AN ELEMENTARY PROOF OF INEQUALITIES FOR GENERALIZED ELEMENTARY SYMMETRIC MEAN

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Abstract. We give an elementary proof for an inequality involving the generalized elementary symmetric means.

1. Introduction

Let $a = (a_1, a_2, \dots, a_n)$, where a_i,$1 \leqslant i \leqslant n$ be non-negative real numbers and let r be an integer, then

$$E_n^{[r]} = E_n^{[r]}(a) = \sum_{i_1, i_2, \cdots, i_n \geqslant 0}^{i_1+i_2+\cdots+i_n=r} \prod_{k=1}^{n} a_k^{i_k} \tag{1.1}$$

is called the rth generalized elementary symmetric function of a,where the sum is over all $\binom{n+r-1}{r}$ n-tuples of non-negative integers with $i_1 + i_2 + \cdots + i_n = r$. In addition we have $E_n^{[0]} = E_n^{[0]}(a) = 1$ for $n \geqslant 0$, $E_n^{[r]} = 0$ for $r < 0$ or $n \leqslant 0$. We note that

$$\sum_n^{[r]} = \sum_n^{[r]}(a) = E_n^{[r]}(a) / \binom{n+r-1}{r} \tag{1.2}$$

is called the rth generalized elementary symmetric mean of a.

In 1968, K.V. Menon [1] proved the following inequality for $n = 2$ or $n \geqslant 3$ and $r = 1, 2, 3$

$$\sum_n^{[r-1]}(a) \cdot \sum_n^{[r+1]}(a) \geqslant \left[\sum_n^{[r]}(a) \right]^2 \tag{1.3}$$

A question in [2] (see also [3]) arises whether the inequality (1.3) holds for arbitrary $n, r \in \mathbb{N}$.

In 1934, I.Schur obtained in [4]

$$\sum_n^{[r]}(a) = (n-1)! \iint \cdots \int \left(\sum_{i=1}^{n} a_i x_i \right)^r dx_1 dx_2 \cdots dx_{n-1}, \tag{1.4}$$

where $x_n = 1 - (x_1 + x_2 + \cdots + x_{n-1})$, and the integral is over $x_k \geqslant 0$ $(k = 1, 2, \dots, n-1)$. By using (1.4), he proved that (1.3) holds by using the Cauchy integral inequality.

Recently, Z.H. Zhang et al. in [5] proved and generalized (1.4), and also proved (1.3) by using the same proof.

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Let $a = (a_1, a_2, \dots, a_n)$ be defined as above, for $1 \leqslant r \leqslant n$,

$$e_r = \sum_{0\leqslant i_1, i_2, \cdots, i_n \leqslant 1}^{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} a_{i_j}$$

is called the rth elementary symmetric function of a, where the sum is over all $\binom{n}{r}$ n-tuples of non-negative integers with $i_1 + i_2 + \cdots + i_n = r$ and $0 \leqslant i_1, i_2, \cdots, i_n \leqslant 1$. In particular, for $e_0 = 1$, we note

$$p_r = e_r \binom{n}{r}, \quad r = 0, 1, \dots, n,$$

is called the rth elementary symmetric mean of a, then

$$\left[\frac{n!}{(r-1)!(n-r-1)!}\right]^2 (p_r^2 - p_{r-1}p_{r+1}) \tag{1.5}$$

$$= (n-1)\sum (a_1 - a_2)^2 (c_{r-1}^{n-2})^2 + \frac{2!(n-3)}{(r-1)(n-r-1)} \sum (a_1 - a_2)^2 (a_3 - a_4)^2 (c_{r-2}^{n-4})^2$$

$$+ \frac{3!(n-5)}{(r-1)(r-2)(n-r-1)(n-r-2)} \sum (a_1 - a_2)^2 (a_3 - a_4)^2 (a_5 - a_6)^2 (c_{r-3}^{n-6})^2 + \cdots$$

where c_{r-1}^{n-2} is the sum of all the possible $(r-1)$ products of the $(n-2)\, a_k$'s other than a_1, a_2. Similarly, one defines $c_{r-2}^{n-4}, c_{r-3}^{n-6}, \cdots$.

