ON INTEGRAL FORMS OF GENERALISED MATHIEU SERIES

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Abstract. Integral representations for generalised Mathieu series are obtained which recapture the Mathieu series as a special case. Bounds are obtained through the use of the integral representations.

1. Introduction

The series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2},\quad r>0 \tag{1.1}$$

is well known in the literature as Mathieu's series. It has been extensively studied in the past since its introduction by Mathieu [11] in 1890, where it arose in connection with work on elasticity of solid bodies. The reader is directed to the references for further illustration.

One of the main questions addressed in relation (1.1) is to obtain sharp bounds. Alzer, Brenner and Ruehr [2] showed that the best constants a and b in

$$\frac{1}{x^2+a}<S(x)<\frac{1}{x^2+b},\qquad x\neq 0$$

are $a=\frac{1}{2\zeta(3)}$ and $b=\frac{1}{6}$ where $\zeta(\cdot)$ denotes the Riemann zeta function defined by

$$\zeta(p)=\sum_{n=1}^{\infty}\frac{1}{n^p}. \tag{1.2}$$

An integral representation for $S(r)$ as given in (1.1) was presented in [6] and [7] as

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{x}{e^x-1}\sin(rx)\,dx. \tag{1.3}$$

Guo [9] utilised (1.3) and the following Lemma 1 to obtain bounds on $S(r)$.

Lemma 1. *([3, pp. 89–90]). If $f\in L([0,\infty])$ with $\lim_{t\to\infty}f(t)=0$ then*

$$\sum_{k=1}^{\infty}(-1)^k f(k\pi)<\int_0^{\infty}f(t)\cos t\,dt<\sum_{k=0}^{\infty}(-1)^k f(k\pi)$$

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and

$$\sum_{k=1}^{\infty}(-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right) < \int_0^{\infty} f(t)\sin t dt < f(0) + \sum_{k=0}^{\infty}(-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right). \tag{1.4}$$

Namely, from (1.4) Guo obtained

$$\frac{\pi}{r^3}\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)}{e^{\left(k+\frac{1}{2}\right)\frac{\pi}{r}}-1} < S(r) < \frac{1}{r^2}\left(1+\frac{\pi}{r}\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)}{e^{\left(k+\frac{1}{2}\right)\frac{\pi}{r}}-1}\right). \tag{1.5}$$

The following results were obtained by Qi and coworkers (see [13, 14, 15])

$$\frac{4\left(1+r^2\right)\left(e^{-\frac{\pi}{r}}+e^{-\frac{\pi}{2r}}\right)-4r^2-1}{\left(e^{-\frac{\pi}{r}}-1\right)\left(1+r^2\right)\left(1+4r^2\right)} \le S(r) \le \frac{\left(1+4r^2\right)\left(e^{-\frac{\pi}{r}}-e^{-\frac{\pi}{2r}}\right)-4\left(1+r^2\right)}{\left(e^{-\frac{\pi}{r}}-1\right)\left(1+r^2\right)\left(1+4r^2\right)} \tag{1.6}$$

$$S(r) < \frac{1}{r}\int_0^{\frac{\pi}{r}}\frac{x}{e^x-1}\sin(rx)\,dx < \frac{1+e^{-\frac{\pi}{2r}}}{r^2+\frac{1}{4}},$$

and

$$S(r) \ge \frac{1}{8r\left(1+r^2\right)^3}\left[16r\left(r^2-3\right)+\pi^3\left(r^2+1\right)^3\operatorname{sech}^2\left(\frac{\pi r}{2}\right)\tanh\left(\frac{\pi r}{2}\right)\right].$$

Guo in [9] poses the interesting problem as to whether there is an integral representation of the generalised Mathieu series

$$S_\mu(r) = \sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^{1+\mu}}, \qquad r>0,\ \mu>0. \tag{1.7}$$

This is solved in Section 2.

Recently in [16] an integral representation was obtained for $S_m(r)$, where $m \in \mathbb{N}$, namely

$$\begin{aligned} S_m(r) = {} & \frac{2}{(2r)^m m!}\int_0^{\infty}\frac{t^m}{e^t-1}\cos\left(\frac{m\pi}{2}-rt\right)dt \\ & -2\sum_{k=1}^{m}\left[\frac{(k-1)(2r)^{k-2m-1}}{k!\,(m-k+1)}\binom{-(m+1)}{m-k}\right. \\ & \left.\times\int_0^{\infty}\frac{t^k\cos\left[\frac{\pi}{2}(2m-k+1)-rt\right]}{e^t-1}dt\right]. \end{aligned} \tag{1.8}$$

Bounds were obtained by Tomovski and Trenčevski [16] using (1.3) and (1.4).

