x) \in (0, \delta_1)$, define $(0, \infty) \ni u \longmapsto \Psi_x(u) = au - b(x)f(u)$. Clearly, $\Psi_x(u)$ is decreasing when $a \le 0$. Suppose now that a is positive. By (A_1), $f'(t) \ge \frac{f(t)}{t}$ for any $t > 0$. Since $f \in C^1[0, \infty)$ and $\lim_{u\to\infty} f(u)/u = \infty$, we deduce that $\frac{f(t)}{t} : (0, \infty) \to (f'(0), \infty)$ is bijective. Let $\delta_2 \in (0, \delta_1)$ be small enough such that

$$b(x) < b_1 := 1 + \lim_{d(x)\to 0} b(x), \quad \forall x \in \Omega \text{ with } d(x) \in (0, \delta_2).$$

Set $C_b := 1 + f'(0)b_1/a$. For each $x \in \Omega$ with $d(x) \in (0, \delta_2)$, let $u_x \in (0, \infty)$ be the unique solution of the equation

$$\frac{b(x)f(u)}{u} = a\, C_b.$$

We see that, for any x with $d(x) \in (0, \delta_2)$, $u \longmapsto \Psi_x(u)$ is decreasing on (u_x, ∞).

Using (40) and Lemma 2 (iii), we have

$$\lim_{d(x)\searrow 0} \frac{b(x)f(u^+(x))}{u^+(x)} = \lim_{d\searrow 0} \frac{k^2(d)f(u^+)}{\xi^+ h''(d)} \frac{h''(d)}{h(d)} = \infty. \tag{41}$$

So, by diminishing δ_2 (if necessary), $u^+(x) > u_x$, for all $x \in \Omega$ with $d \in (0, \delta_2)$. Thus,

$$\Psi_x(u^+(x) + M) \le \Psi_x(u^+(x)), \quad \forall M > 0 \text{ and } \forall x \in \Omega \text{ with } 0 < d(x) < \delta_2. \tag{42}$$

Fix $\sigma \in (0, \delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega : \ \sigma < d(x) < \delta_2/2\}$.

For $M^+ > 0$ to be specified later, we define $u^*_\sigma(x) = u^+(d - \sigma, s) + M^+$, where (d, s) are the local coordinates of $x \in \mathcal{N}_\sigma$. By (ii), (iii), (38) and (42) we obtain

$$\begin{aligned}
-\Delta u^*_\sigma(x) &= -\Delta u^+(d-\sigma, s) \ge au^+(d-\sigma, s) - (1-\varepsilon)k^2(d-\sigma)f(u^+(d-\sigma, s)) \\
&\ge au^+(d-\sigma, s) - (1-\varepsilon)k^2(d)f(u^+(d-\sigma, s)) \\
&\ge au^+(d-\sigma, s) - b(x)f(u^+(d-\sigma, s)) \\
&\ge a(u^+(d-\sigma, s) + M^+) - b(x)f(u^+(d-\sigma, s) + M^+) \\
&= au^*_\sigma(x) - b(x)f(u^*_\sigma(x)) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

So, uniformly with respect to σ, we have the inequality

$$\Delta u^*_\sigma(x) + au^*_\sigma(x) \le b(x)f(u^*_\sigma(x)) \quad \text{in } \mathcal{N}_\sigma. \tag{43}$$

Obviously, $u^*_\sigma(x) \to \infty$ as $d \searrow \sigma$. Let $M^+ > 0$ be large enough such that

$$u^*_\sigma(\delta_2/2, s) = u^+(\delta_2/2 - \sigma, s) + M^+ \ge u_a(\delta_2/2, s), \quad \forall \sigma \in (0, \delta_2/4) \text{ and } \forall s \in \partial\Omega.$$

Thus, for this choice of M^+, u^*_σ becomes a supersolution of the problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \mathcal{N}_\sigma, \\ u = u_a & \text{on } \partial\mathcal{N}_\sigma. \end{cases} \tag{44}$$

Since $b > 0$ on $\overline{\mathcal{N}}_\sigma$, by [10, Lemma 3.2], u_a is the unique positive solution of (44). From [10, Lemma 2.1], $u_a \le u^*_\sigma$ in $\mathcal{N}_\sigma$, for every $\sigma \in (0, \delta_2/4)$. If $\sigma \to 0$ then

$$u_a(x) \le u^+(x) + M^+, \quad \forall x \in \Omega \text{ with } 0 < d < \delta_2/2$$

which achieves the assertion of Step 2 (with $\delta^+ \in (0, \delta_2/2)$ arbitrarily chosen).

Step 3. There exists M^-, $\delta^- > 0$ such that

$$u_a(x) \ge u^-(x) - M^-, \quad \forall x = (d, s) \in \Omega \quad \text{with } 0 < d < \delta^-. \tag{45}$$

For every $r \in (0, \delta)$, define $\Omega_r = \{x \in \Omega :\ 0 < d(x) < r\}$.

Fix arbitrarily $\sigma \in (0, \delta_2/4)$. For $\lambda \in (0,1)$ to be specified by (46), we define $v^*_\sigma(x) = \lambda u^-(d+\sigma, s)$, for $x = (d,s) \in \Omega_{\delta_2/2}$. By (ii), (iii), (38) and (A_1), we have

$$\begin{aligned}
\Delta v^*_\sigma(x) + a v^*_\sigma(x) &= \lambda(\Delta u^-(d+\sigma,s) + a u^-(d+\sigma,s)) \\
&\geq \lambda(1+\varepsilon)k^2(d+\sigma) f(u^-(d+\sigma,s)) \\
&\geq (1+\varepsilon)k^2(d) f(\lambda u^-(d+\sigma,s)) \\
&\geq b(x) f(v^*_\sigma(x)), \quad \forall x = (d,s) \in \Omega_{\delta_2/4}.
\end{aligned}$$

Thus, for each $\lambda \in (0,1)$, v^*_σ is a subsolution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$.