We note that A.E. Jolliffe in [6] proved the Maclaurin symmetric mean inequality $p_r^2 \geqslant p_{r-1}p_{r+1} (1 \leqslant r \leqslant n-1)$. In this paper, we obtain an identity for $\sum_n^{[r]}(a)$ and $E_n^{[r]}(a)$, and give an elementary proof of (1.3).

2. Main Results

Theorem 1. *If $r \in \mathbb{N}$, then*

$$\sum_n^{[r-1]}(a) \cdot \sum_n^{[r+1]}(a) \geqslant \left[\sum_n^{[r]}(a)\right]^2 \tag{2.1}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

In this article, the following generalized result is proved:

Theorem 2. *If $r, s \in \mathbb{N}$, and $r > s$, then*

$$\sum_n^{[s]}(a) \sum_n^{[r+1]}(a) \geqslant \sum_n^{[r]}(a) \sum_n^{[s+1]}(a) \tag{2.2}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

Let $s = r - 1$, then the inequality (2.2) is the inequality (1.3).

3. Proof of the Results

To prove the inequality (2.2), we discussed the following properties for $E_n^{[r]}$:

Property 1. *If $n, r \in \mathbb{N}$, then*

$$E_n^{[r]} = E_{n-1}^{[r]} + a_n E_n^{[r-1]} \tag{3.1}$$

and

$$E_n^{[r]} = \sum_{j=0}^{r} a_n^j E_{n-1}^{[r-j]} \tag{3.2}$$

Proof. If $n = 1$ or $r = 0$, (3.1) holds trivially. When $n > 1$, $r \geqslant 1$, we have

$$\sum_{i_1,i_2,\cdots,i_n \geqslant 0}^{i_1+i_2+\cdots+i_n=r} \prod_{k=1}^{n} a_k^{i_k} = \sum_{i_1,i_2,\cdots,i_{n-1}\geqslant 0, i_n=0}^{i_1+i_2+\cdots+i_n=r} \prod_{k=1}^{n} a_k^{i_k} + a_n \sum_{i_1,i_2,\cdots,i_n\geqslant 0}^{i_1+i_2+\cdots+i_n=r-1} \prod_{k=1}^{n} a_k^{i_k}$$

From the definition of $E_n^{[r]}$, this is just (3.1),and we obtain (3.2) by the recurrence (3.1). ∎

Property 2. *If r be an integer, then*

$$(r+1)E_n^{[r+1]} = \sum_{k=0}^{r} \left(\sum_{i=1}^{n} a_i^{k+1} \right) E_n^{[r-k]} \tag{3.3}$$

Proof. We show this by induction. (3.3) holds trivially for $n = 1$. Now suppose for $n-1$, $n > 1$ and nonnegative integer r, (3.3) is true. We will prove the validity of (3.3) for n.

By (3.2), for $0 \leqslant k \leqslant r$, we have

$$E_n^{[r-k]} = \sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]}$$

and

$$\begin{aligned} &\sum_{j=0}^{r} (j+1) a_n^{j+1} E_{n-1}^{[r-j]} \\ &= a_n E_{n-1}^{[r]} + a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} + a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} \\ &\qquad + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} + a_n^{r+1} E_{n-1}^{[0]} = \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]} \end{aligned}$$

According to the inductive hypothesis, for nonnegative integers r and $0 \leqslant j \leqslant r$, we have

$$(r-j+1)E_{n-1}^{[r+1-j]} = \sum_{k=0}^{r-j} \left(\sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-k-j]}$$