It is the intention of the current paper to investigate further integral representations of the generalised Mathieu series (1.7).

Bounds are obtained in Section 3 for $S_\mu(r)$. In Section 4 the open problem of obtaining an integral representation for

$$S(r;\mu,\gamma) = \sum_{n=1}^{\infty}\frac{2n^\gamma}{\left(n^{2\gamma}+r^2\right)^{\mu+1}}$$

posed by Qi [13] is addressed.

We notice that

$$S(r;1,1) = S_1(r) = S(r),$$

the Mathieu series.

2. Integral Representation of the Generalised Mathieu Series $S_\mu(r)$

Before proceeding to obtain an integral representation for $S_\mu(r)$ as given by (1.7), it is instructive to present an alternative representation in terms of the zeta function $\zeta(p)$ presented in (1.2). Namely, a straightforward series expansion gives

$$S_\mu(r) = 2\sum_{k=0}^{\infty} r^{2k}(-1)^k \binom{\mu+k}{k}\zeta(2\mu+2k+1) \tag{2.1}$$

on using the result $\binom{\alpha}{k} = (-1)^k \binom{k-\alpha-1}{k}$ with $\alpha = -(\mu+1)$.

Theorem 1. *The generalised Mathieu series $S_\mu(r)$ defined by (1.7) may be represented in the integral form*

$$S_\mu(r) = C_\mu(r)\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad \mu > 0, \tag{2.2}$$

where

$$C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \tag{2.3}$$

and $J_\nu(z)$ is the ν^{th} order Bessel function of the first kind.

Proof (A). Consider

$$T_\mu(r) = \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)\,dx. \tag{2.4}$$

Then using the series definition for $J_\nu(z)$ (Gradshtein and Ryzhik [8]),

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

in (2.4) produces after the permissible interchange of summation and integral,

$$T_\mu(r) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{r}{2}\right)^{\mu+2k-\frac{1}{2}}}{k!\Gamma\left(\mu+k+\frac{1}{2}\right)}\int_0^\infty \frac{x^{2(\mu+k)}}{e^x-1}dx. \tag{2.5}$$

Now, the well known representation [8]

$$\int_0^\infty \frac{x^p}{e^x-1}dx = \Gamma(p+1)\zeta(p+1) \tag{2.6}$$

gives from (2.5) with $p = 2(\mu+k)$

$$T_\mu(r) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{r}{2}\right)^{\mu+2k-\frac{1}{2}}\Gamma(2\mu+2k+1)\zeta(2\mu+2k+1)}{k!\Gamma\left(\mu+k+\frac{1}{2}\right)}. \tag{2.7}$$

An application of the duplication identity for the gamma function

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

with $z = \mu + k + \frac{1}{2}$ simplifies the expression in (2.7) to

$$T_\mu(r) = \frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}} 2 \sum_{k=0}^{\infty} (-1)^k r^{2k} \frac{\Gamma(\mu+k+1)}{k!} \zeta(2\mu+2k+1). \tag{2.8}$$

Repeated use of the identity $\Gamma(z+1) = z\Gamma(z)$ gives

$$\frac{\Gamma(\mu+k+1)}{k!} = \binom{\mu+k}{k} \Gamma(\mu+1)$$

and so from (2.8)

$$T_\mu(r) = \frac{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)}{\sqrt{\pi}} 2 \sum_{k=0}^{\infty} (-1)^k r^{2k} \binom{\mu+k}{k} \zeta(2\mu+2k+1)$$

produces the result (2.2) on reference to (2.1), (2.3) and (2.4).

The proof is now complete. □

Proof (B). ¿From (2.4) we have

$$\begin{aligned} T_\mu(r) &= \int_0^\infty \frac{e^{-x}}{1-e^{-x}} x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx \\ &= \sum_{k=1}^{\infty} \int_0^\infty e^{-nx} x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx. \end{aligned} \tag{2.9}$$

Now Gradshtein and Ryzhik [8] on page 712 has the result

$$\int_0^\infty e^{-\alpha x} x^{\nu+1} J_\nu(\beta x)\,dx = \frac{2\alpha(2\beta)^\nu \Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\left[\alpha^2+\beta^2\right]^{\nu+\frac{3}{2}}}, \tag{2.10}$$
$$\operatorname{Re}(\nu) > -1, \quad \operatorname{Re}(\alpha) > |\operatorname{Im}\beta|,$$

which is referred to in Watson [18] whom in turn attributes the result to an 1875 result of Gegenbauer.

