NEW DOUBLE INEQUALITIES FOR MATHIEU TYPE SERIES

ŽIVORAD TOMOVSKI

Abstract. In this paper using the trapezoidal quadrature rule, we established new double inequality, for Mathieu's series of following type:

$$S(a,p,\alpha)=\sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^{\alpha}+a^2)^{p+1}},\ \text{where } a>0,\ p>0,\ \alpha>0.$$

As a corollary of this inequality the solution of the problem posed by Tomovski and Trenčevski in [7], for $\alpha=2$ and $p>0$ is completed.

1. Introduction

In [6] Feng Qi intorduced the series of following type

$$S(a,p,\alpha)=\sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^{\alpha}+a^2)^{p+1}},\ \text{where } a>0,\ \alpha>0 \text{ and } p>0.$$

For $\alpha=2$ this series first was defined in [3] by Diananda.

Concerning the series $S(a,p,2)$ in [3], P.H. Diananda proved the following Theorem:

Theorem A. *If $a,p>0$ then*

$$S(a,p,2)<\frac{1}{pa^{2p}}.$$

For $\alpha=2$, $p=1$ the series $S(a,p,\alpha)$ was introduced in [5] by Mathieu. Thus the series:

$$S(a,1,2)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^2}$$

is called as Mathieu's series.

Mathieu (see [5]) obtained inequality for series $S(a,1,2)$ which is a corollary of Theorem A for $p=1$.

Theorem B. *If $a>0$, then*

$$S(a,1,2)<\frac{1}{a^2}$$

This theorem was refined by several authors (see [2],[3],[8]) but the best result was obtained recently by H. Alzer and J.L. Brenner. Namely they proved the following theorem.

Theorem C. [1] *For all real numbers $a\neq 0$, we have:*

$$\frac{1}{a^2+1/(2\zeta(3))}<S(a,1,2)<\frac{1}{a^2+1/6}$$

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The constants $1/(2\zeta(3))$ and $1/6$ are best possible, where ζ denotes the zeta function. On the other hand, Feng Qi in [6] established double inequality for series $S(a,1,\alpha)$.

Theorem D. [6] *Suppose a is a positive number, then for any positive real number α, we have:*

$$\frac{1}{a^2+\frac{1}{2}} < S(a,1,\alpha) < \frac{1}{a^2}$$

In [7] *we established double inequality for series $S(a,p,2)$, when $a>0$ and $p\in\mathbb{N}$.*

Theorem E. [7] *If $a>0$, $p\in\mathbb{N}$, then*

$$|S(a,p,2)| \le \frac{2(2a)^{-p}}{p!a^{p+1}}\Big[\sum_{k=0}^{\infty}(-1)^k\frac{(k\pi)^p}{\exp(k\pi/a)-1}+\sum_{k=0}^{\infty}(-1)^k\frac{((k+\frac{1}{2})\pi)^p}{\exp\left((k+\frac{1}{2})\frac{\pi}{a}\right)-1}\Big]$$
$$+\sum_{k=1}^{p}\frac{2(2a)^{-2p+k-1}}{k!a^{k+1}}\Big|\binom{-(p+1)}{p-k}\frac{1-k}{p-k+1}\Big|\times\Big[\sum_{j=0}^{\infty}(-1)^j\frac{(j\pi)^k}{exp(j\pi/a)-1}$$
$$+\sum_{j=0}^{\infty}\frac{((j+\frac{1}{2})\pi)^k}{exp((j+\frac{1}{2})\pi/a)-1}\Big]$$

2. The integral expression for series $S(a,p,\alpha)$

In this section, we shall establish an integral expression of $S(a,p,\alpha)$, where $a>0,\ p>0,\ \alpha>0$. This is an open problem, posed by Feng Qi in [6].

Theorem 1. *For $a>0, p>0, \alpha>0$ the following integral expression of $S(a,p,\alpha)$ holds:*

$$S(a,p,\alpha)=\frac{2}{\Gamma(p+1)}\int_0^{\infty}x^pe^{-a^2x}g(x)dx,$$

where

$$g(x)=\sum_{n=1}^{\infty}n^{\alpha/2}e^{-n^{\alpha}x}$$

Proof. Using the well-known formula:

$$\frac{1}{t^{p+1}}=\frac{2}{\Gamma(p+1)}\int_0^{\infty}x^pe^{-xt}dx,$$

we obtain

$$\frac{2n^{\alpha/2}}{(n^{\alpha}+a^2)^{p+1}}=\frac{2}{\Gamma(p+1)}\int_0^{\infty}x^pn^{\alpha/2}e^{-(n^{\alpha}+a^2)x}dx.$$

Applying the Cauchy integration test, we obtain that $\sum_{n=1}^{\infty}n^{\alpha/2}e^{-n^{\alpha}x}$ is convergent for all $x>0$ and $\alpha>0$, i.e. $g(x)=\sum_{n=1}^{\infty}n^{\alpha/2}e^{-n^{\alpha}x}$.

