A NEW APPROACH TO KY FAN-TYPE INEQUALITIES

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ABSTRACT. The study of the behavior of means under equal increments of their variables provides a new approach to Ky Fan-type inequalities. Via this new approach we are able to prove some new results on Ky Fan-type inequalities.

1. INTRODUCTION

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} \omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n} \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume $0 < x_1 \leq x_2 \cdots \leq x_n, m = \min\{x_i\}, M = \max\{x_i\}$. We denote $\sigma_n = \sum_{i=1}^{n} \omega_i (x_i - A_n)^2$.

To any given $\mathbf{x}, t \geq 0$ we associate $\mathbf{x}' = (1 - x_1, 1 - x_2, \cdots, 1 - x_n), \mathbf{x}_t = (x_1 + t, \cdots, x_n + t)$. When there is no risk of confusion, we shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, $P_{n,r,t}$ for $P_{n,r}(\mathbf{x}_t)$ and $P'_{n,r}$ for $P_{n,r}(\mathbf{x}')$ if $1 - x_i \geq 0$ for all i. We also define $A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1}$ and similarly for $A_{n,t}, G_{n,t}, H_{n,t}$.

Recently, the author[10] proved the following result.

Theorem 1.1. *For $r > s, m > 0, t \geq 0$, the following inequalities are equivalent:*

$$\frac{r-s}{2m}\sigma_n \geq \quad P_{n,r} - P_{n,s} \quad \geq \frac{r-s}{2M}\sigma_n, \tag{1.1}$$

$$\frac{M}{1-M}(P_{n,r} - P_{n,s}) \geq \quad P'_{n,r} - P'_{n,s} \quad \geq \frac{m}{1-m}(P_{n,r} - P_{n,s}), \tag{1.2}$$

where in (1.2) we require $M < 1$.

D.Cartwright and M.Field[5] first proved the validity of (1.1) for $r = 1, s = 0$. For other extensions and refinements of (1.1), see [3], [7],[13] and [14]. (1.2) is commonly referred as the additive Ky Fan's inequality. We refer the reader to the survey article[2] and the references therein for an account of Ky Fan's inequality.

The study of the behavior of means under equal increments of their variables was initiated by L. Hoehn and I. Niven[12]. J.Aczél and Zs. Pâles[1] studied the monotonicity of $A_{n,t} - P_{n,s,t}$ as a function of t for any s. The asymptotic behavior of $t(P_{n,r,t} - A_{n,t})$ was studied by J.Brenner and B. Carlson[4]. By studying the monotonicities of $(t + M)(P_{n,r,t} - P_{n,s,t})$ and $(t + m)(P_{n,r,t} - P_{n,s,t})$ as functions of t for $r = 1$ or $s = 1$, the author[9] was able to prove some known results on inequalities of the type (1.1). In fact, the study of the behavior of means under equal increments of their variables can provide us clues on what might be true for inequalities of Ky Fan's type and it is the main goal of this paper to use this approach to give some new results in this direction.

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2. The Main Results

To simplify expressions, we define

$$\Delta_{r,s,t,\alpha}=\frac{P^{\alpha}_{n,r,t}-P^{\alpha}_{n,s,t}}{P^{\alpha}_{n,r}-P^{\alpha}_{n,s}},$$

with $\Delta_{r,s,t,0}=(\ln \frac{P_{n,r,t}}{P_{n,s,t}})/(\ln \frac{P_{n,r}}{P_{n,s}})$. We also write $\Delta_{r,s,t}$ for $\Delta_{r,s,t,1}$. In order to include the case of equality for various inequalities in our discussions, for any given inequality, we define $0/0$ to be the number which makes the inequality an equality.

Suppose we want to prove $A_n-G_n\geq 0$. One way is to show $f(t)=A_{n,t}-G_{n,t}$ is a decreasing function of t, since $\lim_{t\to\infty}f(t)=0$. Since $\mathbf{x}$ is arbitrary, it suffices to show $f'(0)=1-G_n/H_n\leq 0$. How to show this? It is natural to show $g(t)=1-G_{n,t}/H_{n,t}$ is an increasing function of t and this idea leads to

Theorem 2.1. *Let $r>s,t\geq 0, x_1>0$.*
(i). If $\Delta_{r,s,t,\alpha}\leq 1$, then $\Delta_{r,s,t,\beta}\leq 1$ for $\beta\leq\alpha$.
(ii). $\Delta_{r,s,t,\alpha}\leq 1$ for $\alpha\leq 0$.