Let $\lambda \in (0,1)$ be small enough such that

$$v^*_\sigma(\delta_2/4, s) = \lambda u^-(\delta_2/4 + \sigma, s) \leq u_a(\delta_2/4, s), \quad \forall \sigma \in (0, \delta_2/4),\ \forall s \in \partial\Omega. \tag{46}$$

Since $\limsup_{d\searrow 0}(v^*_\sigma - u_a)(x) = -\infty$ and $b > 0$ in $\Omega_{\delta_2/4}$, by [10, Lemma 2.1] we conclude that $v^*_\sigma \leq u_a$ in $\Omega_{\delta_2/4}$. Passing to the limit $\sigma \searrow 0$ we obtain

$$\lambda u^-(x) \leq u_a(x), \quad \forall x \in \Omega_{\delta_2/4}. \tag{47}$$

On the other hand, by (40) and Lemma 2 (iii), $\lim_{d\searrow 0} k^2(d) f(\lambda^2 u^-)/u^- = \infty$, which ensures the existence of some $\tilde{\delta} \in (0, \delta_2/4)$ with the property

$$k^2(d) f(\lambda^2 u^-)/u^- \geq \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \leq \tilde{\delta}. \tag{48}$$

Choose $\delta_* \in (0, \tilde{\delta})$, sufficiently close to $\tilde{\delta}$, such that

$$h(\delta_*)/h(\tilde{\delta}) < 1 + \lambda. \tag{49}$$

We claim that for each $\sigma \in (0, \tilde{\delta} - \delta_*)$, $z_\sigma(x) = u^-(d+\sigma, s) - (1-\lambda)u^-(\delta_*, s)$ is a *positive subsolution* of

$$\Delta u + au = b(x) f(u) \quad \text{in } \Omega_{\delta_*}. \tag{50}$$

By (iv), $u^-(x)$ decreases with d when $d < \tilde{\delta}$. This fact and (49) yield

$$\begin{aligned}
z_\sigma(x) &\geq u^-(\tilde{\delta}, s) - (1-\lambda)u^-(\delta_*, s) = \xi^- \left[h(\tilde{\delta}) - (1-\lambda)h(\delta_*)\right] \\
&> \xi^- \left[h(\tilde{\delta}) - h(\delta_*)/(1+\lambda)\right] > 0, \quad \forall x = (d,s) \in \Omega_{\delta_*}
\end{aligned}$$

i.e., $z_\sigma > 0$ in Ω_{δ_*}. By (38), (ii) and (iii), z_σ is a subsolution of (50) provided that

$$(1+\varepsilon)k^2(d+\sigma)\left[f(u^-(d+\sigma,s)) - f(z_\sigma(d,s))\right] \geq a(1-\lambda)u^-(\delta_*, s), \quad \forall (d,s) \in \Omega_{\delta_*}. \tag{51}$$

The Lagrange mean value theorem and (A_1) show that (51) follows by

$$\frac{k^2(d+\sigma)\, f(z_\sigma(d,s))}{z_\sigma(d,s)} \geq |a|, \quad \forall (d,s) \in \Omega_{\delta_*},$$

which can be rewritten as

$$\frac{k^2(d+\sigma) f\left(u^-(d+\sigma,s)\left(1 - (1-\lambda)u^-(\delta_*,s)/u^-(d+\sigma,s)\right)\right)}{u^-(d+\sigma,s)\left[1 - (1-\lambda)u^-(\delta_*,s)/u^-(d+\sigma,s)\right]} \geq |a|, \quad \forall (d,s) \in \Omega_{\delta_*}.$$

Using (49) and the decreasing character of u^- with d, this last inequality is a consequence of

$$\frac{k^2(d+\sigma) f(\lambda^2 u^-(d+\sigma,s))}{\lambda^2 u^-(d+\sigma,s)} \geq |a|, \quad \forall (d,s) \in \Omega_{\delta_*}$$

which is true, as we can see by (48). Consequently, z_σ is a positive subsolution of (50), for each $\sigma \in (0, \tilde{\delta} - \delta_*)$. This and (47) yield

$$z_\sigma(x) = u^-(\delta_* + \sigma, s) - (1-\lambda)u^-(\delta_*, s) \leq \lambda u^-(\delta_*, s) \leq u_a(x), \quad \forall x = (\delta_*, s) \in \Omega.$$

But $\limsup_{d \searrow 0}(z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*}. Thus, by [10, Lemma 2.1] we have $z_\sigma \leq u_a$ in Ω_{δ_*}, for every $\sigma \in (0, \tilde{\delta} - \delta_*)$. If $\sigma \to 0$ then

$$u^-(d, s) - (1-\lambda)u^-(\delta_*, s) \leq u_a(d, s), \quad \forall (d, s) \in \Omega \text{ with } 0 < d < \delta_*$$

which concludes the assertion of Step 3 (take, for instance, $M^- = (1-\lambda)u^-(\delta_*, s)$ and $\delta^- = \delta_*$).

Thus, by virtue of Steps 2 and 3, for each $\varepsilon \in (0, 1/2)$, we have

$$\xi^- \leq \liminf_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \leq \limsup_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \leq \xi^+.$$

Taking $\varepsilon \to 0$, we obtain (9). This finishes the proof of Theorem 1. ■

8 Proof of Theorem 2

Note that $f \in NRV_{\rho+1}$ ($\rho > 0$) in each of the cases (i)–(iii). Let $a \in (-\infty, \lambda_{\infty,1})$ and u_a denote the unique large solution of (1).

Fix $\varepsilon \in (0, 1/2)$. We take $\delta > 0$ to satisfy (i), (ii) and (iv) stated in the proof of Theorem 1. Using $(\tilde{B})$ and Lemma 4, we can diminish $\delta > 0$ such that

$$\begin{cases} 1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta, \quad \forall x \in \Omega \text{ with } d \in (0, \delta) \\ k^2(t)\left[1 + (\tilde{c} - \varepsilon)t^\theta\right] \text{ is increasing on } (0, \delta). \end{cases} \tag{52}$$

We define $u^\pm(x) = \xi_0 h(d)(1 + \chi_\varepsilon^\pm d^\varpi)$, for $x \in \Omega$ with $d \in (0, \delta)$, where

$$\chi_\varepsilon^\pm = \chi \pm \frac{\varepsilon}{\rho}\left[1 + \text{Heaviside}\,(\zeta - \theta)\right].$$

We take $\delta > 0$ small such that $u^\pm(x) > 0$, for each $x \in \Omega$ with $d \in (0, \delta)$.

By the Lagrange mean value theorem, we find

$$f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d)) \tag{53}$$

where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$ for some $\lambda^\pm(d) \in [0, 1]$. We point out that

$$\lim_{d \searrow 0} \frac{f(\Upsilon^\pm(d))}{f(\xi_0 h(d))} = 1. \tag{54}$$

Fix $\sigma \in (0, 1)$ and $M > 0$ such that $|\chi_\varepsilon^\pm| < M$. Choose $\mu^\star > 0$ small enough so that

$$|(1 \pm Mt)^{\rho+1} - 1| < \sigma/2, \quad \forall t \in (0, 2\mu^\star).$$

Let $\mu_\star \in (0, (\mu^\star)^{1/\varpi})$ be such that, for every $x \in \Omega$ with $d \in (0, \mu_\star)$

$$\left|\frac{f(\xi_0 h(d)(1 \pm M\mu^\star))}{f(\xi_0 h(d))} - (1 \pm M\mu^\star)^{\rho+1}\right| < \sigma/2.$$

Assertion (54) follows now since, for every $x \in \Omega$ with $d \in (0, \mu_\star)$, we have

$$1-\sigma < (1-M\mu^\star)^{\rho+1} - \sigma/2 < \frac{f(\Upsilon^\pm(d))}{f(\xi_0 h(d))} < (1+M\mu^\star)^{\rho+1} + \sigma/2 < 1+\sigma.$$

We now divide our argument into three steps, as we did in the proof of Theorem 1. The assertion of Step 1, even similar in the statement to the previous one, is far more complicated in the proof. Regarding the next two steps, we follow the same pattern as before.