Taking $\alpha = n$, $\nu = \mu - \frac{1}{2}$ and $\beta = r$, all real, in (2.10) and substituting in (2.9) readily produces

$$T_\mu(r) = \frac{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{2n}{[n^2+r^2]^{\mu+1}},$$

giving from (1.7), (2.4) and (2.3) the result (2.2).

We note that the more restrictive condition of $\mu > 0$ needs to be imposed for the convergence of the series although (2.10) requires $\operatorname{Re}(\nu) = \mu - \frac{1}{2} > -1$. □

Remark 1. *If we take $\mu = 1$ in (1.7) and (2.2) – (2.3) then $S_1(r) \equiv S(r)$, the Mathieu series given by (1.1) and its integral representation (1.3). This is easily seen to be the case since $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$ and taking $\mu = 1$ in (2.2) – (2.3) produces (1.3).*

Remark 2. *Gradshtein and Ryzhik* [8] *on page 712 also quote the result*

$$\int_0^\infty e^{-\alpha x} x^\nu J_\nu(\beta x)\,dx = \frac{(2\beta)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\alpha^2+\beta^2\right)^{\nu+\frac{1}{2}}} \tag{2.11}$$
$$\operatorname{Re}(\nu) > -\frac{1}{2}, \qquad \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|,$$

which Watson [18] *again attributes to an 1875 result by Gegenbauer.*

We note that formal differentiation of (2.11) with respect to α *produces the result (2.10).*

Following a similar process as in Proof B above, we may show that

$$\int_0^\infty \frac{x^{\mu-\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)\,dx = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}} \sum_{n=1}^\infty \frac{1}{(n^2+r^2)^\mu}. \tag{2.12}$$

Gradshtein and Ryzhik [8] *have an explicit expression which can be transformed by a simple change of variables to (2.12). Namely,*

$$\int_0^\infty \frac{x^\nu J_\nu(bx)}{e^{\pi x}-1}dx = \frac{(2b)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{n=1}^\infty \frac{1}{(n^2\pi^2+b^2)^{\nu+\frac{1}{2}}}, \tag{2.13}$$

$\mathrm{Re}(\nu) > 0$, $|\mathrm{Im}(b)| < \pi$, *which is attributed by Watson* [18] *to a 1906 result by Kapteyn.*

An explicit integral expression for $S_\mu(r)$ *of the current form does not seem to have been available previously.*

Finally, we note that (2.10) or (2.11) may be looked upon as an integral transform such as the Laplace or Hankel transform and the results may be found in tables of such.

Remark 3. $S_\mu(r)$ *as given in (2.2) – (2.3) may be written in the alternate form*

$$S_\mu(r) = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}} r^{2\mu-1}\Gamma(\mu+1)} \int_0^\infty \frac{x}{e^x-1}\left[(rx)^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\right]dx, \tag{2.14}$$

which, for $\mu = m$*, a positive integer*

$$S_m(r) = \frac{1}{2^{m-1}r^{2m-1}m!}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{x}{e^x-1}R_m(rx)\,dx, \tag{2.15}$$

where

$$\sqrt{\frac{\pi}{2}}R_m(z) = \sqrt{\frac{\pi}{2}} z^{m-\frac{1}{2}} J_{m-\frac{1}{2}}(z). \tag{2.16}$$

For $m = 1, 2, 3, 4$ *we have*

$$\sqrt{\frac{\pi}{2}}R_m(z) = \sin z, \quad \sin z - z\cos z, \quad 3\sin z - 3z\cos z - z^2\sin z,$$

and

$$15\sin z - 15z\cos z - 6z^2\sin z + z^3\cos z,$$

respectively.

