

# GENERALIZATIONS OF WEIGHTED TRAPEZOIDAL INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we establish some generalizations of weighted trapezoid inequality for monotonic mappings, and give several applications for  $r$ -moment, the expectation of a continuous random variable and the Beta mapping.

## 1. INTRODUCTION

The *trapezoid inequality*, states that if  $f''$  exists and is bounded on  $(a, b)$ , then

$$(1.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty,$$

where

$$\|f''\|_\infty := \sup_{x \in (a,b)} |f''| < \infty.$$

Now if we assume that  $I_n : a = x_0 < x_1 < \dots < x_n = b$  is a partition of the interval  $[a, b]$  and  $f$  is as above, then we can approximate the integral  $\int_a^b f(x) dx$  by the *trapezoidal quadrature formula*  $A_T(f, I_n)$ , having an error given by  $R_T(f, I_n)$ , where

$$(1.2) \quad A_T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] l_i,$$

and the remainder satisfies the estimation

$$(1.3) \quad |R_T(f, I_n)| \leq \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} l_i^3,$$

with  $l_i := x_{i+1} - x_i$  for  $i = 0, 1, \dots, n-1$ .

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2] – [8].

Recently, Cerone-Dragomir [3] proved the following two trapezoid type inequalities:

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**Theorem A.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic non-decreasing mapping. Then

$$(1.4) \quad \left| \int_a^b f(t) dt - [(x-a)f(a) + (b-x)f(b)] \right| \\ \leq (b-x)f(b) - (x-a)f(a) + \int_a^b \operatorname{sgn}(x-t)f(t) dt \\ \leq (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \\ \leq \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] [f(b) - f(a)],$$

for all  $x \in [a, b]$ . The above inequalities are sharp.

Let  $I_n, l_i$  ( $i = 0, 1, \dots, n-1$ ) be as above and let  $\xi_i \in [x_i, x_{i+1}]$  ( $i = 0, 1, \dots, n-1$ ) be intermediate points. Define the sum

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})].$$

We have the following result concerning the approximation of the integral  $\int_a^b f(x) dx$  in terms of  $T_P$ .

**Theorem B.** Let  $f$  be defined as in Theorem A, then we have

$$(1.5) \quad \int_a^b f(x) dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi).$$

The remainder term  $R_P(f, I_n, \xi)$  satisfies the estimate

$$(1.6) \quad |R_P(f, I_n, \xi)| \\ \leq \sum_{i=0}^{n-1} [(x_{i+1} - \xi_i)f(x_{i+1}) - (\xi_i - x_i)f(x_i)] \\ + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(\xi_i - t)f(t) dt \\ \leq \sum_{i=0}^{n-1} (\xi_i - x_i)[f(\xi_i) - f(x_i)] + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)[f(x_{i+1}) - f(\xi_i)] \\ \leq \sum_{i=0}^{n-1} \left[ \frac{1}{2}l_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] [f(x_{i+1}) - f(x_i)] \\ \leq \left[ \frac{1}{2}\nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] [f(b) - f(a)] \\ \leq \nu(l) [f(b) - f(a)]$$

where  $\nu(l) := \max \{l_i \mid i = 0, 1, \dots, n-1\}$ .

In this paper, we establish weighted generalizations of Theorems A-B, and give several applications for  $r$ -moments and the expectation of a continuous random variable, the Beta mapping and the Gamma mapping.

## 2. SOME INTEGRAL INEQUALITIES

**Theorem 1.** Let  $g : [a, b] \rightarrow \mathbb{R}$  be non-negative and continuous with  $g(t) > 0$  on  $(a, b)$  and let  $h : [a, b] \rightarrow \mathbb{R}$  be differentiable such that  $h'(t) = g(t)$  on  $[a, b]$ .

(a) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a monotonic non-decreasing mapping. Then

$$\begin{aligned}
 (2.1) \quad & \left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
 & \leq (h(b) - x) f(b) - (x - h(a)) f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t) g(t) dt \\
 & \leq (x - h(a)) \cdot [f(h^{-1}(x)) - f(a)] + (h(b) - x) \cdot [f(b) - f(h^{-1}(x))] \\
 & \leq \left[ \frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] \cdot [f(b) - f(a)]
 \end{aligned}$$

for all  $x \in [h(a), h(b)]$ .

