ON GENERALIZATIONS OF HILBERT'S INEQUALITY

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ABSTRACT. In this paper, by introducing three parameters A, B and λ, and estimating the weight coefficient, we give a new generalization of Hilbert's inequality with a best constant factor. As applications, we consider its equivalent form and some particular results.
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1. INTRODUCTION

If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the famous Hardy-Hilbert's inequality and its equivalent form are given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{sin(\pi/p)}\{\sum_{n=1}^{\infty} a_n^p\}^{1/p}\{\sum_{n=1}^{\infty} b_n^q\}^{1/q}; \tag{1.1}$$

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}\frac{a_m}{m+n})^p < [\frac{\pi}{sin(\pi/p)}]^p \sum_{n=1}^{\infty} a_n^p, \tag{1.2}$$

where the constant factor $\pi/sin(\pi/p)$ and $[\pi/sin(\pi/p)]^p$ are the best possible(see[1]).

For p=q=2, inequality (1.1) reduces to the following Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2)^{1/2}. \tag{1.3}$$

Inequality (1.1),(1.2) and (1.3) are important in analysis and its applications(see [2]). In recent years, by obtaining the inequality of the weight coefficient as follows

$$\varpi_1(r,m) = m^{1-1/r}\sum_{n=1}^{\infty}\frac{1}{(m+n)n^{1-1/r}} < \frac{\pi}{sin(\pi/p)} - \frac{1-\gamma}{n^{1/r}}\,(r=p,q) \tag{1.4}$$

($1-\gamma = 0.42278433^+, \gamma$ is Euler constant), inequality (1.1) had been strengthened by [3,4] as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}}]a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/q}}]b_n^q\}^{\frac{1}{q}}. \tag{1.5}$$

By introducing three parameters A, B and λ, Yang et al. [5] gave a generalization of (1.1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{B(\phi_\lambda(p),\phi_\lambda(q))}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^p\}^{1/p}\{\sum_{n=1}^{\infty} n^{1-\lambda}b_n^q\}^{1/q}, \tag{1.6}$$

where the constant factor

$$\frac{B(\phi_\lambda(p),\phi_\lambda(q))}{A^{\phi_\lambda(p)}B^{\phi_\lambda(q)}}\ \ (\phi_\lambda(r)=\frac{r+\lambda-2}{r},\lambda>2-r,r=p,q;A,B>0)$$

is the best possible (B(u,v) is the β function). For A=B=1, inequality (1.6) reduces to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda}<B\big(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\big)\big\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\big\}^{\frac{1}{p}}\big\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\big\}^{\frac{1}{q}}. \tag{1.7}$$

Both (1.6) and (1.7) are generalizations of (1.1) and (1.3). By introducing a single parameter λ, Yang [6] also gave a generalization of (1.1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}<\frac{\pi}{\lambda sin(\frac{\pi}{p})}\big\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p\big\}^{\frac{1}{p}}\big\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\big\}^{\frac{1}{q}}. \tag{1.8}$$

where the constant factor $\pi/[\lambda sin(\pi/p)]\,(0<\lambda\le min\{p,q\})$ is the best possible.

The main objective of this paper is to estimating the following weight coefficient

$$\omega_\lambda(A,B,q,m)=m^{\lambda(1-1/q)}\sum_{n=1}^{\infty}\frac{1}{(Am^\lambda+Bn^\lambda)n^{1-\lambda/q}}$$

$$(A,B>0,0<\lambda\le q,m\in N), \tag{1.9}$$

and then to obtain a new generalization of inequality (1.3) related to the double series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}1/(Am^\lambda+Bn^\lambda)$ with a best constant factor, which is not a generalization of (1.1). As a particular result, we obtain a new generalization of (1.3) with(p, q)-parameters form other than (1.1). We also consider some equivalent inequalities.

For this, we introduce some lemmas.

2. SOME LEMMAS

Lemma 2.1. If $p>1,\frac{1}{p}+\frac{1}{q}=1,0<\lambda\le q$, and $A,B>0,\omega_\lambda(A,B,q,m)$ is defined by (1.9), then for any $m\in N$, we have

$$\omega_\lambda(A,B,q,m)<\frac{\pi}{A^{1/p}B^{1/q}\lambda sin(\pi/p)}. \tag{2.1}$$

Proof. Since $A,B>0,$ and $0<\lambda\le q$, we have

$$\omega_\lambda(A,B,q,m)<m^{\lambda(1-1/q)}\int_0^\infty\frac{1}{(Am^\lambda+By^\lambda)y^{1-\lambda/q}}dy.$$

Putting $u=(By^\lambda)/(Am^\lambda)$ in the above inequality, we obtain

$$\omega_\lambda(A,B,q,m)<\frac{1}{A^{1/p}B^{1/q}\lambda}\int_0^\infty\frac{u^{-1/p}}{1+u}du.$$

Thus, we have (2.1). The lemma is proved.