Proof. (i). Let $f(t)=|P^{\alpha}_{n,r,t}-P^{\alpha}_{n,s,t}|$, since $\mathbf{x}$ is arbitrary, $\Delta_{r,s,t,\alpha}\leq 1$ is then equivalent to $f'(0)\leq 0$ or the second inequality below

$$\frac{P^{\beta-r}_{n,r}}{P^{\beta-s}_{n,s}}\leq\frac{P^{\alpha-r}_{n,r}}{P^{\alpha-s}_{n,s}}\leq\frac{P^{r-1}_{n,r-1}}{P^{s-1}_{n,s-1}}.$$

Now $\Delta_{r,s,t,\beta}\leq 1$ follows from the first inequality above.

(ii). By part (i), it suffices to show $\Delta_{r,s,t,0}\leq 1$, which is an analogue to the result of J.Chen and Z.Wang [6]. Let $f(t)=\ln P_{n,r,t}-\ln P_{n,s,t}$, it suffices to show $f'(0)\leq 0$ or

$$\frac{\sum_{i=1}^n\omega_i x_i^{r-1}}{\sum_{i=1}^n\omega_i x_i^{r}}\leq\frac{\sum_{i=1}^n\omega_i x_i^{s-1}}{\sum_{i=1}^n\omega_i x_i^{s}}. \tag{2.1}$$

We use the idea of [6] to show (2.1) holds if and only if it holds for $n=2$. Assuming this, and let $0<x_1\leq x_2\leq\cdots\leq x_n, n\geq 3$. Then there exists $\mu>0$ and $\nu=\omega_1\omega_n/\mu>0$ such that

$$S=\frac{\sum_{i=1}^n\omega_i x_i^{r-1}}{\sum_{i=1}^n\omega_i x_i^{r}}=\frac{\mu x_1^{r-1}+\omega_n x_n^{r-1}}{\mu x_1^{r}+\omega_n x_n^{r}}=\frac{\omega_1 x_1^{r-1}+\nu x_n^{r-1}}{\omega_1 x_1^{r}+\nu x_n^{r}}.$$

It's clear $(\omega_1-\mu)(\omega_n-\nu)\leq 0$. Without loss of generality, we may assume $\omega_1\geq\mu$. So

$$S=\frac{(\omega_1-\mu)x_1^{r-1}+\sum_{i=2}^{n-1}\omega_i x_i^{r-1}}{(\omega_1-\mu)x_1^{r}+\sum_{i=2}^{n-1}\omega_i x_i^{r}}\leq\frac{(\omega_1-\mu)x_1^{s-1}+\sum_{i=2}^{n-1}\omega_i x_i^{s-1}}{(\omega_1-\mu)x_1^{s}+\sum_{i=2}^{n-1}\omega_i x_i^{s}}, \tag{2.2}$$

where the inequality follows from induction. Also by induction

$$S=\frac{\mu x_1^{r-1}+\omega_n x_n^{r-1}}{\mu x_1^{r}+\omega_n x_n^{r}}\leq\frac{\mu x_1^{s-1}+\omega_n x_n^{s-1}}{\mu x_1^{s}+\omega_n x_n^{s}}. \tag{2.3}$$

So (2.2), (2.3) imply

$$S\leq\frac{\omega_1 x_1^{s-1}+\sum_{i=2}^{n-1}\omega_i x_i^{s-1}+\omega_n x_n^{s-1}}{\omega_1 x_1^{s}+\sum_{i=2}^{n-1}\omega_i x_i^{s}+\omega_n x_n^{s}},$$

the desired inequality.