Step 1. There exists $\delta_1 \in (0,\delta)$ so that

$$\begin{cases} \Delta u^+ + au^+ - k^2(d)[1+(\tilde{c}-\varepsilon)d^\theta]f(u^+) \le 0, & \forall x \in \Omega \text{ with } d \in (0,\delta_1) \\ \Delta u^- + au^- - k^2(d)[1+(\tilde{c}+\varepsilon)d^\theta]f(u^-) \ge 0, & \forall x \in \Omega \text{ with } d \in (0,\delta_1). \end{cases} \tag{55}$$

To do this, we see that, for every $x \in \Omega$ with $d \in (0,\delta)$, we have

$$\begin{aligned} &\Delta u^\pm + au^\pm - k^2(d)\left[1+(\tilde{c}\mp\varepsilon)d^\theta\right] f(u^\pm) \\ &= \xi_0 d^\varpi h''(d) \left[a\chi_\varepsilon^\pm \frac{h(d)}{h''(d)} + \chi_\varepsilon^\pm \Delta d\, \frac{h'(d)}{h''(d)} + 2\varpi\chi_\varepsilon^\pm \frac{h'(d)}{dh''(d)} + \varpi\chi_\varepsilon^\pm \Delta d\, \frac{h(d)}{dh''(d)} \right. \\ &\qquad \left. +\varpi(\varpi-1)\chi_\varepsilon^\pm \frac{h(d)}{d^2h''(d)} + \Delta d\, \frac{h'(d)}{d^\varpi h''(d)} + \frac{a\,h(d)}{d^\varpi h''(d)} + \sum_{j=1}^4 \mathcal{S}_j^\pm(d) \right] \end{aligned} \tag{56}$$

where, for any $t \in (0,\delta)$, we denote

$$\begin{aligned} &\mathcal{S}_1^\pm(t) := (-\tilde{c}\pm\varepsilon)t^{\theta-\varpi}\, \frac{k^2(t)f(\xi_0 h(t))}{\xi_0 h''(t)}, \quad \mathcal{S}_2^\pm(t) := \chi_\varepsilon^\pm \left(1 - \frac{k^2(t)h(t)f'(\Upsilon^\pm(t))}{h''(t)}\right), \\ &\mathcal{S}_3^\pm(t) := (-\tilde{c}\pm\varepsilon)\chi_\varepsilon^\pm t^\theta\, \frac{k^2(t)h(t)f'(\Upsilon^\pm(t))}{h''(t)}, \quad \mathcal{S}_4^\pm(t) := \frac{1}{t^\varpi}\left(1 - \frac{k^2(t)f(\xi_0 h(t))}{\xi_0 h''(t)}\right). \end{aligned}$$

In what follows, we are going to show that

$$\lim_{t\searrow 0} \mathcal{S}_1^\pm(t) = (-\tilde{c}\pm\varepsilon)\text{Heaviside}\,(\zeta-\theta), \;\; \lim_{t\searrow 0} \mathcal{S}_2^\pm(t) = -\rho\chi_\varepsilon^\pm, \;\; \lim_{t\searrow 0} \mathcal{S}_3^\pm(t) = 0.$$

Indeed, by Lemma 2 (ii), we obtain

$$\lim_{t\searrow 0} \frac{k^2(t)f(\xi_0 h(t))}{\xi_0 h''(t)} = \lim_{t\searrow 0} \frac{f(\xi_0 h(t))}{\xi_0 f(h(t))}\, \frac{k^2(t)f(h(t))}{h''(t)} = \xi_0^\rho \frac{2+\rho}{2+\rho\ell_1} = 1, \tag{57}$$

which yields $\lim_{t\searrow 0} \mathcal{S}_1^\pm(t) = (-\tilde{c}\pm\varepsilon)\text{Heaviside}\,(\zeta-\theta)$.

By Lemma 1, (54) and (57), it turns out that

$$\lim_{t\searrow 0} \frac{k^2(t)h(t)f'(\Upsilon^\pm(t))}{h''(t)} = \lim_{t\searrow 0} \frac{\Upsilon^\pm(t)f'(\Upsilon^\pm(t))}{f(\Upsilon^\pm(t))}\, \frac{f(\Upsilon^\pm(t))}{f(\xi_0 h(t))}\, \frac{k^2(t)f(\xi_0 h(t))}{\xi_0 h''(t)} = \rho+1.$$

Consequently, $\lim_{t\searrow 0} \mathcal{S}_2^\pm(t) = -\rho\chi_\varepsilon^\pm$ and $\lim_{t\searrow 0} \mathcal{S}_3^\pm(t) = 0$.

Using (14), we derive $\mathcal{S}_4^\pm(t) = \frac{k^2(t)f(h(t))}{h''(t)} \sum_{i=1}^3 \mathcal{S}_{4,i}(t)$, for any $t \in (0,\delta)$, where

$$\mathcal{S}_{4,1}(t) = \frac{2\Xi(h(t))}{t^\varpi}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)', \quad \mathcal{S}_{4,2}(t) = \frac{2T_{1,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}, \quad \mathcal{S}_{4,3}(t) = \frac{-T_{2,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}.$$

The definition of T_{1,τ_1} (resp., T_{2,τ_1}) is obtained from (21), by replacing τ with τ_1.