From Property 1 and the above formula, we get

$$\begin{aligned} (r+1)E_n^{[r+1]} &= (r+1) \sum_{j=0}^{r+1} a_n^j E_{n-1}^{[r+1-j]} \\ &= \sum_{j=0}^{r} (r-j+1) a_n^j E_{n-1}^{[r-j+1]} + \sum_{j=1}^{r+1} j a_n^j E_{n-1}^{[r-j+1]} \\ &= \sum_{j=0}^{r} a_n^j \sum_{k=0}^{r-j} \left(\sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-j-k]} + \sum_{j=0}^{r} (j+1) a_n^{j+1} E_{n-1}^{[r-j]} \end{aligned}$$

$$= \sum_{k=0}^{r}\sum_{j=0}^{r-k} a_n^j \left(\sum_{i=1}^{n-1} a_i^{k+1}\right) E_{n-1}^{[r-j-k]} + \sum_{k=0}^{r}\sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]}$$

$$= \sum_{k=0}^{r}\left(\sum_{i=1}^{n-1} a_i^{k+1} + a_n^{k+1}\right)\left(\sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]}\right) = \sum_{k=0}^{r}\left(\sum_{i=1}^{n} a_i^{k+1}\right) E_n^{[r-k]}.$$

This shows (3.3) holds for n and this completes the proof. ∎

Property 3. *If $r, s \in \mathbb{N}$, and $r > s$, then*

$$(3.4) \quad (r+1)(s+1)\binom{n+r}{r+1}\binom{n+s}{s+1}\left(\sum_{n}^{[s]}\sum_{n}^{[r+1]} - \sum_{n}^{[r]}\sum_{n}^{[s+1]}\right)$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\leqslant v<u\leqslant n} \left(E_n^{[s-k]}E_n^{[r-j]} - E_n^{[s-j]}E_n^{[r-k]}\right)\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v - a_u)^2\right]$$

Proof. When $j > k$, we have

$$\sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1} = \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left(a_v^k a_u^{j+1} + a_v^{j+1}a_u^k - a_v^j a_u^{k+1} - a_v^{k+1}a_u^j\right)$$

$$= \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left[a_v^k a_u^k\left(a_v^{j-k+1} + a_u^{j-k+1} - a_v^{j-k}a_u - a_v a_u^{j-k}\right)\right]$$

$$= \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left[a_v^k a_u^k\left(a_v^{j-k} - a_u^{j-k}\right)(a_v - a_u)\right]$$

$$= \sum_{1\leqslant v<u\leqslant n}\left[a_v^k a_u^k\left(a_v^{j-k} - a_u^{j-k}\right)(a_v - a_u)\right]$$

and

$$\left(a_v^{j-k} - a_u^{j-k}\right) = (a_v - a_u)\sum_{t=0}^{j-k-1} a_v^{j-k-1-t}a_u^t$$

Therefore

$$\sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1} = \sum_{1\leqslant v<u\leqslant n}\left[\left(\sum_{t=0}^{k-j-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v - a_u)^2\right]$$

But when $k > j$, we have

$$\sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1} = -\sum_{1\leqslant v<u\leqslant n}$$

$$\left(\sum_{t=0}^{k-j-1} a_v^{j+t}a_u^{k-1-t}\right)(a_v - a_u)^2$$

By the Property 2, we obtain

$$(r+1)E_n^{[r+1]} = \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^{j+1}\right)E_n^{[r-j]}$$

and

$$(n+r)E_n^{[r]} = nE_n^{[r]} + rE_n^{[r]} = \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^j\right)E_n^{[r-j]}$$

From the above formula and note when $k > r, E_n^{[r-k]} = 0$, we have

$$(r+1)(s+1)\left(\begin{array}{c} n+r \\ r+1 \end{array}\right)\left(\begin{array}{c} n+s \\ s+1 \end{array}\right)\left[\sum_n^{[s]}\sum_n^{[r+1]} - \sum_n^{[r]}\sum_n^{[s+1]}\right]$$
$$= (n+s)(r+1)E_n^{[s]}E_n^{[r+1]} - (n+r)(s+1)E_n^{[r]}E_n^{[s+1]}$$
$$= \sum_{k=0}^{s}\left(\sum_{i=1}^{n} a_i^k\right)E_n^{[s-k]} \cdot \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^{j+1}\right)E_n^{[r-j]}$$
$$- \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^j\right)E_n^{[r-j]} \cdot \sum_{k=0}^{s}\left(\sum_{i=1}^{n} a_i^{k+1}\right)E_n^{[s-k]}$$
$$= \sum_{k=0}^{s}\sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1}\right)E_n^{[s-k]} \cdot E_n^{[r-j]}$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\leqslant v<u\leqslant n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$
$$- \sum_{k=0}^{s}\sum_{j=0}^{k}\left[\sum_{1\leqslant v<u\leqslant n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{k-j-1} a_v^{j+t}a_u^{k-1-t}\right)(a_v-a_u)^2\right]$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\leqslant v<u\leqslant n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$
$$- \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\leqslant v<u\leqslant n} E_n^{[s-j]}E_n^{[r-k]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\leqslant v<u\leqslant n} (E_n^{[s-k]}E_n^{[r-j]} - E_n^{[s-j]}E_n^{[r-k]})\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right],$$