Thus it suffices to prove (2.1) for $n=2$. In this case, let

$$g(p)=\frac{\omega_1 x_1^{p-1}+\omega_2 x_2^{p-1}}{\omega_1 x_1^{p}+\omega_2 x_2^{p}},$$

then simple calculation shows

$$g'(p) = \frac{\omega_1\omega_2(x_1x_2)^{p-1}(\ln x_1 - \ln x_2)(x_2 - x_1)}{(\omega_1 x_1^p + \omega_2 x_2^p)^2} < 0$$

for $x_1 \neq x_2$ and it follows $g(r) \leq g(s)$ for $r > s$ and this completes the proof. □

We remark here in general $P_{n,r,t} - P_{n,s,t}$ as a function of t is not monotonic for any $\mathbf{x}, r, s$. For example, when $r = 0, s = -1$ and let $f(t) = G_{n,t} - H_{n,t}$. Then $f'(0) = G_n/H_n - H_n^2/P_{n,-2}^2$. By a change of variables $x_i \to 1/x_{n-i+1}$ we can rewrite $f'(0)$ as $f'(0) = (A_n^3 - G_nP_{n,2}^2)/(A_n^2G_n)$ and by considering the case $n = 2$, it is easy to see that A_n^3 and $G_nP_{n,2}^2$ are not comparable in general.

Now suppose we want to prove the additive Ky Fan's inequality $A_n - G_n \geq \sigma_n/2x_n$. One way is to show $f(t) = (x_n + t)(A_{n,t} - G_{n,t})$ is a decreasing function of t, or $f'(0) = A_n - G_n + x_n(1 - G_n/H_n) \leq 0$. How to show this? It's natural to show $g(t) = A_{n,t} - G_{n,t} + (x_n + t)(1 - G_{n,t}/H_{n,t})$ is a decreasing function of t and this idea leads to

Theorem 2.2. *For $0 < x_1 \leq \cdots \leq x_n$, the following inequalities are equivalent:*
(i). $A_n - G_n \geq \sigma_n/2x_n$; (ii). $A_n - G_n \leq \sigma_n/2x_1$; (iii). $A_n - G_n \leq \frac{x_n}{H_n}(G_n - H_n)$; (iv). $A_n - G_n \geq \frac{x_1}{H_n}(G_n - H_n)$; (v). $G_n - H_n \geq H_n\sigma_n/2x_n^2$; (vi). $G_n - H_n \leq H_n\sigma_n/2x_1^2$.
In particular, since inequality (i) holds, all the inequalities above are valid.

Proof. We first show (ii) ⇒ (iii) ⇒ (i) and similarly one can show (i) ⇒ (iv) ⇒ (ii).
(iii) ⇒ (i): this follows from the discussion above.
(ii) ⇒ (iii): Let $f(t) = A_{n,t} - G_{n,t} + (x_n + t)(1 - G_{n,t}/H_{n,t}), t \geq 0$. It is easy to see that $\lim_{t\to\infty} f(t) = 0$ so it suffices to show $f'(t) \geq 0$ in order to prove (iii). Since $\mathbf{x}$ is arbitrary, it suffices to show $f'(0) \geq 0$. Calculation yields

$$f'(0)/G_n = 2(\frac{1}{G_n} - \frac{1}{H_n}) + x_n(\frac{1}{H_n^2} - \frac{1}{P_{n,-2}^2}). \tag{2.4}$$

By a change of variables $x_i \to 1/x_{n-i+1}$, the right-hand side inequality of (2.4) becomes

$$2(G_n - A_n) + \frac{1}{x_1}\sigma_n \geq 0$$

by (ii).
Now we show (i) and (v) are equivalent, similarly one can show (i) and (vi) are equivalent.
(i) ⇒ (v): We have shown (i) and (iii) are equivalent and hence (v) follows.
(v) ⇒ (i): Let $f(t) = A_{n,t} - G_{n,t} - \sigma_n/2(x_n + t)$. (i) holds if $f'(0) \leq 0$, which is just (v). □

Theorem 2.3. *For $0 < x_1 \leq x_2 \leq \cdots \leq x_n$,*

$$x_1A_n\sigma_n \leq P_{n,3}^3A_n - P_{n,2}^4 \leq x_nA_n\sigma_n \tag{2.5}$$

with equality holding if and only if $x_1 = \cdots = x_n$ and this inequality implies inequality (1.1) *for* $r = 1, s = -1$.

