RATIONAL IDENTITIES AND INEQUALITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper we use integral calculus, complex variable techniques and some classical inequalities to establish rational identities and inequalities involving Fibonacci and Lucas numbers.

1. Introduction

The Fibonacci sequence is a source of many nice interesting identities and inequalities. A similar interpretation exist for Lucas numbers. Many of these identities have been documented in an extensive list that appears in the work of Vajda [1] where they are proved by algebraic means. Even though, combinatorial proofs of many of these interesting algebraic identities are also given (see [2]). However, rational identities and inequalities involving Fibonacci and Lucas numbers seldom have appeared (see [3]). In this paper, integral calculus, complex variable techniques and some classical inequalities are used to obtain several rational Fibonacci and Lucas identities and inequalities.

2. Rational Identities

In what follows several rational identities are considered and proved by using results on contour integrals. We begins with

Theorem 2.1. *Let F_n denote the n^{th} Fibonacci number. That is, $F_0 = 0, F_1 = 1$ and for $n \geq 2, F_n = F_{n-1} + F_{n-2}$. Then, for all positive integer r holds*

$$\sum_{k=1}^{n} \frac{1+F_{r+k}^{\ell}}{F_{r+k}} \left\{ \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{F_{r+k}-F_{r+j}} \right\} = \frac{(-1)^{n+1}}{F_{r+1}F_{r+2}\cdots + F_{r+n}} \tag{2.1}$$

with $0 \leq \ell \leq n-1$.

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Proof. To prove the preceding identity we consider the integral

$$I = \frac{1}{2\pi i}\oint_\gamma \frac{1+z^\ell}{A_n(z)}\,\frac{dz}{z} \tag{2.2}$$

where $A_n(z) = \displaystyle\prod_{j=1}^{n}(z - F_{r+j})$.

Let γ be the curve defined by $\gamma = \{z \in \mathbb{C} : |z| < F_{r+1}\}$. Evaluating the preceding integral in the exterior of γ contour, we obtain

$$I_1 = \frac{1}{2\pi i}\oint_\gamma \left\{\frac{1+z^\ell}{z}\prod_{j=1}^{n}\frac{1}{(z-F_{r+j})}\right\}dz = \sum_{k=1}^{n} R_k$$

where

$$R_k = \lim_{z\to F_{r+k}}\left\{\frac{1+z^\ell}{z}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{(z-F_{r+j})}\right\} = \frac{1+F_{r+k}^\ell}{F_{r+k}}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{(F_{r+k}-F_{r+j})}.$$

Then, I_1 becomes

$$I_1 = \sum_{k=1}^{n}\left\{\frac{1+F_{r+k}^\ell}{F_{r+k}}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{(F_{r+k}-F_{r+j})}\right\}.$$

Evaluating (2.2) in the interior of γ contour, we get

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \left\{\frac{1+z^l}{z}\prod_{j=1}^{n}\frac{1}{(z-F_{r+j})}\right\}dz = \lim_{z\to 0}\left\{\frac{1+z^\ell}{A_n(z)}\right\}$$
$$= \frac{1}{A_n(0)} = \frac{(-1)^n}{F_{r+1}F_{r+2}\cdots F_{r+n}}.$$

According to a result on contour integrals [5] we have that $I_1 + I_2 = 0$ and we are done. □

A similar identity also holds for Lucas numbers. It can be stated as

Corollary 2.2. *Let L_n denote the n^{th} Lucas number. That is, $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Then, for all positive integer r holds*

$$\sum_{k=1}^{n}\frac{1+L_{r+k}^\ell}{L_{r+k}}\left\{\prod_{\substack{j=1\\ j\neq k}}\frac{1}{L_{r+k}-L_{r+j}}\right\} = \frac{(-1)^{n+1}}{L_{r+1}L_{r+2}\cdots + L_{r+n}} \tag{2.3}$$

with $0 \leq \ell \leq n-1$.