Thus

$$S(a,p,\alpha) = \frac{2}{\Gamma(p+1)}\int\limits_0^\infty x^p e^{-a^2x}\Big(\sum_{n=1}^\infty n^{\alpha/2}e^{-n^\alpha x}\Big)$$
$$= \frac{2}{\Gamma(p+1)}\int\limits_0^\infty x^p e^{-a^2x} g(x)dx.$$

□

3. Main results

Our main results are as follows

Theorem 2. *If $a>0, p>0, \alpha>0$, then*

$$\frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2})\Gamma(p-\frac{1}{\alpha}+\frac{1}{2})}{\alpha\Gamma(p+1)a^{2p-\frac{2}{\alpha}+1}} - \frac{2\Gamma(p+\frac{1}{2})}{\sqrt{2e}\Gamma(p+1)a^{2p+1}} < S(a,p,\alpha) <$$
$$< \frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2},1)\Gamma(p-\frac{1}{\alpha}+\frac{1}{2})}{\alpha\Gamma(p+1)a^{2p-2/\alpha+1}} + \frac{2\Gamma(p+\frac{1}{2})}{\sqrt{2e}\Gamma(p+1)a^{2p+1}}$$

Proof. Let $f(x) = x^{\alpha/2}e^{-x^\alpha t},\ x>0,\ t>0$

Since $\int\limits_0^\infty f(x)dx = \dfrac{\Gamma(\dfrac{1}{2}+\dfrac{1}{2})}{\alpha t^{1/\alpha+1/2}}$ and $\max\limits_{x\in\mathbb{R}^+} f(x) = \dfrac{1}{\sqrt{2et}}$, applying the trapezoidal quadrature rule (see[4]):

$$\int\limits_0^\infty f(x)dx - \max_{x\in\mathbb{R}^+} f(x) < \sum_{n=1}^\infty f(n) < \int_0^\infty f(x)dx + \max_{x\in\mathbb{R}^+} f(x) - \int_0^1 f(x)dx,$$

we obtain

$$\frac{\Gamma(\frac{1}{\alpha}+\frac{1}{2})}{\alpha t^{\frac{1}{\alpha}+\frac{1}{2}}} - \frac{1}{\sqrt{2et}} < g(t) < \frac{\Gamma(\frac{1}{\alpha}+\frac{1}{2},1)}{\alpha t^{\frac{1}{\alpha}+\frac{1}{2}}} + \frac{1}{\sqrt{2et}},$$

where g is the function defined in Theorem 1 and $\Gamma(a,x) = \int\limits_x^\infty e^{-t}t^{a-1}dt$.

Hence

$$\frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2})}{\alpha\Gamma(p+1)}\int_0^\infty t^{p-\frac{1}{\alpha}-\frac{1}{2}}e^{-a^2t}dt - \frac{2}{\sqrt{2e}\Gamma(p+1)}\int_0^\infty t^{p-\frac{1}{2}}e^{-a^2t}dt <$$
$$S(a,p,\alpha) < \frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2},1)}{\alpha\Gamma(p+1)}\int_0^\infty t^{p-\frac{1}{\alpha}-\frac{1}{2}}e^{-a^2t}dt + \frac{2}{\sqrt{2e}\Gamma(p+1)}\int_0^\infty t^{p-\frac{1}{2}}e^{-a^2t}dt, \quad \text{i.e.}$$
$$\frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2})\Gamma(p-\frac{1}{\alpha}+\frac{1}{2})}{\alpha\Gamma(p+1)a^{2p-\frac{2}{\alpha}+1}} - \frac{2\Gamma(p+\frac{1}{2})}{\sqrt{2e}\Gamma(p+1)a^{2p+1}} < S(a,p,\alpha) <$$
$$< \frac{2\Gamma(\frac{1}{\alpha}+\frac{1}{2},1)\Gamma(p-\frac{1}{\alpha}+\frac{1}{2})}{\alpha\Gamma(p+1)a^{2p-2/\alpha+1}} + \frac{2\Gamma(p+\frac{1}{2})}{\sqrt{2e}\Gamma(p+1)a^{2p+1}}$$

□

As a corollary of this theorem we obtain new double inequalities for series $S(a, p, 2)$ $a > 0,\ p > 0$, which complete the result of partially solved problem by Tomovski and Trenčevski (see [7]), when $p \in \mathbb{N}$.

Corollary. *If $a,\ p > 0$ then the following inequalities hold:*

$$\frac{2}{\Gamma(p+1)}\Big(\frac{\Gamma(p)}{2a^{2p}} - \frac{\Gamma(p+\frac{1}{2})}{\sqrt{2e}a^{2p+1}}\Big) < S(a,p,2) < \frac{2}{\Gamma(p+1)}\Big(\frac{\Gamma(p)}{2ea^{2p}} + \frac{\Gamma(p+\frac{1}{2})}{\sqrt{2e}a^{2p+1}}\Big)$$

Proof. By putting in Theorem 1, $\alpha = 2$ the proof is completed. □

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Institute of Mathematics, "St. Cyril and Methodius University, P.O. Box 162, Skopje, Macedonia

E-mail address: `tomovski@iunona.pmf.ukim.edu.mk`