Note. If $0<\lambda\le p$, by (2.1), for $B,A>0$ and $n\in N$, we also have

$$\omega_\lambda(B,A,p,n)=n^{\lambda(1-\frac{1}{p})}\sum_{n=1}^{\infty}\frac{1}{(Bn^\lambda+Am^\lambda)m^{1-\frac{\lambda}{p}}}<\frac{\pi}{B^{1/q}A^{1/p}\lambda sin(\frac{\pi}{p})}. \tag{2.2}$$

Lemma 2.2. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq min\{p, q\}$, and $0 < \epsilon < \lambda$, then we have

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{Am^\lambda + Bn^\lambda} m^{\frac{\lambda-p-\epsilon}{p}} n^{\frac{\lambda-q-\epsilon}{q}}$$

$$> A^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} B^{-\frac{1}{q}+\frac{\epsilon}{q\lambda}} \frac{1}{\lambda}\Big[\frac{1}{\epsilon}\int_0^\infty \frac{1}{1+u} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du - \Big(\frac{B}{A}\Big)^{\frac{\lambda-\epsilon}{q\lambda}} \lambda\Big(\frac{q}{\lambda-\epsilon}\Big)^2\Big]. \tag{2.3}$$

Proof. We have

$$\frac{\lambda - r - \epsilon}{r} < 0\,(r = p, q), and\ \lambda - \epsilon > 0.$$

Hence we find

$$I > \int_1^\infty x^{\frac{\lambda-p-\epsilon}{p}} \Big(\int_1^\infty \frac{1}{Ax^\lambda + By^\lambda} y^{\frac{\lambda-q-\epsilon}{q}} dy\Big) dx.$$

Setting $u = (By^\lambda)/(Ax^\lambda)$ in the above integral, we obtain

$$I > A^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} B^{-\frac{1}{q}+\frac{\epsilon}{q\lambda}} \frac{1}{\lambda}\int_1^\infty x^{-1-\epsilon}\Big[\int_{B/(Ax^\lambda)}^\infty \frac{1}{1+u} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du\Big] dx$$

$$= A^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} B^{-\frac{1}{q}+\frac{\epsilon}{q\lambda}} \frac{1}{\lambda}\Big\{\int_1^\infty x^{-1-\epsilon}\Big[\int_0^\infty \frac{1}{1+u} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du\Big] dx$$

$$- \int_1^\infty x^{-1-\epsilon}\Big[\int_0^{B/(Ax^\lambda)} \frac{1}{1+u} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du\Big] dx\Big\}$$

$$> A^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} B^{-\frac{1}{q}+\frac{\epsilon}{q\lambda}} \frac{1}{\lambda}\Big\{\frac{1}{\epsilon}\int_0^\infty \frac{1}{1+u} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du$$

$$- \int_1^\infty x^{-1}\Big[\int_0^{B/(Ax^\lambda)} u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}} du\Big] dx\Big\}.$$

By calculating the last integral, we have (2.3). The lemma is proved.

3. MAIN RESULTS AND APPLICATIONS

Theorem 3.1. If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, and\, 0 < \lambda \leq min\{p, q\}$, such that $0 < \sum_{n=1}^\infty n^{p-1-\lambda} a_n^p < \infty$ and $0 < \sum_{n=1}^\infty n^{q-1-\lambda} b_n^q < \infty$, then for $A, B > 0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{Am^\lambda + Bn^\lambda} < \frac{\pi}{A^{\frac{1}{p}} B^{\frac{1}{q}} \lambda sin(\frac{\pi}{p})} \Big\{\sum_{n=1}^\infty n^{p-1-\lambda} a_n^p\Big\}^{\frac{1}{p}} \Big\{\sum_{n=1}^\infty n^{q-1-\lambda} b_n^q\Big\}^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor $\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$ is the best possible. In particular, for A=B=1, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda sin(\pi/p)} \Big\{\sum_{n=1}^\infty n^{p-1-\lambda} a_n^p\Big\}^{1/p} \Big\{\sum_{n=1}^\infty n^{q-1-\lambda} b_n^q\Big\}^{1/q}. \tag{3.2}$$

