Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ and $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$ two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \text{ for all } n \in \mathbb{N}, n \geq 3.$$ 

The purpose of this note is to give some conditions which guarantee the simultaneous convergence of the sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ and $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$ two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \text{ for all } n \in \mathbb{N}, n \geq 3.$$ 

By continuity of the functions $f$ and $g$, it results that if the sequence $(x_n)_{n \geq 1}$ is convergent, then the sequence $(y_n)_{n \geq 1}$ is convergent. Moreover, if

$$x_\infty = \lim_{n \to \infty} x_n,$$

then

$$\lim_{n \to \infty} y_n = x_\infty + f(x_\infty) + g(x_\infty).$$

Conversely, if the sequence $(y_n)_{n \geq 1}$ is convergent, is the sequence $(x_n)_{n \geq 1}$ convergent? Usually not. Indeed, if $f(x) = 2x$, $g(x) = x$, for all $x \in \mathbb{R}$ and

$$x_n = (-1)^n, \quad y_n = x_n + f(x_{n-1}) + g(x_{n-2}) = (-1)^n + 2(-1)^{n-1} + (-1)^{n-2} = 0,$$

then

$$\lim_{n \to \infty} y_n = 0.$$
for all $n \in \mathbb{N}$, $n \geq 3$, then the sequence $(y_n)_{n \geq 1}$ is convergent, while the sequence $(x_n)_{n \geq 1}$ is not convergent.

The purpose of this note is to give some conditions which guarantee the convergence of the sequence $(x_n)_{n \geq 1}$, when the sequence $(y_n)_{n \geq 1}$ is convergent.

In what follows, we need the next lemma, which can be proved by mathematical induction.

**Lemma 1** Let $a_1 = \alpha$, $a_2 = \alpha^2 + \beta$ and $a_n = \alpha a_{n-1} + \beta a_{n-2}$, for all $n \in \mathbb{N}$, $n \geq 3$. Then

$$a_n = \frac{\alpha (\alpha^2 + 4\beta) - (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{2 (\alpha^2 + 4\beta)} \left( \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2} \right)^{n-1} +$$

$$+ \frac{\alpha (\alpha^2 + 4\beta) + (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{2 (\alpha^2 + 4\beta)} \left( \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \right)^{n-1},$$

for all $n \in \mathbb{N}$, $n \geq 2$.

The main result of this paper is the following theorem.

**Theorem 2** Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$, such that:

(i) there exist two real numbers $\alpha, \beta \in ]0, 1[$ with $\alpha + \beta < 1$ such that

$$|f(x) - f(u)| \leq \alpha |x - u|, \quad |g(x) - g(u)| \leq \beta |x - u|, \quad \text{for all } x, u \in \mathbb{R},$$

(ii) the function $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x) = x + f(x) + g(x), \quad \text{for all } x \in \mathbb{R}$$

is bijective.

Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N}, n \geq 3.$$  \hspace{1cm} (2)

Then the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

**Proof.** If the sequence $(x_n)_{n \geq 1}$ is convergent, then by continuity of the functions $f$ and $g$, we deduce that the sequence $(y_n)_{n \geq 1}$ is convergent. Moreover, if $x_\infty$ is the limit of $(x_n)_{n \geq 1}$, then $x_\infty + f(x_\infty) + g(x_\infty)$ is the limit of $(y_n)_{n \geq 1}$.
Assume now that the sequence \((y_n)_{n \geq 1}\) is convergent and let \(y_\infty\) be the limit of \((y_n)_{n \geq 1}\).

We begin by showing that the sequence \((x_{n+1} - x_n)_{n \geq 1}\) is convergent to 0. Let hence \(\varepsilon > 0\). By convergence of \((y_n)_{n \geq 1}\), we deduce that there exists an integer number \(m \geq 1\), such that

\[
|y_{n+1} - y_n| < \frac{(1 - q)\varepsilon}{2(1 - q + r)}, \text{ for all } n \in \mathbb{N}, n \geq m, \tag{3}
\]

where

\[
q = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}, \quad r = \frac{\alpha (\alpha^2 + 4\beta) + (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{\alpha^2 + 4\beta}. \tag{4}
\]

On the other hand, from (1) and (2), for each \(n \in \mathbb{N}, n \geq 3\), we have

\[
|x_{n+1} - x_n| \leq |y_{n+1} - y_n| + \alpha |x_n - x_{n-1}| + \beta |x_{n-1} - x_{n-2}|.
\]