Proof. We use similar arguments as in the proof of Theorem 2.2, let $f(t) = (x_n + t)(A_{n,t} - H_{n,t}), g(t) = A_{n,t} - H_{n,t} - (x_n + t)(1 - H_{n,t}^2/P_{n,t,-2}^2)$. The right-hand side inequality of (1.1) for $r = 1, s = -1$ holds if $f'(0) \leq 0$, which holds if $g'(0) \geq 0$, by a change of variables $x_i \to 1/x_{n-i+1}$, one checks $g'(0) \geq 0$ is implied by the left-hand side inequality of (2.5). Similarly, one shows the right-hand side inequality of (2.5) implies the left-hand side inequality of (1.1) for $r = 1, s = -1$. This proves the second statement of the theorem.

We now prove the left-hand side inequality of (2.5) and the proof for the right-hand side inequality of (2.5) is similar. We may assume $0 < x_1 = 1 < x_n = b$, $x_i \in (1, b)$ and define two functions($\omega =$

$(\omega_1, \omega_2, \cdots, \omega_n)$):

$$\begin{aligned} f(\omega, \mathbf{x}) &= P_{n,3}^3 A_n - P_{n,2}^4 - x_1 A_n \sigma_n, \\ g(x) &= P_{n,3}^3 x + x^3 A_n - 2P_{n,2}^2 x - x x_1 \sigma_n - x_1 A_n (x^2 - 2A_n x) - \lambda. \end{aligned}$$

Note here in the definition of $g(x)$, $P_{n,3}, P_{n,2}, A_n$ are not functions of x, they take values at some points $(\omega, \mathbf{x})$ to be specified and λ is also a constant to be specified.

We prove the left-hand side inequality of (2.5) by induction on n. It suffices to show $f(\omega, \mathbf{x}) \leq 0$ on the region $R_n \times S_{n-2}$, where $R_n = \{(\omega_1, \omega_2, \cdots, w_n) : 0 \leq w_k \leq 1, 1 \leq k \leq n, \sum_{k=1}^n w_k = 1\}$ and $S_{n-2} = \{(x_2, \cdots, x_{n-1}) : x_k \in [1, b], 2 \leq k \leq n-1\}$. We first show f takes its minimal value at $n \leq 2$. The base case of $n \leq 2$ is clear. Now assume $n \geq 3$.

There is a point $(\omega^*, \mathbf{x}^*)$ of $R_n \times S_{n-2}$ where f is minimized subject to the constraint $\sum_{k=1}^n \omega_k = 1$. If $x_i^* = x_{i+1}^*$ for some $1 \leq i \leq n-1$, by combining x_i^* with x_{i+1}^* and ω_i^* with ω_{i+1}^*, we are back to the case of $n-1$ variables with different weights. Similarly, if $\omega_i^* = 1$ for some i then we are back to the case $n = 1$. If $\omega_i^* = 0$ for some $i > 1$, we are back to the case $n-1$. If $\omega_1^* = 0$, since

$$P_{n,3}^3 A_n - P_{n,2}^4 - x_1^* A_n \sigma_n \geq P_{n,3}^3 A_n - P_{n,2}^4 - x_2^* A_n \sigma_n,$$

we are again back to the case $n-1$. So without loss of generality, from now on we may assume for $1 \leq i, j \leq n, i \neq j$, $\omega_i \neq 0, 1, x_i \neq x_j$ and this implies $(\omega^*, \mathbf{x}^*)$ is an interior point of $R_n \times S_{n-2}$. Thus we may use the Lagrange multiplier method to obtain a real number λ so that at $(\omega^*, \mathbf{x}^*)$:

$$\frac{\partial f}{\partial w_i} = \lambda \frac{\partial}{\partial \omega_i} (\sum_{k=1}^n w_k - 1), \ \frac{1}{\omega_j} \frac{\partial f}{\partial x_j} = 0 \tag{2.6}$$

for all $1 \leq i \leq n$ and $2 \leq j \leq n-1$.