Using Proposition 3 (i) and Corollary 1, we find

$$\lim_{t\searrow 0} \mathcal{S}_{4,1}(t) = \frac{-(1+\zeta)\rho\ell_\star}{\zeta(\rho+2)} \text{ Heaviside}\,(\theta-\zeta). \tag{58}$$

Case (i) (resp., (ii)). From (13) and Remark 3 (resp., Lemma 3 (ii)), we have

$$\lim_{t\searrow 0} S_{4,2}(t) = \lim_{t\searrow 0} S_{4,3}(t) = 0.$$

In view of Lemma 2 (ii) and (58), we derive that

$$\lim_{t\searrow 0} S_4^{\pm}(t) = \frac{-(1+\zeta)\rho\ell_\star}{2\zeta} \text{ Heaviside}\,(\theta-\zeta).$$

Case (iii). By (13) and Lemma 3 (i), we find

$$\lim_{t\searrow 0} S_{4,2}(t) = \frac{-2\ell^\star}{(\rho+2)^2}\left(\frac{-\rho\ell_\star}{2}\right)^{\tau_1} \text{ and } \lim_{t\searrow 0} S_{4,3}(t) = \frac{-2\ell^\star \ln \xi_0}{\rho+2}\left(\frac{-\rho\ell_\star}{2}\right)^{\tau_1}$$

from which we derive

$$\lim_{t\searrow 0} S_4^{\pm}(t) = \frac{-(1+\zeta)\rho\ell_\star}{2\zeta} \text{ Heaviside}\,(\theta-\zeta) - \ell^\star\left(\frac{-\rho\ell_\star}{2}\right)^{\tau_1}\left(\frac{1}{\rho+2}+\ln\xi_0\right).$$

Note that in each of the cases (i)–(iii), the definition of $\chi_\varepsilon^{\pm}$ gives

$$\lim_{t\searrow 0}\sum_{j=1}^{4}\mathcal{S}_j^{+}(t) = -\varepsilon < 0 \;\text{ and }\; \lim_{t\searrow 0}\sum_{j=1}^{4}\mathcal{S}_j^{-}(t) = \varepsilon > 0. \tag{59}$$

By (13), $\lim_{t\searrow 0} \frac{h'(t)}{t^{\varpi}h''(t)} = 0$. But $\lim_{t\searrow 0} \frac{h(t)}{h'(t)} = 0$, so that $\lim_{t\searrow 0} \frac{h(t)}{t^{\varpi}h''(t)} = 0$.
Thus, using (59) and Lemma 2 [(iii), (vi) and (vii)], relation (56) leads to (55).
Step 2. There exists M^+, $\delta^+ > 0$ such that

$$u_a(x) \le u^+(x) + M^+, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

We follow the same line of reasoning as in the proof of the above Step 2. There are only slight changes in the proof of (41) and (43), which will be stated below.

We see that $\lim_{d(x)\searrow 0} u^+(x)/h(d) = \xi_0$. So, for $\bar{\delta} > 0$ small enough, we have $u^+(x) \ge \xi_0 h(d)/2$ for every $x \in \Omega$ with $d(x) \in (0,\bar{\delta})$. This and (A_1) imply

$$\frac{b(x)f(u^+(x))}{u^+(x)} \ge \frac{2b(x)f(\xi_0 h(d)/2)}{\xi_0 h(d)}, \quad \forall x \in \Omega \text{ with } d(x) \in (0,\bar{\delta}). \tag{60}$$

Using Lemma 2 [(ii) and (iii)], we find

$$\lim_{d(x)\searrow 0} \frac{b(x)f(\xi_0 h(d)/2)}{h(d)} = \lim_{d(x)\searrow 0} \frac{k^2(d)f(\xi_0 h(d)/2)}{h''(d)}\,\frac{h''(d)}{h(d)} = \infty$$

which, together with (60), proves that (41) holds.

Using (55), (52) and (42), we obtain

$$\begin{aligned} -\Delta u_\sigma^*(x) &\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)(d-\sigma)^\theta]k^2(d-\sigma)f(u^+(d-\sigma,s)) \\ &\ge au^+(d-\sigma,s) - [1+(\tilde{c}-\varepsilon)d^\theta]k^2(d)f(u^+(d-\sigma,s)) \\ &\ge au^+(d-\sigma,s) - b(x)f(u^+(d-\sigma,s)) \\ &\ge a(u^+(d-\sigma,s)+M^+) - b(x)f(u^+(d-\sigma,s)+M^+) \\ &= au_\sigma^*(x) - b(x)f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma. \end{aligned}$$

Thus, relation (43) is recovered. This concludes the claim of Step 2.

Step 3. There exists M^-, $\delta^- > 0$ such that

$$u_a(x) \geq u^-(x) - M^-, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^-.$$

We proceed in the same way as for proving (45). Further, we present the differences which appear owing to the new meaning of u^-.

To recover (47) (with λ given by (46)), we show that v_σ^* is a subsolution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$. Indeed, using (52), (55) and (A_1), we find

$$\begin{aligned}
\Delta v_\sigma^*(x) + av_\sigma^*(x) &= \lambda(\Delta u^-(d+\sigma, s) + au^-(d+\sigma, s)) \\
&\geq \lambda k^2(d+\sigma)[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta] f(u^-(d+\sigma, s)) \\
&\geq k^2(d)[1 + (\tilde{c}+\varepsilon)d^\theta] f(\lambda u^-(d+\sigma, s)) \\
&\geq b(x) f(v_\sigma^*(x)), \quad \forall x = (d, s) \in \Omega_{\delta_2/4}.
\end{aligned}$$

Since $\lim_{d\searrow 0} u^-(x)/h(d) = \xi_0$, by (A_1) we can assume that, for some $\delta_0 > 0$,

$$\frac{f(\lambda^2 u^-(x))}{u^-(x)} \geq \frac{2f(\lambda^2 \xi_0 h(d)/2)}{\xi_0 h(d)}, \quad \forall x \in \Omega \text{ with } d(x) \in (0, \delta_0).$$

This and Lemma 2 [(ii) and (iii)] yield $\lim_{d\searrow 0} k^2(d) f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)[1 + (\tilde{c}+\varepsilon)d^\theta] f(\lambda^2 u^-)/u^- \geq \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \leq \tilde{\delta}. \tag{61}$$

By Lemma 2 [(i) and (viii)], we easily deduce that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished).

Choose $\delta_* \in (0, \tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$\frac{h(\delta_*)(1 + \chi_\varepsilon^- \delta_*^\varpi)}{h(\tilde{\delta})(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi)} < 1 + \lambda. \tag{62}$$

We now prove that, for each $\sigma \in (0, \tilde{\delta} - \delta_*)$, $z_\sigma(x) = u^-(d+\sigma, s) - (1-\lambda)u^-(\delta_*, s)$ is a positive subsolution of (50). Using (62), we arrive at

$$\begin{aligned}
z_\sigma(x) &\geq u^-(\tilde{\delta}, s) - (1-\lambda)u^-(\delta_*, s) \\
&> \xi_0 \left[h(\tilde{\delta})(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi) - h(\delta_*)(1 + \chi_\varepsilon^- \delta_*^\varpi)/(1+\lambda) \right] > 0, \quad \forall x = (d, s) \in \Omega_{\delta_*}.
\end{aligned}$$

By (52) and (55), z_σ is a subsolution of (50) provided that

$$k^2(d+\sigma)[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta] \left[f(u^-(d+\sigma, s)) - f(z_\sigma(d, s)) \right] \geq a(1-\lambda)u^-(\delta_*, s), \quad \forall (d, s) \in \Omega_{\delta_*}. \tag{63}$$

From Lagrange mean value theorem and (A_1), we infer that (63) is a consequence of

$$\frac{k^2(d+\sigma)[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta]\, f(z_\sigma(d, s))}{z_\sigma(d, s)} \geq |a|, \quad \forall (d, s) \in \Omega_{\delta_*}. \tag{64}$$

By virtue of (61), (62) and the decreasing character of u^- with d, (64) holds. From now on, the argument is the same as before. This proves the claim of Step 3.