(b) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a monotonic non-increasing mapping. Then

$$\begin{aligned}
 (2.2) \quad & \left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
 & \leq (x - h(a)) f(a) - (h(b) - x) f(b) + \int_a^b \operatorname{sgn}(t - h^{-1}(x)) f(t) g(t) dt \\
 & \leq (x - h(a)) \cdot [f(a) - f(h^{-1}(x))] + (h(b) - x) \cdot [f(h^{-1}(x)) - f(b)] \\
 & \leq \left[ \frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] \cdot [f(a) - f(b)]
 \end{aligned}$$

for all  $x \in [h(a), h(b)]$ .

The above inequalities are sharp.

*Proof.* (1)

(a) Let  $x \in [h(a), h(b)]$ . Using integration by parts, we have the following identity

$$\begin{aligned}
 (2.3) \quad & \int_a^b (x - h(t)) df(t) \\
 & = (x - h(t)) f(t) \Big|_a^b + \int_a^b f(t)g(t) dt \\
 & = \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)].
 \end{aligned}$$

It is well known [3, p. 813] that if  $\mu, \nu : [a, b] \rightarrow \mathbb{R}$  are such that  $\mu$  is continuous on  $[a, b]$  and  $\nu$  is monotonic non-decreasing on  $[a, b]$ , then

$$(2.4) \quad \left| \int_a^b \mu(t) d\nu(t) \right| \leq \int_a^b |\mu(t)| d\nu(t).$$

Now, using identity (2.3) and inequality (2.4), we have

$$\begin{aligned}
(2.5) \quad & \left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
& \leq \int_a^b |x - h(t)| df(t) \\
& = \int_a^{h^{-1}(x)} (x - h(t)) df(t) + \int_{h^{-1}(x)}^b (h(t) - x) df(t) \\
& = (x - h(t)) f(t) \Big|_a^{h^{-1}(x)} + \int_a^{h^{-1}(x)} f(t) g(t) dt \\
& \quad + (h(t) - x) \Big|_{h^{-1}(x)}^b - \int_{h^{-1}(x)}^b f(t) g(t) dt \\
& = (h(b) - x) f(b) - (x - h(a)) f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t) g(t) dt
\end{aligned}$$

and the first inequalities in (2.1) are proved.

As  $f$  is monotonic non-decreasing on  $[a, b]$ , we obtain

$$\begin{aligned}
\int_a^{h^{-1}(x)} f(t) g(t) dt & \leq f(h^{-1}(x)) \int_a^{h^{-1}(x)} g(t) dt \\
& = (x - h(a)) f(h^{-1}(x))
\end{aligned}$$

and

$$\begin{aligned}
\int_{h^{-1}(x)}^b f(t) g(t) dt & \geq f(h^{-1}(x)) \int_{h^{-1}(x)}^b g(t) dt \\
& = (h(b) - x) f(h^{-1}(x)),
\end{aligned}$$

then

$$\int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t) g(t) dt \leq (x - h(a)) f(h^{-1}(x)) + (x - h(b)) f(h^{-1}(x)).$$

Therefore,

$$\begin{aligned}
(2.6) \quad & (h(b) - x) f(b) - (x - h(a)) f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t) g(t) dt \\
& \leq (h(b) - x) f(b) - (x - h(a)) f(a) \\
& \quad + (x - h(a)) f(h^{-1}(x)) + (x - h(b)) f(h^{-1}(x)) \\
& = (x - h(a)) \cdot [f(h^{-1}(x)) - f(a)] + (h(b) - x) \cdot [f(b) - f(h^{-1}(x))]
\end{aligned}$$

which proves that the second inequality in (2.1).

As  $f$  is monotonic non-decreasing on  $[a, b]$ , we have

$$f(a) \leq f(h^{-1}(x)) \leq f(b)$$

and

$$\begin{aligned}
(2.7) \quad & (x - h(a)) \cdot [f(h^{-1}(x)) - f(a)] + (h(b) - x) \cdot [f(b) - f(h^{-1}(x))] \\
& \leq \max\{x - h(a), h(b) - x\} \cdot [f(h^{-1}(x)) - f(a) + f(b) - f(h^{-1}(x))] \\
& = \left[ \frac{h(b) - h(a)}{2} + \left| x - \frac{h(a) + h(b)}{2} \right| \right] \cdot [f(b) - f(a)] \\
& = \left[ \frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] \cdot [f(b) - f(a)].
\end{aligned}$$

Thus, by (2.5), (2.6) and (2.7), we obtain (2.1).