By (2.6), a computation shows each x_k^* $(2 \leq k \leq n-1)$ is a common root of the equations $g(x)$ and $g'(x)$(where $P_{n,3}, P_{n,2}, A_n$ takes their values at $(\omega^*, \mathbf{x}^*)$). Now $n \geq 3$ implies $g(x)$ and $g'(x)$ have in common at least one distinct, positive root,$1 < x_2^* < b$. Moreover, $g(1) = g(b) = 0$ by (2.6) and it follows from Rolle's Theorem that there must be at least three positive roots of $g'(x)$, but $g'(x)$ is a quadratic polynomial and this contradiction implies it suffices to prove the left-hand side inequality of (2.5) for the case $n = 2$. Now for $n = 2$, let $0 < x_1 = x \leq x_2 = 1, \omega_1 = q$ and $\omega_2 = 1 - q$, we have

$$P_{2,3}^3 A_2 - P_{2,2}^4 - x_1 A_2 \sigma_2 = q^2 (1-q) x (1-x)^3 \geq 0$$

and this completes the proof. □

Theorem 2.4. *For $0 < x_1 \leq \cdots \leq x_n$, the following inequalities are equivalent:*

(i). $A_n - H_n \geq \frac{H_n}{x_n A_n} \sigma_n$; (ii). $A_n - H_n \leq \sigma_n / x_1$.

In particular, $A_n - H_n \geq \sigma_n / x_n$ implies $A_n - H_n \leq \sigma_n / x_1$. Moreover, we also have

$$A_n - H_n \geq \frac{P_{n,2}^2 - 2A_n H_n + H_n^2}{x_n}, \tag{2.7}$$

with equality holding if and only if $x_1 = \cdots = x_n$, which implies

$$A_n - H_n \leq \frac{H_n}{x_1 A_n} \sigma_n, \tag{2.8}$$

and (2.8) *further implies $A_n - H_n \leq \sigma_n / x_1$.*

Proof. We first show inequality(i) is equivalent to (ii). Let $f(t) = (x_1 + t)(A_{n,t} - H_{n,t}), g(t) = (x_n + t) A_{n,t} (A_{n,t} - H_{n,t}) / H_{n,t}$.

(i) $\Rightarrow$ (ii): By using similar arguments as in the proof of Theorem 2.2, (ii) holds if $f'(0) \geq 0$, by a change of variables $x_i \to 1/x_{n-i+1}$, one checks $f'(0) \geq 0$ is equivalent to (i).

(ii) $\Rightarrow$ (i): Similarly, (ii) holds if $g'(0) \le 0$. By a change of variables $x_i \to 1/x_{n-i+1}$, one checks $g'(0) \le 0$ is equivalent to

$$A_n - H_n \le \frac{P_{n,2}^2 - 2A_nH_n + H_n^2}{x_1} = \frac{\sigma_n}{x_1} + \frac{(A_n - H_n)^2}{x_1}. \tag{2.9}$$

Thus (ii) implies (2.9), hence (i).

Similarly, one can show (2.7) implies (2.8) and hence $A_n - H_n \le \sigma_n/x_1$. It now remains to show (2.7).

We may assume $0 < x_1 = a < x_n = 1$, $x_i \in (a, 1)$ and define two functions($\omega = (\omega_1, \omega_2, \cdots, \omega_n)$):

$$\begin{aligned} f(\omega, \mathbf{x}) &= x_n(A_n - H_n) - (P_{n,2}^2 - 2A_nH_n + H_n^2), \\ g(x) &= x_n(x + \frac{H_n^2}{x}) - (x^2 - 2xH_n + \frac{2A_nH_n^2}{x} - \frac{2H_n^3}{x}) - \lambda, \end{aligned}$$

then by using a similar method as in the proof of Theorem 2.3 while noting $x^2g'(x)$ is a cubic polynomial with no linear term, hence can have at most two positive roots, we reduce the proof of (2.7) to the case $n = 2$. In this case let $0 < x_1 = x < x_2 = 1, \omega_1 = q, \omega_2 = 1 - q$, one checks easily:

$$A_2 - H_2 - (\frac{P_{2,2}^2 - 2A_2H_2 + H_2^2}{x_2}) = \frac{q(1-q)^2x(1-x)^3}{(q + (1-q)x)^2} \ge 0,$$

and this completes the proof. ∎

3. Some Refinements of Ky Fan-type Inequalities

Theorem 3.1. *For $-1 \le r \ne 1 \le 2$, $0 < x_1 \le x_2 \cdots \le x_n$,*

$$|A_n - P_{n,r}| \ge \frac{|1-r|\sigma_n}{(2 - c_r)x_n + c_rB_r}, \tag{3.1}$$

where $B_r = \min\{A_n, P_{n,r}\}, c_r = \min\{(2+2r)/3, (4-2r)/3\}$ and equality holds if and only if $x_1 = \cdots = x_n$.