Thus, for example,

$$\begin{aligned}
S_1(r) &= \frac{1}{r}\int_0^\infty \frac{x}{e^x-1}\sin(rx)\,dx,\\
S_2(r) &= \frac{1}{4r^3}\int_0^\infty \frac{x}{e^x-1}\left[\sin(rx) - (rx)\cos(rx)\right]dx,\\
S_3(r) &= \frac{1}{24r^5}\int_0^\infty \frac{x}{e^x-1}\left[3\sin(rx) - 3(rx)\cos(rx) - (rx)^2\sin(rx)\right]dx,
\end{aligned}$$

and

$$S_4(r) = \frac{1}{192r^7}\int_0^\infty \frac{x}{e^x-1}\Big[15\sin(rx) - 15(rx)\cos(rx) \\ -6(rx)^2\sin(rx) + (rx)\cos(rx)\Big]\,dx.$$

The above results for integer m can also be obtained using the relationship from (1.1) and (1.3)

$$S_1(r) = S(r) = \sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2} = \frac{1}{r}\int_0^\infty \frac{x}{e^x-1}\sin(rx)\,dx. \tag{2.17}$$

Formal differentiation with respect to r of (2.17) gives

$$\begin{aligned}(-4r)S_2(r) &= \int_0^\infty \frac{x}{e^x-1}\left[\frac{x\cos rx}{r} - \frac{\sin rx}{r^2}\right]dx \\ &= -\frac{1}{r^2}\int_0^\infty \frac{x}{e^x-1}(\sin rx - rx\cos rx)\,dx\end{aligned}$$

producing the result above. Continuing in this manner would produce further representations for $S_m(r)$.

The following theorem gives an explicit representation for $S_m(r)$, $m \in \mathbb{N}$.

Theorem 2. *For m a positive integer we have*

$$S_m(r) = \frac{1}{2^{m-1}}\cdot\frac{1}{r^{2m-1}}\cdot\frac{1}{m}\sum_{k=0}^{m-1}\frac{(-1)^{\left\lfloor\frac{3k}{2}\right\rfloor}}{k!}r^k\left[\delta_{k\ even}A_k(r) + \delta_{k\ odd}B_k(r)\right], \tag{2.18}$$

where

$$A_k(r) = \int_0^\infty \frac{x^{k+1}}{e^x-1}\sin(rx)\,dx, \quad B_k(r) = \int_0^\infty \frac{x^{k+1}}{e^x-1}\cos(rx)\,dx, \tag{2.19}$$

with $\delta_{condition} = 1$ if condition holds and zero otherwise and $\lfloor x\rfloor$ is the smallest integer part of x.

Proof. ¿From (2.17) we may differentiate $m-1$ times with respect to r to produce

$$\begin{aligned}S_1^{(m-1)}(r) &= (-1)^{m-1}m!\,(2r)^{m-1}S_m(r) \\ &= \int_0^\infty \frac{x}{e^x-1}\cdot\frac{d^{m-1}}{dr^{m-1}}\left(\frac{\sin rx}{r}\right)dx.\end{aligned} \tag{2.20}$$

Now,

$$\frac{d^{m-1}}{dr^{m-1}}\left(\frac{\sin rx}{r}\right) = \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{d^{m-1-k}}{dr^{m-1-k}}\left(r^{-1}\right)\cdot\frac{d^k}{dr^k}(\sin rx) \tag{2.21}$$

and

$$\begin{aligned}\frac{d^l}{dr^l}\left(r^{-1}\right) &= (-1)^l\, l!\,r^{-(l+1)}, \\ \frac{d^k}{dr^k}(\sin rx) &= (-1)^{\left\lfloor\frac{k}{2}\right\rfloor}x^k\left[\delta_{k\ \mathrm{even}}\sin(rx) + \delta_{k\ \mathrm{odd}}\cos(rx)\right]\end{aligned}$$

where $\delta_{\mathrm{condition}} = 1$ if condition is true and zero otherwise.

Thus from (2.21)

$$\frac{d^{m-1}}{dr^{m-1}}\left(\frac{\sin(rx)}{r}\right) \tag{2.22}$$
$$= \frac{1}{r^m}\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{m-1-k+\lfloor\frac{k}{2}\rfloor}$$
$$\times (m-1-k)! r^k x^k \left[\delta_{k\text{ even}}\sin(rx)+\delta_{k\text{ odd}}\cos(rx)\right]$$
$$= \frac{(-1)^{m-1}}{r^m}(m-1)!\sum_{k=0}^{m-1}\frac{(-1)^{\lfloor\frac{3k}{2}\rfloor}}{k!}r^k x^k\left[\delta_{k\text{ even}}\sin(rx)+\delta_{k\text{ odd}}\cos(rx)\right].$$

Substitution of (2.22) into (2.20) and simplifying produces the stated result (2.18). □

Remark 4. *The integral representation for $S_m(r)$ given in Theorem 2 is simpler than that obtained in* [16] *as given by (1.8). Further, the derivation here is much more straight forward.*

3. Bounds for $S_\mu(r)$

It was stated in the introduction that considerable effort has been expended in determining bounds for the generalised Mathieu series. More recently, bounds for the generalised Mathieu series (1.7) has been investigated in particular by Qi and coworkers and by Tomovski and Trenčevski [16].