Let

$$\begin{aligned}
g(t) & \equiv 1, \quad t \in [a, b] \\
h(t) & = t, \quad t \in [a, b] \\
f(t) & = \begin{cases} 0, & t \in [a, b) \\ 1, & t = b \end{cases}
\end{aligned}$$

and  $x = \frac{a+b}{2}$ . Then

$$\begin{aligned}
& \left| \int_a^b f(t)g(t) dt - [(x - h(a))f(a) + (h(b) - x)f(b)] \right| \\
& = (h(b) - x)f(b) - (x - h(a))f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \\
& = (x - h(a)) \cdot [f(h^{-1}(x)) - f(a)] + (h(b) - x) \cdot [f(b) - f(h^{-1}(x))] \\
& = \left[ \frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] \cdot [f(b) - f(a)] \\
& = \frac{b-a}{2}
\end{aligned}$$

which proves that the inequalities (2.1) are sharp.

(b) If  $f$  is replaced by  $-f$  in (a), then (2.2) is obtained from (2.1).

This completes the proof. ■

**Remark 1.** If we choose  $g(t) \equiv 1, h(t) = t$  on  $[a, b]$ , then the inequalities (2.1) reduce to (1.4).

**Corollary 1.** If we choose  $x = \frac{h(a)+h(b)}{2}$ , then we get

$$\begin{aligned}
(2.8) \quad & \left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \\
& \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(b) - f(a)] \\
& \quad + \int_a^b \operatorname{sgn} \left( h^{-1} \left( \frac{h(a) + h(b)}{2} \right) - t \right) f(t)g(t) dt \\
& \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(b) - f(a)]
\end{aligned}$$

where  $f$  and  $g$  are defined as in (a) of Theorem 1, and

$$\begin{aligned}
 (2.9) \quad & \left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(a) - f(b)] \\
 & \quad + \int_a^b \operatorname{sgn} \left( t - h^{-1} \left( \frac{h(a) + h(b)}{2} \right) \right) f(t) g(t) dt \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(a) - f(b)]
 \end{aligned}$$

where  $f$  and  $g$  are defined as in (b) of Theorem 1.

The inequalities (2.8) and (2.9) are the “weighted trapezoid” inequalities.

Note that the trapezoid inequality (2.8) and (2.9) are, in a sense, the best possible inequalities we can obtain from (2.1) and (2.2). Moreover, the constant  $\frac{1}{2}$  is the best possible for both inequalities in (2.8) and (2.9), respectively.

**Remark 2.** The following inequality is well-known in the literature as the Fejér inequality (see for example [9]):

$$(2.10) \quad f \left( \frac{a+b}{2} \right) \int_a^b g(t) dt \leq \int_a^b f(t)g(t) dt \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt,$$

where  $f : [a, b] \rightarrow \mathbb{R}$  is convex and  $g : [a, b] \rightarrow \mathbb{R}$  is positive integrable and symmetric to  $\frac{a+b}{2}$ .

Using the above results and (2.8) – (2.9), we obtain the following error bound of the second inequality in (2.10):

$$\begin{aligned}
 (2.11) \quad & 0 \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(b) - f(a)] \\
 & \quad + \int_a^b \operatorname{sgn} \left( h^{-1} \left( \frac{h(a) + h(b)}{2} \right) - t \right) f(t) g(t) dt \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(b) - f(a)]
 \end{aligned}$$

provided that  $f$  is monotonic non-decreasing on  $[a, b]$ .

$$\begin{aligned}
 (2.12) \quad & 0 \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(a) - f(b)] \\
 & \quad + \int_a^b \operatorname{sgn} \left( t - h^{-1} \left( \frac{h(a) + h(b)}{2} \right) \right) f(t) g(t) dt \\
 & \leq \frac{1}{2} \int_a^b g(t) dt \cdot [f(a) - f(b)]
 \end{aligned}$$

provided that  $f$  is monotonic non-increasing on  $[a, b]$ .

## 3. APPLICATIONS FOR QUADRATURE FORMULA

Throughout this section, let  $g$  and  $h$  be defined as in Theorem 1.