Proof. First let $n = 2$, $0 < x_1 = x \le x_2 = 1, \omega_1 = q, \omega_2 = 1 - q$, we will show for $-1 \le r \ne 1 \le 2$ and c_r as given above

$$((2 - c_r) + c_rx)|A_n - P_{n,r}| \ge |1 - r|\sigma_n. \tag{3.2}$$

This will then prove (3.1) for $n = 2$. Let $f(x) = ((2 - c_r) + c_rx)(qx + 1 - q - (qx^r + 1 - q)^{1/r}) - (1 - r)q(1 - q)(x - 1)^2$, $0 < x \le 1$. We need to show $f(x) \ge 0$ for $-1 \le r < 1$ and $f(x) \le 0$ for $1 < r \le 2$. It's easy to check that $f(1) = f'(1) = f''(1) = 0$ and

$$f'''(x) = q(1-q)(1-r)(q + (1-q)x^{-r})^{\frac{1-3r}{r}}x^{-2r-2}g(x),$$

where

$$g(x) = qx^r(c_r(2-r)x - (2-c_r)(1+r)) + (1-q)(c_r(1+r)x - (2-c_r)(2-r)).$$

One checks easily for the c_r as defined above, we have $c_r(2 - r)x - (2 - c_r)(1 + r) \le 0$ and $c_r(1 + r)x - (2 - c_r)(2 - r) \le 0$ for $0 < x \le 1$. Hence $g(x) \le 0$ and $f'''(x) \le 0$ for $0 < x \le 1$ with respect to the choice of c_r. Thus by the mean value theorem, $f(x) = f'''(\eta)(x - 1)^3 \ge 0$ for $0 < x \le 1$ and some $x < \eta < 1$ and (3.2) then follows.

Now for the general case, we treat the case $-1 \le r < 1$ here and the other cases are similar. We may assume $0 < x_1 = a < x_n = 1$, $x_i \in (a, 1)$ and define two functions($\omega = (\omega_1, \omega_2, \cdots, \omega_n)$):

$$\begin{aligned} f(\omega, \mathbf{x}) &= ((2 - c_r)x_n + c_rP_{n,r})(A_n - P_{n,r}) - (1 - r)\sigma_n, \\ g(x) &= c_rP_{n,r}^{1-r}x^r(A_n - P_{n,r})/r + ((2 - c_r)x_n + c_rP_{n,r})(x - P_{n,r}^{1-r}x^r/r) \\ &\quad -(1 - r)(x^2 - 2xA_n) - \lambda, \end{aligned}$$

then by using a similar method as in the proof of Theorem 2.3, we can reduce the general case to the case $n = 2$ and this completes the proof. $\square$

Theorems 2.2 and 2.4 suggest that there should exist some relations between the right-hand side inequality of (1.1) and the left-hand side inequality of (1.1). We now raise the following

Conjecture 3.1. *For $0 < x_1 \leq \cdots \leq x_n \leq 1/2$, $q = \min\{\omega_i\}$*

$$2((1-q)x_n + qx_1)(A_n - G_n) \geq \sigma_n. \tag{3.3}$$

One checks by direct calculation(see the proof of Theorem 3.1, replacing c_r by $2q$ there) that the above conjecture holds for $n = 2$, we don't know whether it holds for all n. We now give a weaker result.