In particular (2.1) and (2.3) can be used (see [3]) to obtain

Corollary 2.3. *For* $n \geq 2$,

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)} + \frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})}$$
$$+\frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})} = 1$$

Corollary 2.4. *For* $n \geq 2$,

$$\frac{L_{n+1}L_{n+2}}{(L_{n+1}-L_n)(L_{n+2}-L_n)} + \frac{L_{n+2}L_n}{(L_n-L_{n+1})(L_{n+2}-L_{n+1})}$$
$$+\frac{L_nL_{n+1}}{(L_n-L_{n+2})(L_{n+1}-L_{n+2})} = 1.$$

In the sequel F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Theorem 2.5. *If* $n \geq 3$, *then holds*

$$\sum_{i=1}^{n} \frac{1}{L_i^{n-2}} \left[\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} + L_i^{n-1} \right] = L_{n+2} - 3.$$

Proof. First, we observe that the given statement can be written as

$$\sum_{i=1}^{n} \left[\frac{1}{L_i^{n-2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} \right] + \sum_{i=1}^{n} L_i = L_{n+2} - 3.$$

Since $\sum_{i=1}^{n} L_i = L_{n+2}-3$, as can be easily established by mathematical induction, then will be suffice to prove

$$\sum_{i=1}^{n} \left[\frac{1}{L_i^{n-2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} \right] = 0. \tag{2.4}$$

We will argue by using residue techniques. We consider the monic complex polynomial $A(z) = \prod_{k=1}^{n}(z - L_k)$ and we evaluate the integral

$$I = \frac{1}{2\pi i} \oint_{\gamma} \frac{z}{A(z)}\, dz$$

over the interior and exterior domains limited by γ, a circle centered at the origin and radius L_{n+1}, i.e., $\gamma = \{z \in \mathbb{C} : |z| < L_{n+1}\}$. Integrating in the region inside the γ contour we have

$$I_1 = \frac{1}{2\pi i}\oint_\gamma \frac{z}{A(z)}\,dz = \sum_{i=1}^{n} \operatorname{Res}\left\{\frac{z}{A(z)}, z = L_i\right\}$$

$$= \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{L_i}{L_i - L_j}\right) = \sum_{i=1}^{n}\left[\frac{1}{L_i^{n-2}}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1 - \frac{L_j}{L_i}\right)^{-1}\right].$$

Integrating in the region outside of the γ contour we get

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \frac{z}{A(z)}\,dz = 0.$$

According to a well known result on contour integrals [5] we have $I_1 + I_2 = 0$. This completes the proof of (2.4) and we are done. $\square$

Note that (2.4) can also be established by using routine algebra.

3. Inequalities

Next, several inequalities are considered and proved with the aid of integral calculus and the use of classical inequalities. First, we state and prove two nice inequalities involving circular powers of Lucas numbers similar to those obtained for Fibonacci numbers in [4].

Theorem 3.1. *Let n be a positive integer, then hold the following inequalities*

$$(a)\ L_n^{L_{n+1}} + L_{n+1}^{L_{n+2}} + L_{n+2}^{L_n} < L_n^{L_n} + L_{n+1}^{L_{n+1}} + L_{n+2}^{L_{n+2}}$$
$$(b)\ L_n^{L_{n+1}} L_{n+1}^{L_{n+2}} L_{n+2}^{L_n} < L_n^{L_n} L_{n+1}^{L_{n+1}} L_{n+2}^{L_{n+2}}.$$

Proof. Part (a) trivially holds if $n = 1, 2$. To prove the general statement we observe that

$$\left(L_n^{L_n} + L_{n+1}^{L_{n+1}} + L_{n+2}^{L_{n+2}}\right) - \left(L_n^{L_{n+1}} + L_{n+1}^{L_{n+2}} + L_{n+2}^{L_n}\right)$$

$$= \left[\left(L_n^{L_n} + L_{n+1}^{L_{n+1}}\right) - \left(L_n^{L_{n+1}} + L_{n+1}^{L_n}\right)\right] + \left[\left(L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n}\right) - \left(L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n}\right)\right].$$

Therefore, our statement will be established if we prove that for $n \geq 3$,

$$L_n^{L_{n+1}} + L_{n+1}^{L_n} < L_n^{L_n} + L_{n+1}^{L_{n+1}} \tag{3.1}$$

and

$$L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n} < L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n}. \tag{3.2}$$

hold. In fact, we consider the integral

$$I_1 = \int_{L_n}^{L_{n+1}} \Big(L_{n+1}^x \log L_{n+1} - L_n^x \log L_n \Big)\, dx.$$

Since $L_n < L_{n+1}$ if $n \geq 3$, then for $L_n \leq x \leq L_{n+1}$ we have

$$L_n^x \log L_n < L_{n+1}^x \log L_n < L_{n+1}^x \log L_{n+1}$$

and $I_1 > 0$.