By Steps 2 and 3, it follows that

$$\begin{aligned}
\chi_\varepsilon^+ &\geq \frac{-1 + u_a(x)/[\xi_0 h(d)]}{d^\varpi} - \frac{M^+}{\xi_0 d^\varpi h(d)}, \qquad \forall x \in \Omega \text{ with } d \in (0, \delta^+), \\
\chi_\varepsilon^- &\leq \frac{-1 + u_a(x)/[\xi_0 h(d)]}{d^\varpi} + \frac{M^-}{\xi_0 d^\varpi h(d)}, \qquad \forall x \in \Omega \text{ with } d \in (0, \delta^-).
\end{aligned}$$

Passing to the limit $d \searrow 0$ and using Lemma 2 (ix), we obtain

$$\chi_\varepsilon^- \leq \liminf_{d\searrow 0} \frac{-1 + u_a(x)/[\xi_0 h(d)]}{d^\varpi} \leq \limsup_{d\searrow 0} \frac{-1 + u_a(x)/[\xi_0 h(d)]}{d^\varpi} \leq \chi_\varepsilon^+.$$

By sending ε to 0, the proof of Theorem 2 is concluded. ■

9 Proof of Theorem 3

The presentation is closely related to that in §8. To make easier the comparison, we use the same notation even though its meaning is sometimes different.

We see that, in each of the cases (i) and (ii), $f \in NRV_{\rho+1}$ ($\rho > 0$) and $\alpha^2 + \ell_\sharp^2 \neq 0$.

Let $a \in (-\infty, \lambda_{\infty,1})$ and u_a be the corresponding unique large solution of (1).

Fix $\varepsilon \in (0, 1/2)$. Let $\delta > 0$ be such that (i), (ii), (iv) from §7 and (52) hold.

We define $u^\pm(x) = \xi_0 h(d)\,[1 + \chi_\varepsilon^\pm (-\ln d)^{-\tau}]$, for $x \in \Omega$ with $d \in (0, \delta)$, where

$$\chi_\varepsilon^\pm = \tilde{\chi} \pm \varepsilon\,. \tag{65}$$

We can assume $u^\pm(x) > 0$ for every $x \in \Omega$ with $d(x) \in (0, \delta)$.

Relation (53) reads now as

$$f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm \frac{h(d)}{(-\ln d)^\tau}\, f'(\Upsilon^\pm(d))$$

where $\Upsilon^\pm(d) = \xi_0 h(d)\,[1 + \chi_\varepsilon^\pm \lambda^\pm(d)(-\ln d)^{-\tau}]$, for some $\lambda^\pm(d) \in [0, 1]$.

Relation (54) and its proof remain as before by choosing $\mu_\star > 0$ such that $(-\ln \mu_\star)^{-\tau} < \mu^\star$ (instead of $\mu_\star < (\mu^\star)^{1/\varpi}$).

Step 1. Proof of (55).

For every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\begin{aligned}
&\Delta u^\pm + a u^\pm - k^2(d)\left[1 + (\tilde{c} \mp \varepsilon) d^\theta\right] f(u^\pm) \\
&= \xi_0 \frac{h''(d)}{(-\ln d)^\tau} \left[a \chi_\varepsilon^\pm \frac{h(d)}{h''(d)} + \chi_\varepsilon^\pm \Delta d\, \frac{h'(d)}{h''(d)} + \frac{h'(d)}{h''(d)} (-\ln d)^\tau \Delta d \right. \\
&\quad + a\, \frac{h(d)}{h''(d)} (-\ln d)^\tau - 2\tau \chi_\varepsilon^\pm \frac{h'(d)}{d h''(d) \ln d} + \tau \chi_\varepsilon^\pm \frac{h(d)}{d^2 h''(d) \ln d} \\
&\quad \left. + \tau(\tau+1) \chi_\varepsilon^\pm \frac{h(d)}{d^2 h''(d) \ln^2 d} - \tau \chi_\varepsilon^\pm \Delta d\, \frac{h(d)}{d h''(d) \ln d} + \sum_{j=1}^4 \mathcal{S}_j^\pm(d) \right]
\end{aligned} \tag{66}$$

where, for each $t \in (0, \delta)$, $\mathcal{S}_2^\pm(t)$ and $\mathcal{S}_3^\pm(t)$ are defined as in §8, but

$$\begin{aligned}
\mathcal{S}_1^\pm(t) &:= (-\tilde{c} \pm \varepsilon) t^\theta (-\ln t)^\tau \frac{k^2(t) f(\xi_0 h(t))}{\xi_0 h''(t)}, \\
\mathcal{S}_4^\pm(t) &:= (-\ln t)^\tau \left(1 - \frac{k^2(t) f(\xi_0 h(t))}{\xi_0 h''(t)}\right).
\end{aligned}$$

By (57), we have $\lim_{t\searrow 0} S_1^\pm(t) = \lim_{t\searrow 0} S_3^\pm(t) = 0$ and $\lim_{t\searrow 0} S_2^\pm(t) = -\rho\chi_\varepsilon^\pm$.

