SIMULTANEOUS CONVERGENCE OF TWO SEQUENCES

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Abstract

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ and $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N},\ n \geq 3.$$

The purpose of this note is to give some conditions which guarantee the simultaneous convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ and $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N},\ n \geq 3.$$

By continuity of the functions f and g, it results that if the sequence $(x_n)_{n\geq 1}$ is convergent, then the sequence $(y_n)_{n\geq 1}$ is convergent. Moreover, if

$$x_\infty = \lim_{n\to\infty} x_n,$$

then

$$\lim_{n\to\infty} y_n = x_\infty + f(x_\infty) + g(x_\infty).$$

Conversely, if the sequence $(y_n)_{n\geq 1}$ is convergent, is the sequence $(x_n)_{n\geq 1}$ convergent? Usually not. Indeed, if $f(x) = 2x$, $g(x) = x$, for all $x \in \mathbb{R}$ and

$$x_n = (-1)^n,\ y_n = x_n + f(x_{n-1}) + g(x_{n-2}) = (-1)^n + 2(-1)^{n-1} + (-1)^{n-2} = 0,$$

for all $n \in \mathbb{N}$, $n \geq 3$, then the sequence $(y_n)_{n\geq 1}$ is convergent, while the sequence $(x_n)_{n\geq 1}$ is not convergent.

The purpose of this note is to give some conditions which guarantee the convergence of the sequence $(x_n)_{n\geq 1}$, when the sequence $(y_n)_{n\geq 1}$ is convergent.

In what follows, we need the next lemma, which can be proved by mathematical induction.

Lemma 1 *Let $a_1 = \alpha$, $a_2 = \alpha^2 + \beta$ and $a_n = \alpha a_{n-1} + \beta a_{n-2}$, for all $n \in \mathbb{N}$, $n \geq 3$. Then*

$$a_n = \frac{\alpha\left(\alpha^2+4\beta\right) - \left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{2\left(\alpha^2+4\beta\right)}\left(\frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}\right)^{n-1} +$$

$$+\frac{\alpha\left(\alpha^2+4\beta\right) + \left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{2\left(\alpha^2+4\beta\right)}\left(\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}\right)^{n-1},$$

for all $n \in \mathbb{N}$, $n \geq 2$.

The main result of this paper is the following theorem.

Theorem 2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$, such that:*

(i) there exist two real numbers $\alpha, \beta \in]0,1[$ with $\alpha + \beta < 1$ such that

$$|f(x) - f(u)| \leq \alpha |x-u|, \quad |g(x) - g(u)| \leq \beta |x-u|, \text{ for all } x, u \in \mathbb{R}, \tag{1}$$

(ii) the function $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x) = x + f(x) + g(x), \text{ for all } x \in \mathbb{R}$$

is bijective.

Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two sequences such that

$$y_n = x_n + f(x_{n-1}) + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N},\ n \geq 3. \tag{2}$$

Then the sequence $(x_n)_{n\geq 1}$ is convergent if and only if the sequence $(y_n)_{n\geq 1}$ is convergent.

Proof. If the sequence $(x_n)_{n\geq 1}$ is convergent, then by continuity of the functions f and g, we deduce that the sequence $(y_n)_{n\geq 1}$ is convergent. Moreover, if x_∞ is the limit of $(x_n)_{n\geq 1}$, then $x_\infty + f(x_\infty) + g(x_\infty)$ is the limit of $(y_n)_{n\geq 1}$.

Assume now that the sequence $(y_n)_{n\geq 1}$ is convergent and let y_∞ be the limit of $(y_n)_{n\geq 1}$.

We begin by showing that the sequence $(x_{n+1}-x_n)_{n\geq 1}$ is convergent to 0. Let hence $\varepsilon > 0$. By convergence of $(y_n)_{n\geq 1}$, we deduce that there exists an integer number $m \geq 1$, such that

$$|y_{n+1}-y_n| < \frac{(1-q)\,\varepsilon}{2\,(1-q+r)}, \text{ for all } n \in \mathbb{N},\ n \geq m, \tag{3}$$

where

$$q = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2},\ \ r = \frac{\alpha\left(\alpha^2+4\beta\right)+\left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{\alpha^2+4\beta}. \tag{4}$$