Let  $f : [a, b] \rightarrow \mathbb{R}$ , and let  $I_n : a = x_0 < x_1 < \dots < x_n = b$  be a partition of  $[a, b]$  and  $\xi_i \in [h(x_i), h(x_{i+1})]$  ( $i = 0, 1, \dots, n-1$ ) be intermediate points. Put  $l_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t) dt$  and define the sum

$$T_P(f, g, h, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})].$$

We have the following result concerning the approximation of the integral  $\int_a^b f(t)g(t) dt$  in terms of  $T_P$ .

**Theorem 2.** *Let  $\nu(l) := \max\{l_i \mid i = 0, 1, \dots, n-1\}$ ,  $f$  be defined as in Theorem 1 and let*

$$(3.1) \quad \int_a^b f(t)g(t) dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi).$$

*Then, the remainder term  $R_P(f, g, h, I_n, \xi)$  satisfies the following estimates:*

(a) *Suppose  $f$  is monotonic non-decreasing on  $[a, b]$ , then*

$$(3.2) \quad \begin{aligned} & |R_P(f, g, h, I_n, \xi)| \\ & \leq \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i)] \\ & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t) g(t) dt \\ & \leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(h^{-1}(\xi_i)) - f(x_i)] \\ & \quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\ & \leq \sum_{i=0}^{n-1} \left[ \frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(x_{i+1}) - f(x_i)] \\ & \leq \left[ \frac{1}{2} \nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(b) - f(a)] \\ & \leq \nu(l) [f(b) - f(a)]. \end{aligned}$$

(b) *Suppose  $f$  is monotonic non-increasing on  $[a, b]$ , then*

$$(3.3) \quad \begin{aligned} & |R_P(f, g, h, I_n, \xi)| \\ & \leq \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) - (h(x_{i+1}) - \xi_i) f(x_{i+1})] \\ & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - h^{-1}(\xi_i)) f(t) g(t) dt \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(x_i) - f(h^{-1}(\xi_i))] \\
&\quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(h^{-1}(\xi_i)) - f(x_{i+1})] \\
&\leq \sum_{i=0}^{n-1} \left[ \frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(x_i) - f(x_{i+1})] \\
&\leq \left[ \frac{1}{2}\nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(a) - f(b)] \\
&\leq \nu(l) [f(a) - f(b)].
\end{aligned}$$

*Proof.* (1)

(a) Apply Theorem 1 on the intervals  $[x_i, x_{i+1}]$  ( $i = 0, 1, \dots, n-1$ ) to get

$$\begin{aligned}
&\left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right| \\
&\leq (h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i) \\
&\quad + \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t) g(t) dt \\
&\leq (\xi_i - h(x_i)) \cdot [f(h^{-1}(\xi_i)) - f(x_i)] \\
&\quad + (h(x_{i+1}) - \xi_i) \cdot [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\
&\leq \left[ \frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] \cdot [f(x_{i+1}) - f(x_i)]
\end{aligned}$$

for all  $i \in \{0, 1, \dots, n-1\}$ .



Using this and the generalized triangle inequality, we have

$$\begin{aligned}
(3.4) \quad & |R_P(f, g, h, I_n, \xi)| \\
& \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right| \\
& \leq \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i)] \\
& \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t)g(t) dt \\
& \leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(h^{-1}(\xi_i)) - f(x_i)] \\
& \quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\
& \leq \sum_{i=0}^{n-1} \left[ \frac{1}{2} l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(x_{i+1}) - f(x_i)] \\
& \leq \left[ \frac{1}{2} \nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(b) - f(a)].
\end{aligned}$$

Next, we observe that

$$(3.5) \quad \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2} l_i \quad (i = 0, 1, \dots, n-1);$$

and then

$$(3.6) \quad \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2} \nu(l).$$

Thus, by (3.4), (3.5) and (3.6), we obtain (3.2).

(b) The proof is similar as (a) and we omit the details.

This completes the proof. ■

**Remark 3.** If we choose  $g(t) \equiv 1, h(t) = t$  on  $[a, b]$ , then the inequalities (3.2) reduce to (1.6).

Now, let  $\xi_i = \frac{h(x_i) + h(x_{i+1})}{2}$  ( $i = 0, 1, \dots, n-1$ ) and let  $T_{PW}(f, g, h, I_n)$  and  $R_P(f, g, h, I_n)$  be defined as

$$T_{PW}(f, g, h, I_n) = T_P(f, g, h, I_n, \xi) = \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] \int_{x_i}^{x_{i+1}} g(t) dt$$

and

$$\begin{aligned}
R_{PW}(f, g, h, I_n) &= R_P(f, g, h, I_n, \xi) \\
&= \int_a^b f(t)g(t) dt - \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] \int_{x_i}^{x_{i+1}} g(t) dt.
\end{aligned}$$