Theorem 3.2. *For $0 < x_1 \leq x_2 \cdots \leq x_n$, $q = \min\{\omega_i\}$*

$$2((1-q)x_n + qG_n)(A_n - G_n) \geq \sigma_n \tag{3.4}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. Let $f(x_n) = 2((1-q)x_n + qG_n)(A_n - G_n) - \sigma_n$, then

$$\begin{aligned} \frac{f'(x_n)}{2\omega_n} &= (\frac{1-q}{\omega_n} + q\frac{G_n}{x_n})(A_n - G_n) + ((1-q)x_n + qG_n)(1 - \frac{G_n}{x_n}) - (x_n - A_n) \\ &\geq (1 - q + q\frac{G_n}{x_n})(A_n - G_n) + ((1-q)x_n + qG_n)(1 - \frac{G_n}{x_n}) - (x_n - A_n). \end{aligned} \tag{3.5}$$

We may assume $0 \leq x_1 \leq x_2 \cdots \leq x_n = 1$ and rewrite the right-hand side of (3.5) as

$$g_n(x_1, \cdots, x_{n-1}) = (1 - q + qG_n)(A_n - G_n) + (1 - q + qG_n)(1 - G_n) - (1 - A_n). \tag{3.6}$$

We want to show $g_n \geq 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0, 1]^{n-1}$ be the point in which the absolute minimum of g_n is reached.

We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_i = a_{i+1}$ for some $1 \leq i \leq n-2$ or $a_{n-1} = 1$, by combining a_i with a_{i+1} and ω_i with ω_{i+1} or a_{n-1} with 1 and ω_{n-1} with ω_n, while noticing increasing q will decrease the value of g_n, we can reduce the determination of the absolute minimum of g_n to that of g_{n-1} with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$. If $\mathbf{a}$ is a boundary point of $[0, 1]^{n-1}$, then $a_1 = 0$, (3.6) is reduced to

$$g_n = (1-q)A_n + (1-q) - (1 - A_n) = (2-q)A_n - q \geq (2-q)\omega_n - q \geq (2-q)q - q \geq 0.$$

Now we may assume $a_1 > 0$ and $\mathbf{a}$ is an interior point of $[0, 1]^{n-1}$, then we obtain

$$\nabla g_n(a_1, \cdots, a_{n-1}) = 0$$

such that $a_1, \cdots, a_{n-1}$ solve the equation

$$q\frac{G_n}{x}(A_n - G_n) + (1 - q + qG_n)(1 - \frac{G_n}{x}) + q\frac{G_n}{x}(1 - G_n) - (1 - q + qG_n)\frac{G_n}{x} + 1 = 0.$$

The above equation has at most one root(regarding G_n as a constant), so we only need to show $g_n \geq 0$ for the case $n = 2$. Now by letting $0 < x_1 = x \leq x_2 = 1$, we will actually show

$$h(x) = (1 - q + qx)(A_2 - G_2) + (1 - q + qx)(1 - G_2) - (1 - A_2) \geq 0.$$

It will then imply $g_2 \geq 0$ since $G_2 \geq x$.

It's easy to check $h(1) = h'(1) = 0$ and $h''(x) = 2\omega_1[q + x^{-2+\omega_1}((1-q)(1-\omega_1) - q(1+\omega_1)x)]$. Since $h'''(x) = 2\omega_1(1-\omega_1)x^{-3+\omega_1}a(x)$ with $a(x) = (2-\omega_1)(q-1) + q(\omega_1+1)x$ and $a(0) = (2-\omega_1)(q-1) \leq 0, a(1) = \omega_1 + 3q - 2 \leq 0$. We know $h'''(x) \leq 0$ and hence $h''(x) \geq h''(1) \geq 0$. Hence by the mean value theorem, $h(x) = h''(\eta)(x-1)^3 \geq 0$ for $0 < x \leq 1$ and some $x < \eta < 1$ and the theorem then follows. $\square$

The author has shown[8] for $x_1 \neq x_n, n \geq 2$ and $1 > r \geq 0$

$$A_n - P_{n,r} > \frac{x_n^{1-r} - P_{n,r}^{1-r}}{2x_n^{1-r}(x_n - A_n)}\sigma_n + q\frac{(A_n - P_{n,r})^2}{2(x_n - A_n)}. \tag{3.7}$$