On the other hand, from (1) and (2), for each $n \in \mathbb{N}$, $n \geq 3$, we have

$$|x_{n+1}-x_n| \leq |y_{n+1}-y_n| + \alpha\,|x_n-x_{n-1}| + \beta\,|x_{n-1}-x_{n-2}|\,.$$

From this it follows that for each $p \in \mathbb{N}$, $p \geq 3$,

$$\begin{gathered}
|x_{m+p+1}-x_{m+p}| \leq |y_{m+p+1}-y_{m+p}| + \\
+\alpha\,|x_{m+p}-x_{m+p-1}| + \beta\,|x_{m+p-1}-x_{m+p-2}| \leq \\
\leq |y_{m+p+1}-y_{m+p}| + \alpha(|y_{m+p}-y_{m+p-1}| + \alpha\,|x_{m+p-1}-x_{m+p-2}| + \\
+\beta\,|x_{m+p-2}-x_{m+p-3}|) + \beta\,|x_{m+p-1}-x_{m+p-2}| = \\
= |y_{m+p+1}-y_{m+p}| + \alpha\,|y_{m+p}-y_{m+p-1}| + \\
+\left(\alpha^2+\beta\right)|x_{m+p-1}-x_{m+p-2}| + \alpha\beta\,|x_{m+p-2}-x_{m+p-3}| \leq ... \\
\leq |y_{m+p+1}-y_{m+p}| + a_1\,|y_{m+p}-y_{m+p-1}| + a_2\,|y_{m+p-1}-y_{m+p-2}| + ... \\
... + a_p\,|y_{m+1}-y_m| + a_{p+1}\,|x_m-x_{m-1}| + \beta a_p\,|x_{m-1}-x_{m-2}|\,,
\end{gathered}$$

where

$$a_1 = \alpha,\ \ a_2 = \beta+\alpha^2,$$

and

$$a_{k+1} = \alpha a_k + \beta a_{k-1},\ \text{ for all } k \in \mathbb{N},\ k \geq 2.$$

Now, relation (3) implies

$$\begin{aligned}
|x_{m+p+1}-x_{m+p}| \leq (1+a_1+a_2+...+a_p)\,&\frac{1-q}{2\,(1-q+r)}\varepsilon + \\
&+ a_{p+1}\,|x_m-x_{m-1}| + \beta a_p\,|x_{m-1}-x_{m-2}|\,.
\end{aligned} \tag{5}$$

On the other hand, by lemma 1, we have

$$a_k = \frac{\alpha\left(\alpha^2+4\beta\right)-\left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{2\left(\alpha^2+4\beta\right)}\left(\frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}\right)^{k-1}+$$

$$+\frac{\alpha\left(\alpha^2+4\beta\right)+\left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{2\left(\alpha^2+4\beta\right)}\left(\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}\right)^{k-1},$$

for all $k \in \mathbb{N}$, $k \geq 2$.

From this it follows that

$$0 \leq a_k \leq \frac{\alpha\left(\alpha^2+4\beta\right)+\left(\alpha^2+2\beta\right)\sqrt{\alpha^2+4\beta}}{2\left(\alpha^2+4\beta\right)}\left(\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}\right)^{k-1},$$

for all $k \in \mathbb{N}$, $k \geq 2$. Then

$$1 + a_1 + a_2 + ... + a_p \leq 1 + \frac{1}{1-q} r = \frac{1-q+r}{1-q}, \tag{6}$$

where q and r are given by (4). From (5) and (6) we obtain

$$|x_{m+p+1} - x_{m+p}| \leq \frac{\varepsilon}{2} + a_{p+1}|x_m - x_{m-1}| + \beta a_p |x_{m-1} - x_{m-2}|. \tag{7}$$

Since the sequence $(a_n)_{n\geq 1}$ converges to zero, there is an integer number $p_0 \geq 1$ such that, for each integer number $p \geq p_0$, we have

$$a_{p+1}|x_m - x_{m-1}| < \frac{\varepsilon}{4} \text{ and } \beta a_p |x_{m-1} - x_{m-2}| < \frac{\varepsilon}{4}. \tag{8}$$

From (5), (7) and (8), it follows that

$$|x_{n+1} - x_n| < \varepsilon, \text{ for all } n \geq m + p_0.$$

Consequently, the sequence $(x_{n+1} - x_n)_{n\geq 1}$ converges to zero.