Using (14), we write $S_4^\pm(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{i=1}^3 S_{4,i}(t)$ for any $t \in (0, \delta)$, where

$$\begin{aligned}
S_{4,1}(t) &= 2\Xi(h(t))(-\ln t)^\tau \left[\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' - \ell_1\right], \\
S_{4,2}(t) &= 2(1 - \ell_1)\left(\frac{-\ln t}{\ln h(t)}\right)^\tau T_{1,\tau}(h(t)), \quad S_{4,3}(t) = -\left(\frac{-\ln t}{\ln h(t)}\right)^\tau T_{2,\tau}(h(t)).
\end{aligned}$$

By Proposition 3 (ii) and Corollary 1, we arrive at

$$\lim_{t\searrow 0} S_{4,1}(t) = \frac{\rho \ell_\sharp}{(\rho+2)(\alpha+1)^2}.$$

Case (i). By Lemmas 3 (ii) and 2 [(ii), (viii)], it turns out that

$$\lim_{t\searrow 0} S_{4,2}(t) = \lim_{t\searrow 0} S_{4,3}(t) = 0 \quad \text{and} \quad \lim_{t\searrow 0} S_4^{\pm}(t) = \frac{\rho \ell_\sharp}{(\alpha+1)(\rho+2\alpha+2)}.$$

Case (ii). By Lemmas 3 (i) and 2 [(ii), (viii)], we deduce

$$\begin{aligned} \lim_{t\searrow 0} S_{4,2}(t) &= \frac{-2\alpha\ell^\star}{(\rho+2)^2(\alpha+1)} \left(\frac{\rho}{2(\alpha+1)}\right)^\tau, \\ \lim_{t\searrow 0} S_{4,3}(t) &= \frac{-\ell^\star(\rho+2\alpha+2)}{(2+\rho)(\alpha+1)} \left(\frac{\rho}{2(\alpha+1)}\right)^\tau \ln \xi_0 \end{aligned}$$

from which we conclude that

$$\lim_{t\searrow 0} S_4^{\pm}(t) = \frac{\rho\ell_\sharp}{(\alpha+1)(\rho+2\alpha+2)} - \ell^\star \left(\frac{\rho}{2(\alpha+1)}\right)^\tau \left[\frac{2\alpha}{(\rho+2)(\rho+2\alpha+2)} + \ln \xi_0\right].$$

Therefore, in both cases the definition (65) of $\chi_\varepsilon^{\pm}$ leads to

$$\lim_{t\searrow 0} \sum_{j=1}^{4} S_j^+(t) = -\rho\varepsilon < 0 \quad \text{and} \quad \lim_{t\searrow 0} \sum_{j=1}^{4} S_j^-(t) = \rho\varepsilon > 0.$$

By virtue of (66) and Lemma 2 [(iii), (vi), (vii)], relation (55) is deduced.

Step 2. See Step 2 in the proof of Theorem 2 for the same claim and proof apply here.

Step 3. The assertion is exactly the same as in §8. The proof goes as before with only one exception. We choose $\delta_* \in (0, \tilde{\delta})$, sufficiently close to $\tilde{\delta}$, such that

$$\frac{h(\delta_*)(1+\chi_\varepsilon^-(-\ln \delta_*)^{-\tau})}{h(\tilde{\delta})(1+\chi_\varepsilon^-(-\ln \tilde{\delta})^{-\tau})} < 1+\lambda. \tag{67}$$

In the rest of our reasoning, (67) takes place of (62).

By Steps 2 and 3, it follows that

$$\begin{aligned} \chi_\varepsilon^+ &\geq \left[-1+\frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau - \frac{M^+(-\ln d)^\tau}{\xi_0 h(d)}, \qquad \forall x \in \Omega \text{ with } d \in (0, \delta^+), \\ \chi_\varepsilon^- &\leq \left[-1+\frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau + \frac{M^-(-\ln d)^\tau}{\xi_0 h(d)}, \qquad \forall x \in \Omega \text{ with } d \in (0, \delta^-). \end{aligned} \tag{68}$$

Using Lemma 2 (viii), we have

$$\lim_{t\searrow 0} \frac{(-\ln t)^\tau}{h(t)} = \lim_{t\searrow 0} \left(\frac{-\ln t}{\ln h(t)}\right)^\tau \frac{(\ln h(t))^\tau}{h(t)} = \left(\frac{\rho\ell_1}{2}\right)^\tau \lim_{u\to\infty} \frac{(\ln u)^\tau}{u} = 0.$$

Passing to the limit $d \searrow 0$ in (68), we obtain

$$\chi_\varepsilon^- \leq \liminf_{d\searrow 0} \left[-1+\frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \leq \limsup_{d\searrow 0} \left[-1+\frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \leq \chi_\varepsilon^+.$$

By sending ε to 0, the proof of Theorem 3 is finished. ■

10 About Theorem 1

Recall that, assuming (A_1), the existence of large solutions of (1) takes place only if (A_0) holds and $a \in (-\infty, \lambda_{\infty,1})$. Furthermore, if $f \in NRV_{\rho+1}$ with $\rho \neq 0$ and $b(x) \sim k^2(d)$ as $d(x) \searrow 0$ for some $k \in \mathcal{K}$, then the same asymptotic behavior (9) is shown by any large solution u_a, irrespectiv of $a \in (-\infty, \lambda_{\infty,1})$. As a consequence, the uniqueness of a large solution u_a is concluded.

From now on, when we refer to u_a, we understand that a is arbitrary within the above range. By $f_1(x) \sim f_2(x)$ as $d(x) \searrow 0$ we mean that $\lim_{d(x)\searrow 0} f_1(x)/f_2(x) = 1$.

The aim of this section is to show that, for a wide class of functions f and k, the behavior of the unique large solution u_a may be illustrated.

10.1 Case $f(u) = u^{\rho+1}$, $\rho > 0$

Consider first the logistic equation (1) for a superlinear power in nonlinearity f. This is because of its significance in Riemannian Geometry and population dynamics, as shown in §1.

By virtue of (9) and (10), we have

$$u_a(x) \sim \left[\frac{2(2+\ell_1\rho)}{\rho^2}\right]^{1/\rho} \left\{\int_0^{d(x)} k(s)\, ds\right\}^{-2/\rho} = \xi_0 h(d(x)) \text{ as } d(x) \searrow 0. \tag{69}$$

The explosion rate of $u_a(x)$ when $x \to \partial\Omega$ can be estimated from that of $\xi_0 h(t)$, $t \searrow 0$.

Example 1 Let $k(t) = -1/\ln t$ be defined in a small neighborhood on the right of the origin. By Proposition 1, $k \in \mathcal{K}$ with $\ell_1 = 1$ and

$$\int_0^t k(s)\, ds = \int_0^t -\frac{ds}{\ln s} = \mathrm{Ei}\,(1, -\ln t) = \int_1^\infty \frac{t^s}{s}\, ds,$$

where $\mathrm{Ei}\,(1, z)$ denotes the exponential integral defined for $z > 0$ as follows

$$\mathrm{Ei}\,(1, z) = \int_1^\infty \frac{\exp(-zs)}{s}\, ds.$$

Hence, relation (69) yields

$$u_a(x) \sim \left[\frac{2(2+\rho)}{\rho^2}\right]^{1/\rho} \left(\int_1^\infty \frac{[d(x)]^s}{s}\, ds\right)^{-2/\rho} \quad \text{as } d(x) \searrow 0.$$

Figure 1 illustrates $\xi_0 h(t)$ when $t \in (10^{-5}, 1.2 \times 10^{-5})$, for each $\rho \in (0.4, 0.405)$.