In a recent article Landau [10] obtained the best possible uniform bounds for Bessel functions using monotonicity arguments. Of particular interest to us here is that he showed that

$$|J_\nu(x)| < \frac{b_L}{\nu^{\frac{1}{3}}} \tag{3.1}$$

uniformly in the argument x and is best possible in the exponent $\frac{1}{3}$ and constant

$$b_L = 2^{\frac{1}{3}}\sup_x Ai(x) = 0.674885\cdots, \tag{3.2}$$

where $Ai(x)$ is the Airy function satisfying

$$w'' - xw = 0.$$

Landau also showed that

$$|J_\nu(x)| \le \frac{c_L}{x^{\frac{1}{3}}} \tag{3.3}$$

uniformly in the order $\nu > 0$ and the exponent $\frac{1}{3}$ is best possible with

$$c_L = \sup_x x^{\frac{1}{3}} J_0(x) = 0.78574687\ldots. \tag{3.4}$$

The following theorem is based on the Landau bounds (3.1) – (3.4).

Theorem 3. *The generalised Mathieu series $S_\mu(r)$ satisfies the bounds for $\mu > \frac{1}{2}$ and $r > 0$*

$$S_\mu(r) \le b_L\frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}}\cdot\frac{1}{\left(\mu-\frac{1}{2}\right)^{\frac{1}{3}}}\cdot\frac{\Gamma\left(\mu+\frac{3}{2}\right)}{\Gamma(\mu+1)}\zeta\left(\mu+\frac{3}{2}\right), \tag{3.5}$$

and

$$S_\mu(r) \le c_L \cdot \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}} r^{\mu-\frac{1}{6}}} \cdot \Gamma\left(\mu+\frac{7}{6}\right)\zeta\left(\mu+\frac{7}{6}\right), \tag{3.6}$$

where b_L and c_L are given by (4.2) and (4.4) respectively.

Proof. ¿From (2.2) and (2.3) we have

$$S_\mu(r) \le C_\mu(r) \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1}\left|J_{\mu-\frac{1}{2}}(rx)\right| dx, \quad r>0 \tag{3.7}$$

and so from (3.1) we obtain, on utilising (2.6)

$$S_\mu(r) \le C_\mu(r) \cdot \frac{b_L}{\left(\mu-\frac{1}{2}\right)^{\frac{1}{3}}}\Gamma\left(\mu+\frac{3}{2}\right)\zeta\left(\mu+\frac{3}{2}\right),$$

which simplifies down to (3.5).

Further, using (3.3) into (3.7) gives

$$\begin{aligned} S_\mu(r) &\le C_\mu(r) \cdot c_L \cdot \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1} \cdot \frac{1}{|rx|^{\frac{1}{3}}} dx \\ &= C_\mu(r) \cdot \frac{c_L}{r^{\frac{1}{3}}} \int_0^\infty \frac{x^{\mu+\frac{1}{6}}}{e^x-1} dx \end{aligned}$$

which upon using (2.6) produces

$$S_\mu(r) \le C_\mu(r) \cdot \frac{c_L}{r^{\frac{1}{3}}} \cdot \Gamma\left(\mu+\frac{7}{6}\right)\zeta\left(\mu+\frac{7}{6}\right). \tag{3.8}$$

Simplifying (3.8) and using (2.3) gives the stated result (3.6). □

Corollary 1. *The Mathieu series $S(r)$ satisfies the following bounds*

$$S(r) \le \frac{3\pi}{2^{\frac{11}{12}}} b_L \zeta\left(\frac{5}{2}\right) \tag{3.9}$$

and

$$S(r) \le \frac{7c_L}{36} \cdot \sqrt{\frac{\pi}{2}} \cdot \Gamma\left(\frac{1}{6}\right)\zeta\left(\frac{13}{6}\right) \cdot r^{-\frac{5}{6}}, \tag{3.10}$$

where b_L and c_L are given by (3.2) and (3.4) respectively.

Proof. Taking $\mu = 1$ in (3.5) and (3.6), noting that $S(r) = S_1(r)$ gives the stated results after some simplification. □

The following corollary gives coarser bounds than Theorem 3 without the presence of the zeta function.