We now show in general (3.1) and (3.7) are not comparable for $0 \leq r < 1$. It suffices to show $(x_n^{1-r} - P_{n,r}^{1-r})((2-c_r)x_n + c_rP_{n,r})x_n^{r-1}\sigma_n + q(A_n - P_{n,r})^2((2-c_r)x_n + c_rP_{n,r})$ is not comparable to $2(1-r)(x_n - A_n)\sigma_n$. Consider the case $n = 2, 0 < x_1 = x \leq x_2 = 1, \omega_1 = q_1, \omega_2 = 1 - q_1$, let

$$f(x) = (1 - P_{n,r}^{1-r})(2 - c_r + c_rP_{n,r})\sigma_n + q(A_n - P_{n,r})^2(2 - c_r + c_rP_{n,r}) - 2(1-r)(1 - A_n)\sigma_n$$

where we regard $P_{n,r}, A_n, \sigma_n$ as functions of x. Calculation yields $f(1) = f'(1) = f''(1) = f'''(1) = 0$ and

$$f^{(4)}(1) = 12q_1(1-q_1)^2(1-r)g(c_r, q_1),$$

where

$$g(c_r, q_1) = q(1-q_1)(1-r) + 4q_1r + 2(1 - q_1 - c_rq_1 - r). \tag{3.8}$$

We then have $g(c_r, 0) = 2(1-r) > 0$ and $g(c_r, 1) = 2(r - c_r) < 0$ for $r < 4/5$. This shows (3.1) and (3.7) are not comparable at least for $r < 4/5$.

We note also here if we take $c_r = 2q$ in (3.8) for the case $r = 0$, we see $g(2q, 0) = 2$ and if we choose $q_1 > 3/5$ then $q = 1 - q_1$ and $g(2q, q_1) = (1-q_1)(3-5q_1) \leq 0$ and this shows (3.4) and (3.7) are also not comparable.

4. A Result on Symmetric Means

Let $\mathbf{x} = (x_1, \cdots, x_n)$ be an n-tuple of positive real numbers, $r \in \{0, 1, \cdots, n\}$ and

$$E_r(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{i=1}^{r} x_{i_j}, \ E_0 = 1; P_r(x) = \frac{E_r(x)}{\binom{n}{r}}.$$

$E_r(\mathbf{x})$ is called the rth symmetric function of $\mathbf{x}$ and $P_r(\mathbf{x})$ the mean of $E_r(\mathbf{x})$. The following result is known(see [11], Theorems 51 and 52).

Theorem 4.1.

$$p_n^{1/n} \leq p_{n-1}^{1/(n-1)} \leq \cdots \leq p_2^{1/2} \leq p_1. \tag{4.1}$$

and for $0 < r < n$ *an integer,*

$$p_{r-1}p_{r+1} \leq p_r^2. \tag{4.2}$$

In fact (4.2) implies (4.1)(see also [11]). We now use (4.2) to show

Theorem 4.2. *For* $t \geq 0, 0 < r < n$,

$$p_r^{1/r}(\mathbf{x}_t)/p_{r+1}^{1/(r+1)}(\mathbf{x}_t) \tag{4.3}$$

is a decreasing function of t. *In particular,* (4.1) *follows.*

Proof. Let $f(t) = \ln(p_r^{1/r}(\mathbf{x}_t)/p_{r+1}^{1/(r+1)}(\mathbf{x}_t))$, it suffices to show $f'(0) \leq 0$. One checks this is equivalent to (4.2). Since $\lim_{t\to\infty} p_r^{1/r}(\mathbf{x}_t)/p_{r+1}^{1/(r+1)}(\mathbf{x}_t) = 1$, (4.1) hence follows and this completes the proof. □

We note the above theorem is similar to the following result of P.F.Wang and W.L.Wang[15].

Theorem 4.3. *If* $x_i \in (0, 1/2] (i = 1, \cdots, n)$, *then*

$$\frac{E_n^{1/n}(\mathbf{x})}{E_n^{1/n}(\mathbf{x}')} \leq \frac{E_{n-1}^{1/(n-1)}(\mathbf{x})}{E_{n-1}^{1/(n-1)}(\mathbf{x}')} \leq \cdots \leq \frac{E_2^{1/2}(\mathbf{x})}{E_2^{1/2}(\mathbf{x}')} \leq \frac{E_1(\mathbf{x})}{E_1(\mathbf{x}')}. \tag{4.4}$$

In fact the method used to prove the above theorem can be extended easily to give a proof of Theorem 4.2, see [2] for the details.

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