Figure 1: *Graph of $\xi_0 h(t)$, when $k(t) = -1/\ln t$, $t \in (10^{-5}, 1.2 \times 10^{-5})$ and $\rho \in (0.4, 0.405)$*

Example 2 Let $k(t) = \exp(-1/t)$, for $t > 0$. It can be seen that $k \in \mathcal{K}_0$ and

$$\int_0^t k(s)\, ds = \int_0^t \exp(-1/s)\, ds = t\exp(-1/t) - \mathrm{Ei}(1, 1/t).$$

In view of (69), we find

$$u_a(x) \sim \left(\frac{4}{\rho^2}\right)^{1/\rho} \left[d(x)\exp(-1/d(x)) - \mathrm{Ei}(1, 1/d(x))\right]^{-2/\rho} \quad \text{as } d(x) \searrow 0.$$

The graph of $\xi_0 h(t)$ when $t \in (0.1, 0.1005)$ and $\rho \in (0.4, 0.4002)$ is given by Figure 2.

Figure 2: *Graph of* $\xi_0 h(t)$, *when* $k(t) = \exp(-1/t)$, $t \in (0.1, 0.1005)$ *and* $\rho \in (0.4, 0.4002)$

Example 3 Let $k(t) = \exp(-1/t^2)$, for $t > 0$. We have $k \in \mathcal{K}_0$ and

$$\int_0^t k(s)\, ds = \int_0^t \exp(-1/s^2)\, ds = \frac{t}{\exp(1/t^2)} + \sqrt{\pi}\,\mathrm{erf}(1/t) - \sqrt{\pi},$$

where $\mathrm{erf}(z)$ denotes the error function defined by $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)\, dt$.

In this case, relation (69) reads as

$$u_a(x) \sim \left(\frac{4}{\rho^2}\right)^{1/\rho} \left[\frac{d(x)}{\exp(1/d^2(x))} + \sqrt{\pi}\,\mathrm{erf}(1/d(x)) - \sqrt{\pi}\right]^{-2/\rho} \quad \text{as } d(x) \searrow 0.$$

Figure 3 illustrates $\xi_0 h(t)$ when $t \in (0.11, 0.1101)$, for each $\rho \in (2, 2.01)$.

10.2 Other cases

Let $k \in \mathcal{K}$ be arbitrary and $b(x) \sim k^2(d)$ as $d(x) \searrow 0$. We apply here Theorem 1 for other nonlinearities $f \in NRV_{\rho+1}$, $\rho > 0$ in order to express the asymptotic behavior of the unique large solution $u_a(x)$ when $x \to \partial\Omega$.

Example 4 The following assertions take place:

(i) If $f(u) = 4u(u^2+1)\arctan^2 u + 2(u^2+1)\arctan u$ for $u \geq 0$, then (A_1) holds, $\rho = 2$ and $F(u) = (u^2+1)^2 \arctan^2 u$. Consequently,

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}}\, \tan\left[\frac{\pi}{2\exp\left(\sqrt{2}\int_0^{d(x)} k(s)\, ds\right)}\right] \quad \text{as } d(x) \searrow 0.$$

Figure 3: *Graph of* $\xi_0 h(t)$*, when* $k(t) = \exp(-1/t^2)$*,* $t \in (0.11, 0.1101)$ *and* $\rho \in (2, 2.01)$

(ii) If $f(u) = 4u^3 + 2u$ for $u \geq 0$, then

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}} \, \frac{1}{\sinh(\sqrt{2}\int_0^{d(x)} k(s)\, ds)} \quad \text{as } d(x) \searrow 0.$$

(iii) If $f(u) = 8u^7 + 12u^5 + 4u^3$ for $u \geq 0$, then

$$u_a(x) \sim \left(\frac{1+3\ell_1}{4}\right)^{1/6} \tilde{h}^{-1}\left(\sqrt{2}\int_0^{d(x)} k(s)\, ds\right) \quad \text{as } d(x) \searrow 0,$$

where $\tilde{h}^{-1}$ denotes the inverse of $\tilde{h}(t) = 1/t + \arctan t - \pi/2$, $t > 0$.

(iv) If $f(u) = (4u^7 + 6u^5)/(u^2+1)^2$ for $u \geq 0$, then $F(u) = u^6/(u^2+1)$ and

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}} \, \tilde{h}^{-1}\left(2\sqrt{2}\int_0^{d(x)} k(s)\, ds\right) \quad \text{as } d(x) \searrow 0,$$

where $\tilde{h}^{-1}$ is the inverse of $\tilde{h}(t) = \sqrt{t^2+1}/t^2 + \operatorname{arctanh}(1/\sqrt{t^2+1})$, $t > 0$. Here, $\operatorname{arctanh} u$ denotes the inverse of the hyperbolic function $\tanh u$.

(v) If $f(u) = \dfrac{4u^3}{\ln^2(u+1)} - \dfrac{2u^4}{(u+1)\ln^3(u+1)}$ for $u > 0$ and $f(0) = 0$, then $f \in C^1[0,\infty)$, (A_1) holds, $\rho = 2$ and $F(u) = u^4/\ln^2(u+1)$. It follows that

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}} \, \tilde{h}^{-1}\left(\sqrt{2}\int_0^{d(x)} k(s)\, ds\right) \quad \text{as } d(x) \searrow 0,$$

where $\tilde{h}^{-1}$ is the inverse of $\tilde{h}(t) = (1 + 1/t)\ln(t+1) - \ln t$, $t > 0$.

(vi) If $f(u) = \dfrac{4u^3}{\arctan^2 u} - \dfrac{2u^4}{(u^2+1)\arctan^3 u}$ for $u > 0$ and $f(0) = 0$, then $f \in C^1[0,\infty)$, $\rho = 2$, (A_1) holds and $F(u) = u^4/\arctan^2 u$. Hence,

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}} \, \tilde{h}^{-1}\left(\sqrt{2}\int_0^{d(x)} k(s)\, ds\right) \quad \text{as } d(x) \searrow 0,$$

where $\tilde{h}^{-1}$ is the inverse of $\tilde{h}(t) = (\arctan t)/t - \ln(t/\sqrt{t^2+1})$, $t > 0$.

(vii) If $f(u) = \dfrac{4u^3}{\ln^2(u+\sqrt{u^2+1})} - \dfrac{2u^4}{\sqrt{u^2+1}\,\ln^3(u+\sqrt{u^2+1})}$ for $u > 0$ and $f(0) = 0$, then $f \in C^1[0,\infty)$, $\rho = 2$, (A_1) holds and $F(u) = u^4/\ln^2(u+\sqrt{u^2+1})$. So,

$$u_a(x) \sim \sqrt{\frac{1+\ell_1}{2}}\,\tilde{h}^{-1}\left(\sqrt{2}\int_0^{d(x)} k(s)\,ds\right) \quad \text{as } d(x) \searrow 0,$$

where $\tilde{h}^{-1}$ is the inverse of $\tilde{h}(t) = [\ln(t+\sqrt{t^2+1})]/t + \operatorname{arctanh}(1/\sqrt{t^2+1})$, $t > 0$.