If we consider the *weighted trapezoidal formula*  $T_{PW}(f, g, h, I_n)$ , then we have the following corollary:

**Corollary 2.** *Let  $f, g, h$  be defined as in Theorem 2 and let  $\xi_i = \frac{h(x_i)+h(x_{i+1})}{2}$  ( $i = 0, 1, \dots, n-1$ ). Then*

$$\int_a^b f(t)g(t)dt = T_{PW}(f, g, h, I_n) + R_{PW}(f, g, h, I_n)$$

where the remainder satisfies the following estimates:

(a) *Suppose  $f$  is monotonic non-decreasing on  $[a, b]$ , then*

$$\begin{aligned} (3.7) \quad & |R_{PW}(f, g, h, I_n)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_{i+1}) - f(x_i)] \\ & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn} \left( h^{-1} \left( \frac{h(x_i) + h(x_{i+1})}{2} \right) - t \right) f(t) g(t) dt \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} g(t) dt \right) \cdot [f(x_{i+1}) - f(x_i)] \\ & \leq \frac{\nu(l)}{2} \cdot [f(b) - f(a)]. \end{aligned}$$

(b) *Suppose  $f$  is monotonic non-increasing on  $[a, b]$ , then*

$$\begin{aligned} (3.8) \quad & |R_{PW}(f, g, h, I_n)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_i) - f(x_{i+1})] \\ & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn} \left( t - h^{-1} \left( \frac{h(x_i) + h(x_{i+1})}{2} \right) \right) f(t) g(t) dt \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} g(t) dt \right) \cdot [f(x_i) - f(x_{i+1})] \\ & \leq \frac{\nu(l)}{2} \cdot [f(a) - f(b)]. \end{aligned}$$

**Remark 4.** *In Corollary 2, suppose  $f$  is monotonic on  $[a, b]$ ,*

$$x_i = h^{-1} \left[ h(a) + \frac{i(h(b) - h(a))}{n} \right] \quad (i = 0, 1, \dots, n),$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b) - h(a)}{n} = \frac{1}{n} \int_a^b g(t) dt. \quad (i = 0, 1, \dots, n-1).$$

If we want to approximate the integral  $\int_a^b f(t)g(t)dt$  by  $T_{PW}(f, g, h, I_n)$  with an accuracy less than  $\varepsilon > 0$ , we need at least  $n_\varepsilon \in \mathbb{N}$  points for the partition  $I_n$ , where

$$n_\varepsilon := \left\lceil \frac{1}{2\varepsilon} \int_a^b g(t) dt \cdot |f(b) - f(a)| \right\rceil + 1$$

and  $[r]$  denotes the Gaussian integer of  $r$  ( $r \in \mathbb{R}$ ).

## 4. SOME INEQUALITIES FOR RANDOM VARIABLES

Throughout this section, let  $0 < a < b$ ,  $r \in \mathbb{R}$ , and let  $X$  be a continuous random variable having the continuous probability density mapping  $g : [a, b] \rightarrow \mathbb{R}$  with  $g(t) > 0$  on  $(a, b)$ ,  $h : [a, b] \rightarrow \mathbb{R}$  with  $h'(t) = g(t)$  for  $t \in (a, b)$  and the  $r$ -moment

$$E_r(X) := \int_a^b t^r g(t) dt,$$

which is assumed to be finite.

**Theorem 3.** *The inequalities*

$$(4.1) \quad \left| E_r(X) - \frac{a^r + b^r}{2} \right| \leq \frac{1}{2} (b^r - a^r) + \int_a^b \operatorname{sgn} \left( h^{-1} \left( \frac{1}{2} \right) - t \right) t^r g(t) dt \\ \leq \frac{1}{2} (b^r - a^r) \quad \text{as } r \geq 0$$

and

$$(4.2) \quad \left| E_r(X) - \frac{a^r + b^r}{2} \right| \leq \frac{1}{2} (a^r - b^r) + \int_a^b \operatorname{sgn} \left( t - h^{-1} \left( \frac{1}{2} \right) \right) t^r g(t) dt \\ \leq \frac{1}{2} (a^r - b^r) \quad \text{as } r < 0,$$

hold.