Corollary 2. *The generalised Mathieu series $S_\mu(r)$ satisfies the bounds for $\mu > \frac{1}{2}$ and $r > 0$*

$$S_\mu(r) \le 2\sqrt{\pi} \cdot \frac{b_L}{\left(\mu-\frac{1}{2}\right)^{\frac{1}{3}}} \cdot \frac{1}{r^{\mu-\frac{1}{2}}} \cdot \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)} \tag{3.11}$$

and

$$S_\mu(r) \le 2^{\frac{2}{3}}\sqrt{\pi} \cdot \frac{c_L}{r^{\mu-\frac{1}{6}}} \cdot \frac{\Gamma\left(\mu+\frac{1}{6}\right)}{\Gamma(\mu+1)} \tag{3.12}$$

with b_L and c_L given by (3.2) and (3.4).

Proof. We use the well known inequality

$$e^{-x} < \frac{x}{e^x - 1} < e^{-\frac{x}{2}}$$

to produce from (3.7)

$$S_\mu(r) \le C_\mu(r) \int_0^\infty e^{-\frac{x}{2}} x^{\mu-\frac{1}{2}} \left| J_{\mu-\frac{1}{2}}(rx) \right| dx. \tag{3.13}$$

We know from Laplace transforms or the definition of the gamma function that

$$\int_0^\infty e^{-\alpha x} x^s dx = \frac{\Gamma(s+1)}{\alpha^{s+1}}. \tag{3.14}$$

Hence, placing (3.1) into (3.13) and utilising (3.14) we obtain (3.11) after simplification. A similar approach produces (3.12) starting from (3.3) rather than (3.1). □

4. Further Integral Expressions for Generalised Mathieu Series

In [16], Tomovski and Trenčevski gave the integral representation

$$S_\mu(r) = \frac{2}{\Gamma(\mu+1)} \int_0^\infty x^\mu e^{-r^2 x} f(x)\, dx, \tag{4.1}$$

where

$$f(x) = \sum_{n=1}^\infty n e^{-n^2 x}, \quad \text{convergent for finite } x > 0, \tag{4.2}$$

by effectively utilising the result (3.14).

They leave the summation of the series in (4.2) as an open problem.

If we place $\nu = \mu - \frac{1}{2}$ and $\beta = r$, all real in (2.9) then we obtain the identity

$$C_\mu(r) \int_0^\infty e^{-\alpha x} x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx = \frac{2\alpha}{[\alpha^2 + r^2]^{\mu+1}}, \tag{4.3}$$

where $C_\mu(r)$ is as given by (2.3).

Proof B of Theorem 1 takes $\alpha = n$ and sums to produce the identity (2.1) – (2.2).

If we take $\alpha = n^\gamma$ then we have from (4.3) on summing

$$\begin{aligned} S(r;\mu,\gamma) &= \sum_{n=1}^\infty \frac{2n^\gamma}{(n^{2\gamma} + r^2)^{\mu+1}} \\ &= C_\mu(r) \int_0^\infty \left(\sum_{n=1}^\infty e^{-n^\gamma x} \right) x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx, \end{aligned} \tag{4.4}$$

giving an integral representation that was left as an open problem by Qi [13].

As a matter of fact, if we take $\alpha = a_n$ where $\mathbf{a} = (a_1, a_2, \dots, a_n, \dots)$ is a positive sequence, then

$$\begin{aligned} \mathcal{S}(r;\mu;\mathbf{a}) &= \sum_{n=1}^\infty \frac{2a_n}{(a_n^2 + r^2)^{\mu+1}} \\ &= C_\mu(r) \int_0^\infty \left(\sum_{n=1}^\infty e^{-a_n x} \right) x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx. \end{aligned} \tag{4.5}$$

We note that for $\mathbf{a}^{+} = (1^{\gamma}, 2^{\gamma}, \dots)$ then

$$\mathcal{S}\left(r;\mu;\mathbf{a}^{+}\right) = S\left(r;\mu,\gamma\right).$$

The series

$$\sum_{n=1}^{\infty} \frac{2a_n}{\left(a_n^2 + r^2\right)^2}$$

has been investigated in [14].

A closed form expression for

$$F\left(\mathbf{a}\right) = \sum_{n=1}^{\infty} e^{-a_n x}, \qquad x > 0$$

where a_n is a positive sequence, remains an open problem.

If $\mathbf{a}^{*} = (1, 2, 3, \dots, n, \dots)$, then

$$F\left(\mathbf{a}^{*}\right) = \frac{1}{e^{x} - 1}.$$

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