From this it follows that for each \(p \in \mathbb{N}, p \geq 3\),

\[
|x_{m+p+1} - x_{m+p}| \leq |y_{m+p+1} - y_{m+p}| + \alpha |x_{m+p} - x_{m+p-1}| + \beta |x_{m+p-1} - x_{m+p-2}|
\leq |y_{m+p+1} - y_{m+p}| + \alpha (|y_{m+p} - y_{m+p-1}| + \alpha |x_{m+p-1} - x_{m+p-2}| + 
+ \beta |x_{m+p-2} - x_{m+p-3}|) + \beta |x_{m+p-1} - x_{m+p-2}| = 
= |y_{m+p+1} - y_{m+p}| + \alpha |y_{m+p} - y_{m+p-1}| + 
+ (\alpha^2 + \beta) |x_{m+p-1} - x_{m+p-2}| + \alpha \beta |x_{m+p-2} - x_{m+p-3}| \leq \ldots 
\leq |y_{m+p+1} - y_{m+p}| + a_1 |y_{m+p} - y_{m+p-1}| + a_2 |y_{m+p-1} - y_{m+p-2}| + \ldots 
\ldots + a_p |y_{m+1} - y_m| + a_{p+1} |x_m - x_{m-1}| + \beta a_p |x_{m-1} - x_{m-2}|,
\]

where

\[
a_1 = \alpha, \quad a_2 = \beta + \alpha^2,
\]

and

\[
a_{k+1} = \alpha a_k + \beta a_{k-1}, \text{ for all } k \in \mathbb{N}, k \geq 2.
\]

Now, relation (3) implies

\[
|x_{m+p+1} - x_{m+p}| \leq (1 + a_1 + a_2 + \ldots + a_p) \frac{1 - q}{2(1 - q + r)} \varepsilon + 
+ a_{p+1} |x_m - x_{m-1}| + \beta a_p |x_{m-1} - x_{m-2}|. \tag{5}
\]
On the other hand, by lemma 1, we have

\[ a_k = \frac{\alpha (\alpha^2 + 4\beta) - (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{2 (\alpha^2 + 4\beta)} \left( \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2} \right)^{k-1} + \]

\[ + \frac{\alpha (\alpha^2 + 4\beta) + (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{2 (\alpha^2 + 4\beta)} \left( \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \right)^{k-1}, \]

for all \( k \in \mathbb{N}, k \geq 2. \)

From this it follows that

\[ 0 \leq a_k \leq \frac{\alpha (\alpha^2 + 4\beta) + (\alpha^2 + 2\beta) \sqrt{\alpha^2 + 4\beta}}{2 (\alpha^2 + 4\beta)} \left( \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \right)^{k-1}, \]

for all \( k \in \mathbb{N}, k \geq 2. \) Then

\[ 1 + a_1 + a_2 + \ldots + a_p \leq 1 + \frac{1}{1 - q} r = \frac{1 - q + r}{1 - q}, \quad (6) \]

where \( q \) and \( r \) are given by (4). From (5) and (6) we obtain

\[ |x_{m+p+1} - x_{m+p}| \leq \frac{\varepsilon}{2} + a_{p+1} |x_m - x_{m-1}| + \beta a_p |x_{m-1} - x_{m-2}|. \quad (7) \]

Since the sequence \((a_n)_{n \geq 1}\) converges to zero, there is an integer number \( p_0 \geq 1 \) such that, for each integer number \( p \geq p_0, \) we have

\[ a_{p+1} |x_m - x_{m-1}| < \frac{\varepsilon}{4} \quad \text{and} \quad \beta a_p |x_{m-1} - x_{m-2}| < \frac{\varepsilon}{4}. \quad (8) \]

From (5), (7) and (8), it follows that

\[ |x_{n+1} - x_n| < \varepsilon, \text{ for all } n \geq m + p_0. \]

Consequently, the sequence \((x_{n+1} - x_n)_{n \geq 1}\) converges to zero.

Now, from

\[ |g(x_{n+1}) - g(x_n)| \leq \beta |x_{n+1} - x_n|, \text{ for all } n \in \mathbb{N}, \]

and by the fact that the sequence \((x_{n+1} - x_n)_{n \geq 1}\) converges to zero, we obtain

\[ \lim_{n \to \infty} (g(x_{n+1}) - g(x_n)) = 0. \quad (9) \]