By the other hand, evaluating the integral, we obtain

$$\begin{aligned} I_1 &= \int_{L_n}^{L_{n+1}} \Big(L_{n+1}^x \log L_{n+1} - L_n^x \log L_n \Big)\, dx = \big[L_{n+1}^x - L_n^x \big]_{L_n}^{L_{n+1}} \\ &= \Big(L_n^{L_n} + L_{n+1}^{L_{n+1}} \Big) - \Big(L_n^{L_{n+1}} + L_{n+1}^{L_n} \Big) \end{aligned}$$

and (3.1) is proved.

To prove (3.2) we consider the integral

$$I_2 = \int_{L_n}^{L_{n+2}} \Big(L_{n+2}^x \log L_{n+2} - L_{n+1}^x \log L_{n+1} \Big)\, dx.$$

Since $L_{n+1} < L_{n+2}$, then for $L_n \leq x \leq L_{n+2}$ we have

$$L_{n+1}^x \log L_{n+1} < L_{n+2}^x \log L_{n+2}$$

and $I_2 > 0$.

On the other hand, evaluating I_2, we get

$$\begin{aligned} I_2 &= \int_{L_n}^{L_{n+2}} \Big(L_{n+2}^x \log L_{n+2} - L_{n+1}^x \log L_{n+1} \Big)\, dx = \big[L_{n+2}^x - L_{n+1}^x \big]_{L_n}^{L_{n+2}} \\ &= \Big(L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n} \Big) - \Big(L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n} \Big). \end{aligned}$$

This completes the proof of part (a).

We will prove part (b) using the weighted AM-GM-HM inequality (see [6]). The proof will be done in two steps. First, we will prove

$$L_n^{L_{n+1}} L_{n+1}^{L_{n+2}} L_{n+2}^{L_n} < \Big(\frac{L_n + L_{n+1} + L_{n+2}}{3} \Big)^{L_n + L_{n+1} + L_{n+2}}. \tag{3.3}$$

In fact, setting $x_1 = L_n, x_2 = L_{n+1}, x_3 = L_{n+2}$ and

$$w_1 = \frac{L_{n+1}}{L_n + L_{n+1} + L_{n+2}}, w_2 = \frac{L_{n+2}}{L_n + L_{n+1} + L_{n+2}}, w_3 = \frac{L_n}{L_n + L_{n+1} + L_{n+2}}$$

and applying the AM-GM inequality, we have

$$L_n^{L_{n+1}/(L_n+L_{n+1}+L_{n+2})} L_{n+1}^{L_{n+2}/(L_n+L_{n+1}+L_{n+2})} L_{n+2}^{L_n/(L_n+L_{n+1}+L_{n+2})}$$

$$< \frac{L_nL_{n+1}}{L_n+L_{n+1}+L_{n+2}} + \frac{L_{n+1}L_{n+2}}{L_n+L_{n+1}+L_{n+2}} + \frac{L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}}$$

or

$$L_n^{L_{n+1}}L_{n+1}^{L_{n+2}}L_{n+2}^{L_n} < \left(\frac{L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}}\right)^{L_n+L_{n+1}+L_{n+2}}.$$

Inequality (3.3) will be established if we prove that

$$\left(\frac{L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}}\right)^{L_n+L_{n+1}+L_{n+2}} < \left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}}$$

or equivalently

$$\frac{L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}} < \frac{L_n+L_{n+1}+L_{n+2}}{3}.$$

That is,

$$L_n^2+L_{n+1}^2+L_{n+2}^2 > L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n.$$

The last inequality immediately follws by adding up the inequalities $L_n^2+L_{n+1}^2 \ge 2L_nL_{n+1}, L_{n+1}^2+L_{n+2}^2 > 2L_{n+1}L_{n+2}, L_{n+2}^2+L_n^2 > 2L_{n+2}L_n$ and the result is proved.