*Proof.* If we put  $f(t) = t^r$  ( $t \in [a, b]$ ),  $h(t) = \int_a^t g(x) dx$  ( $t \in [a, b]$ ) and  $x = \frac{h(a)+h(b)}{2} = \frac{1}{2}$  in Corollary 1, then we obtain (4.1) and (4.2). This completes the proof. ■

The following corollary which is a special case of Theorem 3.

**Corollary 3.** *The inequalities*

$$(4.3) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{b-a}{2} + \int_a^b \operatorname{sgn} \left( h^{-1} \left( \frac{1}{2} \right) - t \right) t g(t) dt \leq \frac{b-a}{2}$$

hold where  $E(X)$  is the expectation of the random variable  $X$ .

## 5. INEQUALITIES FOR BETA MAPPING AND GAMMA MAPPING

The following two mappings are well-known in the literature as the *Beta mapping* and the *Gamma mapping*, respectively:

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, y > 0. \\ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0.$$

The following inequality which is an application of Theorem 1 for the Beta mapping holds:

**Theorem 4.** *Let  $p, q > 0$ . Then we have the inequality*

$$\begin{aligned}
(5.1) \quad & |\beta(p+1, q+1) - x| \\
& \leq x + \int_a^b \operatorname{sgn} \left[ t - ((p+1)x)^{\frac{1}{p+1}} \right] t^p (1-t)^q dt \\
& \leq x + \left( \frac{1}{p+1} - 2x \right) \left[ 1 - ((p+1)x)^{\frac{1}{p+1}} \right]^q \\
& \leq \frac{1}{2(p+1)} + \left| x - \frac{1}{2(p+1)} \right|
\end{aligned}$$

for all  $x \in \left[ 0, \frac{1}{p+1} \right]$ .

*Proof.* If we put  $a = 0$ ,  $b = 1$ ,  $f(t) = (1-t)^q$ ,  $g(t) = t^p$  and  $h(t) = \frac{t^{p+1}}{p+1}$  ( $t \in [0, 1]$ ) in Theorem 1, we obtain the inequality (5.1) for all  $x \in \left[ 0, \frac{1}{p+1} \right]$ . This completes the proof. ■

The following remark which is an applicaton of Theorem 4 for the Gamma mapping holds:

**Remark 5.** *Taking into account that  $\beta(p+1, q+1) = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$ , the inequality (5.1) is equivalent to*

$$\begin{aligned}
\left| \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} - x \right| & \leq x + \int_a^b \operatorname{sgn} \left[ t - ((p+1)x)^{\frac{1}{p+1}} \right] t^p (1-t)^q dt \\
& \leq x + \left( \frac{1}{p+1} - 2x \right) \left[ 1 - ((p+1)x)^{\frac{1}{p+1}} \right]^q \\
& \leq \frac{1}{2(p+1)} + \left| x - \frac{1}{2(p+1)} \right|
\end{aligned}$$

*i.e.,*

$$\begin{aligned}
& |(p+1)\Gamma(p+1)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\
& \leq \left[ x + \int_a^b \operatorname{sgn} \left[ t - ((p+1)x)^{\frac{1}{p+1}} \right] t^p (1-t)^q dt \right] (p+1)\Gamma(p+q+2) \\
& \leq \left[ x + \left( \frac{1}{p+1} - 2x \right) \left[ 1 - ((p+1)x)^{\frac{1}{p+1}} \right]^q \right] (p+1)\Gamma(p+q+2) \\
& \leq \left[ \frac{1}{2} + \left| x(p+1) - \frac{1}{2} \right| \right] \cdot \Gamma(p+q+2)
\end{aligned}$$

and as  $(p+1)\Gamma(p+1) = \Gamma(p+2)$ , we get

$$\begin{aligned}
(5.2) \quad & |\Gamma(p+2)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\
& \leq \left[ x + \int_a^b \operatorname{sgn} \left[ t - ((p+1)x)^{\frac{1}{p+1}} \right] t^p (1-t)^q dt \right] (p+1)\Gamma(p+q+2) \\
& \leq \left[ x + \left( \frac{1}{p+1} - 2x \right) \left[ 1 - ((p+1)x)^{\frac{1}{p+1}} \right]^q \right] (p+1)\Gamma(p+q+2) \\
& \leq \left[ \frac{1}{2} + \left| x(p+1) - \frac{1}{2} \right| \right] \cdot \Gamma(p+q+2).
\end{aligned}$$

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