11 Appendix

We give here a brief account of the definitions and properties of regularly varying functions involved in our paper (see [6] or [41] for details).

Definition 1 *A positive measurable function Z defined on $[A,\infty)$, for some $A > 0$, is called* regularly varying (at infinity) with index $\rho \in \mathbf{R}$, *written $Z \in RV_\rho$, provided that*

$$\lim_{u\to\infty} \frac{Z(\xi u)}{Z(u)} = \xi^\rho, \qquad \textit{for all } \xi > 0.$$

When the index ρ is zero, we say that the function is slowly varying.

Remark 4 *Let $Z : [A,\infty) \to (0,\infty)$ be a measurable function. Then*

(i) *Z is regularly varying if and only if $\lim_{u\to\infty} Z(\xi u)/Z(u)$ is finite and positive for each ξ in a set $S \subset (0,\infty)$ of positive measure (see [41, Lemma 1.6 and Theorem 1.3]).*

(ii) *Set*

$$Z(u) = u^\rho L(u). \tag{70}$$

Then $\lim_{u\to\infty} Z(\xi u)/Z(u) = u^\rho$ if and only if $\lim_{u\to\infty} L(\xi u)/L(u) = 1$, for every $\xi > 0$. Thus the transformation (70) reduces regular variation to slow variation.

Example 5 Any measurable function on $[A,\infty)$ which has a positive limit at infinity is slowly varying. The logarithm $\log u$, its iterates $\log\log u$ $(= \log_2 u)$, $\log_m u$ $(= \log\log_{m-1} u)$ and powers of $\log_m u$ are nontrivial examples of slowly varying functions. Nonlogarithmic examples are given by $\exp\{(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2}\dots(\log_m u)^{\alpha_m}\}$, where $\alpha_i \in (0,1)$ and $\exp\{(\log u)/\log\log u\}$.

In what follows L denotes a slowly varying function defined on $[A,\infty)$. For details on Propositions 4–8, we refer to [6] (pp. 6, 12, 14, 16, 28, 30).

Proposition 4 (Uniform Convergence Theorem). *The convergence $\frac{L(\xi u)}{L(u)} \to 1$ as $u \to \infty$ holds uniformly on each compact ξ-set in $(0,\infty)$.*

Proposition 5 (Representation Theorem). *The function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u) = M(u)\exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\} \quad (u \ge B) \tag{71}$$

for some $B > A$, where $y \in C[B,\infty)$ satisfies $\lim_{u\to\infty} y(u) = 0$ and $M(u)$ is measurable on $[B,\infty)$ such that $\lim_{u\to\infty} M(u) := \overline{M} \in (0,\infty)$.

The Karamata representation (71) is non-unique because we can adjust one of $M(u)$, $y(u)$ and modify properly the other one. Thus, the function y may be assumed arbitrarily smooth, but the smothness properties of $M(u)$ can ultimately reach those of $L(u)$. If $M(u)$ is replaced by its limit at infinity $\overline{M} > 0$, we obtain a slowly varying function $L_0 \in C^1[B,\infty)$ of the form

$$L_0(u) = \overline{M}\exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\} \quad (u \geq B),$$

where $y \in C[B,\infty)$ vanishes at infinity. Such a function $L_0(u)$ is called *normalised* slowly varying function.

As an important subclass of RV_ρ, we distinguish NRV_ρ defined as

$$NRV_\rho = \left\{Z \in RV_\rho :\ Z(u)u^{-\rho} \text{ is a } \textit{normalised} \text{ slowly varying function}\right\}. \tag{72}$$

Notice that $L(u)$ given by (71) is asymptotic equivalent to $L_0(u)$, which has much enhanced properties. For instance, we see that

$$y(u) = \frac{uL_0'(u)}{L_0(u)}, \qquad \forall u \geq B.$$

Conversely, any function $L_0 \in C^1[B,\infty)$ which is positive and satisfies

$$\lim_{u\to\infty} \frac{uL_0'(u)}{L_0(u)} = 0 \tag{73}$$

is a normalised slowly varying. Moreover, if the right hand side of (73) is $\rho \in \mathbf{R}$, then $L_0 \in NRV_\rho$.

Proposition 6 (Elementary properties of slowly varying functions).
(i) *For any* $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha}L(u) \to 0$ *as* $u \to \infty$.
(ii) $(L(u))^\alpha$ *varies slowly for every* $\alpha \in \mathbf{R}$.
(iii) *If* L_1, L_2 *vary slowly, so do* $L_1(u)L_2(u)$ *and* $L_1(u) + L_2(u)$.

From Proposition 6 (i) and Remark 4 (ii), $\lim_{u\to\infty} Z(u) = \infty$ (resp., 0) for any function $Z \in RV_\rho$ with $\rho > 0$ (resp., $\rho < 0$).

We point out that the behavior at infinity for a slowly varying function cannot be predicted. For instance, $L(u) = \exp\left\{(\log u)^{1/2}\cos((\log u)^{1/2})\right\}$ exhibits infinite oscillation in the sense that $\liminf_{u\to\infty} L(u) = 0$ and $\limsup_{u\to\infty} L(u) = \infty$.

Proposition 7 (Karamata's Theorem; direct half). *Let* $Z \in RV_\rho$ *be locally bounded in* $[A,\infty)$. *Then*
(i) *for any* $j \geq -(\rho+1)$,

$$\lim_{u\to\infty} \frac{u^{j+1}Z(u)}{\int_A^u x^j Z(x)\,dx} = j+\rho+1. \tag{74}$$

(ii) *for any* $j < -(\rho+1)$ *(and for* $j = -(\rho+1)$ *if* $\int^\infty x^{-(\rho+1)}Z(x)\,dx < \infty$*)*

$$\lim_{u\to\infty} \frac{u^{j+1}Z(u)}{\int_u^\infty x^j Z(x)\,dx} = -(j+\rho+1). \tag{75}$$

Proposition 8 (Karamata's Theorem; converse half). *Let* Z *be positive and locally integrable in* $[A,\infty)$.
(i) *If* (74) *holds for some* $j > -(\rho+1)$, *then* $Z \in RV_\rho$.
(ii) *If* (75) *is satisfied for some* $j < -(\rho+1)$, *then* $Z \in RV_\rho$.

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