Finally, we will prove

$$\left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}} < L_n^{L_n}L_{n+1}^{L_{n+1}}L_{n+2}^{L_{n+2}}. \tag{3.4}$$

In fact, setting $x_1 = L_n, x_2 = L_{n+1}, x_3 = L_{n+2}, w_1 = L_n/(L_n+L_{n+1}+L_{n+2}), w_2 = L_{n+1}/(L_n+L_{n+1}+L_{n+2})$, and $w_3 = L_{n+2}/(L_n+L_{n+1}+L_{n+2})$ and using GM-HM inequality, we haven

$$\frac{L_n+L_{n+1}+L_{n+2}}{3} = \left(\frac{3}{L_n+L_{n+1}+L_{n+2}}\right)^{-1}$$

$$= \frac{1}{\frac{1}{L_n+L_{n+1}+L_{n+2}} + \frac{1}{L_n+L_{n+1}+L_{n+2}} + \frac{1}{L_n+L_{n+1}+L_{n+2}}}$$

$$< L_n^{L_n/(L_n+L_{n+1}+L_{n+2})}L_{n+1}^{L_{n+1}/(L_n+L_{n+1}+L_{n+2})}.$$

Hence,

$$\left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}} < L_n^{L_n}L_{n+1}^{L_{n+1}}L_{n+2}^{L_{n+2}}$$

and (3.4) is proved. This completes the proof of part (b) and we are done. □

Stronger inequalities for second order recurrence sequences, generalizing the ones given in [4] have been obtained by Stanica in [7].

Finally, we state and prove an inequality involving Fibonacci and Lucas numbers.

Theorem 3.2. *Let n be a positive integer, then the following inequality*

$$\sum_{k=1}^{n} \frac{F_{k+2}}{F_{2k+2}} \geq \frac{n^{n+1}}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{F_{k+1}^{-\frac{n+1}{n}} - L_{k+1}^{-\frac{n+1}{n}}}{F_{k+1}^{-1} - L_{k+1}^{-1}} \right\}$$

holds.

Proof. From the AM-GM inequality, namely,

$$\frac{1}{n} \sum_{k=1}^{n} x_k \geq \prod_{k=1}^{n} x_k^{\frac{1}{n}}, \quad \text{where} \quad x_k > 0,\ k = 1, 2, \ldots, n,$$

and taking into account that for all $j \geq 2$, is $0 < L_j^{-1} < F_j^{-1}$, we get

$$\int_{L_2^{-1}}^{F_2^{-1}} \int_{L_3^{-1}}^{F_3^{-1}} \cdots \int_{L_{n+1}^{-1}}^{F_{n+1}^{-1}} \left(\frac{1}{n} \sum_{\ell=2}^{n+1} x_\ell \right) dx_2 dx_3 \ldots dx_{n+1}$$

$$\geq \int_{L_2^{-1}}^{F_2^{-1}} \int_{L_3^{-1}}^{F_3^{-1}} \cdots \int_{L_{n+1}^{-1}}^{F_{n+1}^{-1}} \left(\prod_{\ell=1}^{n+1} x_\ell^{\frac{1}{n}} \right) dx_2 dx_3 \ldots dx_{n+1}. \tag{3.5}$$

Evaluating the preceding integrals (3.5) becomes

$$\sum_{\ell=2}^{n+1} (F_2^{-1} - L_2^{-1}) \ldots (F_{\ell-1}^{-1} - L_{\ell-1}^{-1})(F_\ell^{-2} - L_\ell^{-2})(F_{\ell+1}^{-1} - L_{\ell+1}^{-1}) \ldots (F_{n+1}^{-1} - L_{n+1}^{-1})$$

$$\geq \frac{2n^{n+1}}{(n+1)^n} \prod_{\ell=2}^{n+1} \Big(F_\ell^{-\frac{n+1}{n}} - L_\ell^{-\frac{n+1}{n}} \Big) \tag{3.6}$$

or equivalently

$$\prod_{\ell=2}^{n+1} (F_\ell^{-1} - L_\ell^{-1}) \sum_{\ell=2}^{n+1} (F_\ell^{-1} + L_\ell^{-1}) \geq \frac{2n^{n+1}}{(n+1)^n} \prod_{\ell=2}^{n+1} \Big(F_\ell^{-\frac{n+1}{n}} - L_\ell^{-\frac{n+1}{n}} \Big).$$

Setting $k = \ell - 1$ in the preceding inequality, taking into account that $F_k + L_k = 2F_{k+1}, F_k L_k = F_{2k}$ and after simplification, we obtain

$$\sum_{k=1}^{n} \frac{F_{k+2}}{F_{2k+2}} \geq \frac{n^{n+1}}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{F_{k+1}^{-\frac{n+1}{n}} - L_{k+1}^{-\frac{n+1}{n}}}{F_{k+1}^{-1} - L_{k+1}^{-1}} \right\}$$

and the proof is completed. ∎

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