A GENERALISATION OF AN OSTROWSKI INEQUALITY IN INNER PRODUCT SPACES

SEVER S. DRAGOMIR AND ANCA C. GOŞA

Abstract. A generalisation of inner product spaces of an inequality due to Ostrowski and applications for sequences and integrals are given.

1. Introduction

In 1951, A.M. Ostrowski [2, p. 289] obtained the following result (see also [1, p. 92]).

Theorem 1. *Suppose that* $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)$ *and* $\mathbf{x}=(x_1,\dots,x_n)$ *are real* $n-$*tuples such that* $\mathbf{a}\neq 0$ *and*

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x_i = 1. \tag{1.1}$$

Then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}, \tag{1.2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}, \quad k=1,\dots,n. \tag{1.3}$$

Another similar result due to Ostrowski which is far less known and obtained in the same work [2, p. 130] (see also [1, p. 94]), is the following one.

Theorem 2. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{x}$ *be* $n-$*tuples of real numbers with* $\mathbf{a}\neq 0$ *and*

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = 1. \tag{1.4}$$

Then

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2}{\sum_{i=1}^{n} a_i^2} \geq \left(\sum_{i=1}^{n} b_i x_i\right)^2. \tag{1.5}$$

If $\mathbf{a}$ *and* $\mathbf{b}$ *are not proportional, then the equality holds in (1.5) iff*

$$x_k = q \cdot \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right]^{\frac{1}{2}}}, \quad k \in \{1,\dots,n\}, \tag{1.6}$$

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with $q \in \{-1, 1, \}$.

The case of equality which was neither mentioned in [1] nor in [2] is considered in Remark 1.

In the present paper, by the use of an elementary argument based on Schwarz's inequality, a natural generalisation in inner-product spaces of (1.5) is given. The case of equality is analyzed. Applications for sequences and integrals are also provided.

2. The Results

The following theorem holds.

Theorem 3. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $a, b \in H$ *two linearly independent vectors. If* $x \in H$ *is such that*

$$\langle x, a \rangle = 0 \text{ and } \|x\| = 1, \tag{(i)}$$

then

$$\frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \geq |\langle x, b \rangle|^2. \tag{2.1}$$

The equality holds in (2.1) iff

$$x = \nu \left(b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right), \tag{2.2}$$

where $\nu \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$ *is such that*

$$|\nu| = \frac{\|a\|}{\left[\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right]^{\frac{1}{2}}}. \tag{2.3}$$

Proof. We use Schwarz's inequality in the inner product space H, i.e.,

$$\|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2, \quad u, v \in H \tag{2.4}$$

with equality iff there is a scalar $\alpha \in \mathbb{K}$ such that

$$u = \alpha v. \tag{2.5}$$

If we apply (2.4) for $u = z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c$, $v = d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c$, where $c \neq 0$ and $c, d, z \in H$, and taking into account that

$$\left\| z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2}{\|c\|^2},$$

$$\left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 = \frac{\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2}{\|c\|^2}$$

and

$$\left\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\rangle = \frac{\langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle}{\|c\|^2},$$

we deduce the inequality

$$\left[\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \geq \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^2 \tag{2.6}$$

with equality iff there is a $\beta \in \mathbb{K}$ such that

$$z = \frac{\langle z, c\rangle}{\|c\|^2} \cdot c + \beta \left(d - \frac{\langle d, c\rangle}{\|c\|^2} \cdot c \right). \tag{2.7}$$

If in (2.6) we choose $z = x$, $c = a$ and $d = b$, where a and x satisfy (i), then we deduce

$$\|a\|^2 \left[\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2 \right] \geq \left[\langle x, b\rangle \|a\|^2 \right]^2$$

which is clearly equivalent to (2.1).

The equality holds in (2.1) iff

$$x = \nu \left(b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a \right),$$

where $\nu \in \mathbb{K}$ satisfies the condition

$$1 = \|x\| = |\nu| \left\| b - \frac{\overline{\langle a, b\rangle}}{\|a\|^2} \cdot a \right\| = |\nu| \left[\frac{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}{\|a\|^2} \right]^{\frac{1}{2}}, \tag{2.8}$$

and the theorem is thus proved. □

The following particular cases hold.

1. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x} \in \ell^2(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, where

$$\ell^2(\mathbb{K}) := \left\{ x = (x_i)_{i \in \mathbb{N}},\ \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with $\mathbf{a}$, $\mathbf{b}$ linearly independent and

$$\sum_{i=1}^{\infty} x_i \overline{a_i} = 0, \quad \sum_{i=1}^{\infty} |x_i|^2 = 1, \tag{a}$$

then

$$\frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2}{\sum_{i=1}^{\infty} |a_i|^2} \geq \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right|^2. \tag{2.9}$$

The equality holds in (2.9) iff

$$x_i = \nu \left[b_i - \frac{\sum_{k=1}^{\infty} a_k \overline{b_k}}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \{1, 2, \dots\} \tag{2.10}$$

with $\nu \in \mathbb{K}$ is such that

$$|\nu| = \frac{\left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}}{\left[\sum_{k=1}^{\infty} |a_k|^2 \sum_{k=1}^{\infty} |b_k|^2 - \left| \sum_{k=1}^{\infty} a_k \overline{b_k} \right|^2 \right]^{\frac{1}{2}}}. \tag{2.11}$$

Remark 1. *The case of equality in (1.5) is obviously a particular case of the above. We omit the details.*

2. If $f, g, h \in L^2(\Omega, m)$, where Ω is an $m-$measurable space and

$$L^2(\Omega, m) := \left\{ f : \Omega \to \mathbb{K}, \int_\Omega |f(x)|^2 \, dm(x) < \infty \right\},$$

with f, g being linearly independent and

$$\int_\Omega h(x) \overline{f(x)} dm(x) = 0, \quad \int_\Omega |h(x)|^2 \, dm(x) = 1, \tag{2.12}$$

then

$$\frac{\int_\Omega |f(x)|^2 \, dm(x) \int_\Omega |g(x)|^2 \, dm(x) - \left| \int_\Omega f(x) \overline{g(x)} dm(x) \right|^2}{\int_\Omega |f(x)|^2 \, dm(x)} \geq \left| \int_\Omega h(x) \overline{g(x)} dm(x) \right|^2 . \tag{2.13}$$

The equality holds in (2.13) iff

$$h(x) = \nu \left[g(x) - \frac{\int_\Omega g(x) \overline{f(x)} dm(x)}{\int_\Omega |f(x)|^2 \, dm(x)} f(x) \right] \quad \text{for a.e. } x \in \Omega$$

and $\nu \in \mathbb{K}$ with

$$|\nu| = \frac{\left(\int_\Omega |f(x)|^2 \, dm(x) \right)^{\frac{1}{2}}}{\left[\int_\Omega |f(x)|^2 \, dm(x) \int_\Omega |g(x)|^2 \, dm(x) - \left| \int_\Omega f(x) \overline{g(x)} dm(x) \right|^2 \right]^{\frac{1}{2}}}.$$

References

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School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`

College No. 12, Reşiţa, Jud. Caraş-Severin, R0-1700, Reşiţa, Romania.

E-mail address: `ancagosa@hotmail.com`