Let now \((t_n)_{n \geq 1}\) be the sequence with

\[ t_n = x_n + f(x_n) + g(x_n), \text{ for all } n \in \mathbb{N}. \]
Since for each \( n \in \mathbb{N}, n \geq 2 \), we have

\[
  t_n = x_n + y_{n+1} - x_{n+1} - g(x_{n-1}) + g(x_n),
\]

by (9) and by the fact that the sequence \( (x_{n+1} - x_n)_{n \geq 1} \) converges to zero, we deduce that the sequence \( (t_n)_{n \geq 1} \) converges to \( y_\infty \). Then the sequence \( (\varphi^{-1}(t_n))_{n \geq 1} \) is convergent to \( \varphi^{-1}(y_\infty) \). Since

\[
  \varphi^{-1}(t_n) = x_n, \text{ for all } n \in \mathbb{N},
\]

it results that the sequence \( (x_n)_{n \geq 1} \) is convergent to \( \varphi^{-1}(y_\infty) \). The theorem is proved. ■

Some examples are interesting.

**Example 3** Let \( \alpha \) and \( \beta \) be two real numbers such that \( \alpha, \beta \in [0, 1] \), with \( \alpha + \beta < 1 \) and let \( (x_n)_{n \geq 1} \) and \( (y_n)_{n \geq 1} \) be two sequences such that

\[
  y_n = x_n + \alpha \sin x_{n-1} + \beta \arctan x_{n-2}, \text{ for all } n \in \mathbb{N}, n \geq 3.
\]

Then the sequence \( (x_n)_{n \geq 1} \) is convergent if and only if the sequence \( (y_n)_{n \geq 1} \) is convergent.

Moreover, if the sequence \( (x_n)_{n \geq 1} \) converges to \( x_\infty \), then the sequence \( (y_n)_{n \geq 1} \) converges to \( x_\infty + \alpha \sin x_\infty + \beta \arctan x_\infty \), and conversely.

**Proof.** Apply theorem 2, with \( f, g : \mathbb{R} \to \mathbb{R} \) defined by

\[
  f(x) = \alpha \sin x \text{ and } g(x) = \beta \arctan x, \text{ for all } x \in \mathbb{R}.
\]

■

**Example 4** Let \( \alpha \) and \( \beta \) be two real numbers such that \( \alpha, \beta \in [0, 1] \), with \( \alpha + \beta < 1 \) and let \( (x_n)_{n \geq 1} \) and \( (y_n)_{n \geq 1} \) be two sequences such that

\[
  y_n = x_n + \alpha \cos x_{n-1} + \beta \frac{1}{1 + (x_{n-2})^2}, \text{ for all } n \in \mathbb{N}, n \geq 3.
\]

Then the sequence \( (x_n)_{n \geq 1} \) is convergent if and only if the sequence \( (y_n)_{n \geq 1} \) is convergent.

Moreover, if the sequence \( (x_n)_{n \geq 1} \) converges to \( x_\infty \), then the sequence \( (y_n)_{n \geq 1} \) converges to \( x_\infty + \alpha \cos x_\infty + \beta \frac{1}{1 + x_\infty^2} \), and conversely.
Proof. Apply theorem 2, with \( f, g : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \alpha \cos x \quad \text{and} \quad g(x) = \beta \frac{1}{1 + x^2}, \text{ for all } x \in \mathbb{R}.
\]

\[\blacksquare\]

Example 5 Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which satisfies the following two properties:

(i) there is a real number \( \alpha \in [0,1] \) such that

\[
|f(x) - f(u)| \leq \alpha |x - u|, \text{ for all } x, u \in \mathbb{R},
\]

(ii) the function \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(x) = x + f(x) \), for all \( x \in \mathbb{R} \) is bijective.

Let \((x_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) be two sequences such that

\[
y_n = x_n + f(x_{n-1}), \quad \text{for all } n \in \mathbb{N}, n \geq 2.
\]

Then the sequence \((x_n)_{n \geq 1}\) is convergent if and only if the sequence \((y_n)_{n \geq 1}\) is convergent.

Proof. Apply theorem 2. \(\blacksquare\)

Example 6 Let \( g : \mathbb{R} \to \mathbb{R} \) be a function which satisfies the following two properties:

(i) there is a real number \( \beta \in [0,1] \) such that

\[
|g(x) - g(u)| \leq \beta |x - u|, \text{ for all } x, u \in \mathbb{R},
\]

(ii) the function \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(x) = x + g(x) \), for all \( x \in \mathbb{R} \) is bijective.

Let \((x_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) be two sequences such that

\[
y_n = x_n + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N}, n \geq 3.
\]

Then the sequence \((x_n)_{n \geq 1}\) is convergent if and only if the sequence \((y_n)_{n \geq 1}\) is convergent.

Proof. Apply theorem 2. \(\blacksquare\)
References