Now, from

$$|g(x_{n+1}) - g(x_n)| \leq \beta |x_{n+1} - x_n|, \text{ for all } n \in \mathbb{N},$$

and by the fact that the sequence $(x_{n+1} - x_n)_{n\geq 1}$ converges to zero, we obtain

$$\lim_{n\to\infty}\left(g(x_{n+1}) - g(x_n)\right) = 0. \tag{9}$$

Let now $(t_n)_{n\geq 1}$ be the sequence with

$$t_n = x_n + f(x_n) + g(x_n), \text{ for all } n \in \mathbb{N}.$$

Since for each $n \in \mathbb{N}$, $n \geq 2$, we have

$$t_n = x_n + y_{n+1} - x_{n+1} - g(x_{n-1}) + g(x_n),$$

by (9) and by the fact that the sequence $(x_{n+1} - x_n)_{n \geq 1}$ converges to zero, we deduce that the sequence $(t_n)_{n \geq 1}$ converges to y_∞. Then the sequence $\left(\varphi^{-1}(t_n)\right)_{n \geq 1}$ is convergent to $\varphi^{-1}(y_\infty)$. Since

$$\varphi^{-1}(t_n) = x_n, \text{ for all } n \in \mathbb{N},$$

it results that the sequence $(x_n)_{n \geq 1}$ is convergent to $\varphi^{-1}(y_\infty)$. The theorem is proved. ■

Some examples are interesting.

Example 3 *Let α and β be two real numbers such that $\alpha, \beta \in [0, 1[$, with $\alpha + \beta < 1$ and let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two sequences such that*

$$y_n = x_n + \alpha \sin x_{n-1} + \beta \arctan x_{n-2}, \text{ for all } n \in \mathbb{N},\ n \geq 3.$$

Then the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

Moreover, if the sequence $(x_n)_{n \geq 1}$ converges to x_∞, then the sequence $(y_n)_{n \geq 1}$ converges to $x_\infty + \alpha \sin x_\infty + \beta \arctan x_\infty$, and conversely.

Proof. Apply theorem 2, with $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \alpha \sin x \quad \text{and} \quad g(x) = \beta \arctan x, \text{ for all } x \in \mathbb{R}.$$

■

Example 4 *Let α and β be two real numbers such that $\alpha, \beta \in [0, 1[$, with $\alpha + \beta < 1$ and let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two sequences such that*

$$y_n = x_n + \alpha \cos x_{n-1} + \beta \frac{1}{1 + (x_{n-2})^2}, \text{ for all } n \in \mathbb{N},\ n \geq 3.$$

Then the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

Moreover, if the sequence $(x_n)_{n \geq 1}$ converges to x_∞, then the sequence $(y_n)_{n \geq 1}$ converges to $x_\infty + \alpha \cos x_\infty + \beta \frac{1}{1+x_\infty^2}$, and conversely.

Proof. Apply theorem 2, with $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \alpha \cos x \quad \text{and} \quad g(x) = \beta \frac{1}{1+x^2}, \text{ for all } x \in \mathbb{R}.$$

■

Example 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the following two properties:*

(i) there is a real number $\alpha \in [0, 1[$ such that

$$|f(x) - f(u)| \leq \alpha |x - u|, \text{ for all } x, u \in \mathbb{R},$$

(ii) the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x + f(x)$, for all $x \in \mathbb{R}$ is bijective.

Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two sequences such that

$$y_n = x_n + f(x_{n-1}), \quad \text{for all } n \in \mathbb{N},\ n \geq 2.$$

Then the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

Proof. Apply theorem 2. ■

Example 6 *Let $g : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the following two properties:*

(i) there is a real number $\beta \in [0, 1[$ such that

$$|g(x) - g(u)| \leq \beta |x - u|, \text{ for all } x, u \in \mathbb{R},$$

(ii) the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x + g(x)$, for all $x \in \mathbb{R}$ is bijective.

Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two sequences such that

$$y_n = x_n + g(x_{n-2}), \quad \text{for all } n \in \mathbb{N},\ n \geq 3.$$

Then the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

Proof. Apply theorem 2. ■

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