which gives expression (3.4). ∎

Property 4. *If $r, s \in \mathbb{N}$, and $r > s$, then*

$$E_n^{[r-1]} \cdot E_n^{[s]} \geqslant E_n^{[r]} \cdot E_n^{[s-1]} \tag{3.5}$$

with equality holding if and only if among $a_1, a_2, \cdots, a_n$, we have $(n-1)\, a_k = 0$'s.

Proof. From the Property 1, we have

$$
\begin{aligned}
&E_n^{[r-1]} \cdot E_n^{[s]} - E_n^{[r]} \cdot E_n^{[s-1]} \\
&= E_n^{[r-1]} \left(E_{n-1}^{[s]} + a_n E_n^{[s-1]} \right) - \left(E_{n-1}^{[r]} + a_n E_n^{[r-1]} \right) E_n^{[s-1]} \\
&= E_n^{[r-1]} \cdot E_{n-1}^{[s]} - E_{n-1}^{[r]} \cdot E_n^{[s-1]} \\
&= \left(\sum_{j=0}^{r-1} a_n^j E_{n-1}^{[r-1-j]} \right) E_{n-1}^{[s]} - E_{n-1}^{[r]} \left(\sum_{j=0}^{s-1} a_n^j E_{n-1}^{[s-1-j]} \right) \\
&= \sum_{j=0}^{s-1} a_n^j \left(E_{n-1}^{[r-1-j]} \cdot E_{n-1}^{[s]} - E_{n-1}^{[r]} \cdot E_{n-1}^{[s-1-j]} \right) + E_{n-1}^{[s]} \left(\sum_{j=s}^{r-1} a_n^j E_{n-1}^{[r-1-j]} \right).
\end{aligned}
$$

Since (3.5) holds $n = 1$, it follow induction that (3.5)holds for n. ∎

Property 5. *If $r, s, j, k \in \mathbb{N}$, and $r > s > j > k$, then*

$$E_n^{[s-k]} \cdot E_n^{[r-j]} \geqslant E_n^{[s-j]} \cdot E_n^{[r-k]} \tag{3.6}$$

with equality holding if and only if among $a_1, a_2, \cdots, a_n$ we have $(n-1)\, a_k = 0$'s.

Proof. From the Property 4, if $r-k+1 > s-k+1, r-k+2 > s-k+2, \cdots, r-j > s-j$, then

$$\prod_{m=k+1}^{j} \left(E_n^{[s-m]} \cdot E_n^{[j-m+1]} \right) \geqslant \prod_{m=k+1}^{j} \left(E_n^{[s-m+1]} \cdot E_n^{[j-m]} \right)$$

This gives(3.6) with the right condition equality. ∎

Combining Property 3 and Property 5, we obtain the Theorem 2.

References

[1] K.V. Menon: Inequalities for symmetric functions. *Duke Math J.*, 35, 37-45 (1968).

[2] D.S. Mitrinovic: *Analytic Inequalities.* Spring-Verlag,1970.

[3] Kuang Jichang:*Applied Inequalities.*2nd ed. Hunan Education Press, Changsha, China, 1993 (Chinese).

[4] G.H. Hardy, J.E. Littlewood, and G. Polya: *Inequalities,* 2nd ed. Cambridge Univ. Press, 1952, 182.

[5] Zhang Zhihua:Three classes of new mean in variables and their applications. *J. Hunan Edu. Ins.*,1997, 15(5):130–136 (Chinese).

[6] A.E. Jolliffe: An identity connected with a polynomial algebraic equation. *J. London Math. Soc.*, **8**(1933),82–85.

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