Proof. By Holder's inequality, in view of (1.9) and (2.2), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{Am^\lambda+Bn^\lambda}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\frac{a_m}{(Am^\lambda+Bn^\lambda)^{1/p}}\Big(\frac{m^{(1-\lambda)/q+(\lambda/q^2)}}{n^{(1-\lambda)/p+(\lambda/p^2)}}\Big)\Big]$$
$$\times\Big[\frac{b_n}{(Am^\lambda+Bn^\lambda)^{1/q}}\Big(\frac{n^{(1-\lambda)/p+(\lambda/p^2)}}{m^{(1-\lambda)/q+(\lambda/q^2)}}\Big)\Big]$$
$$\leq\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{Am^\lambda+Bn^\lambda}\Big(\frac{m^{(p-1)(1-\lambda)+(\lambda p)/q^2}}{n^{1-\lambda/q}}\Big)\Big\}^{1/p}$$
$$\times\Big\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^q}{Am^\lambda+Bn^\lambda}\Big(\frac{n^{(q-1)(1-\lambda)+(\lambda q)/p^2}}{m^{1-\lambda/p}}\Big)\Big\}^{1/q}$$
$$=\Big\{\sum_{m=1}^{\infty}\omega_\lambda(A,B,q,m)m^{p-1-\lambda}a_m^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}\omega_\lambda(B,A,p,n)n^{q-1-\lambda}b_n^q\Big\}^{\frac{1}{q}}. \qquad (3.3)$$

Hence by (2.1) and (2.2), we have (3.1).

For $0<\epsilon<\lambda$, setting $\overline{a}_m and\ \overline{b'}_n$ as:

$$\overline{a}_m=m^{\frac{\lambda-p-\epsilon}{p}},\overline{b}_n=n^{\frac{\lambda-q-\epsilon}{q}}\ (m,n\in N),$$

then we have

$$\Big\{\sum_{n=1}^{\infty}n^{p-1-\lambda}\overline{a}_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty}n^{q-1-\lambda}\overline{b}_n^q\Big\}^{1/q}=\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}$$
$$=1+\sum_{n=2}^{\infty}\frac{1}{n^{1+\epsilon}}<1+\int_2^\infty\frac{1}{t^{1+\epsilon}}dt=1+\frac{1}{\epsilon}. \qquad (3.4)$$

If there exists $A,B>0$ and $0<\lambda\leq min\{p,q\}$, such that the constant factor $\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$ in (3.1) is not the best possible, then, there exists a positive number $K<\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$, such that (3.1) is valid if we replace $\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$ by K. In particular, we have

$$\epsilon I=\epsilon\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{a}_m\overline{b}_n}{Am^\lambda+Bn^\lambda}<\epsilon K\Big\{\sum_{n=1}^{\infty}n^{p-1-\lambda}\overline{a}_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}n^{q-1-\lambda}\overline{b}_n^q\Big\}^{\frac{1}{q}},$$

and by (2.3) and (3.4), we find

$$A^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}}B^{-\frac{1}{q}+\frac{\epsilon}{q\lambda}}\frac{1}{\lambda}\Big[\int_0^\infty\frac{1}{1+u}u^{-\frac{1}{p}-\frac{\epsilon}{q\lambda}}du-\epsilon\Big(\frac{B}{A}\Big)^{\frac{\lambda-\epsilon}{q\lambda}}\lambda\Big(\frac{q}{\lambda-\epsilon}\Big)^2\Big]<K(\epsilon+1).$$

Setting $\epsilon\to 0^+$ in the above inequality, we conclude that $\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]\leq K$. This contradicts the fact that $K<\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$.Thus, the constant factor $\pi/[A^{1/p}B^{1/q}\lambda sin(\pi/p)]$ in (3.1) is the best possible. The theorem is proved.

Theorem 3.2. If $a_n\geq 0, p>1, \frac{1}{p}+\frac{1}{q}=1$, and $0<\lambda\leq min\{p,q\}$, such that $0<\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p<\infty$, then for $A,B>0$, we have

$$\sum_{n=1}^{\infty}n^{\lambda(p-1)-1}\Big(\sum_{m=1}^{\infty}\frac{a_m}{Am^\lambda+Bn^\lambda}\Big)^p<\frac{1}{AB^{p-1}}\Big[\frac{\pi}{\lambda sin(\frac{\pi}{p})}\Big]^p\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p, \qquad (3.5)$$

where the constant factor $\frac{1}{AB^{p-1}}\big[\frac{\pi}{\lambda sin(\pi/p)}\big]^p$ is the best possible; Inequality (3.5) is equivalent to (3.1). In particular, for A=B=1, we have the equivalent form of (3.2) as:

$$\sum_{n=1}^{\infty}n^{\lambda(p-1)-1}\Big(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\Big)^p<\Big[\frac{\pi}{\lambda sin(\pi/p)}\Big]^p\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p. \qquad (3.6)$$

Proof. Since $0 < \sum_{n=1}^{\infty} n^{p-1-\lambda} a_n^p < \infty$, there exists $k_0 \geq 1$, such for any $k \geq k_0$, that $0 < \sum_{n=1}^{k} n^{p-1-\lambda} a_n^p < \infty$. We set $b_n(k) = n^{\lambda(p-1)-1} \big(\sum_{m=1}^{k} \frac{a_m}{Am^\lambda + Bn^\lambda} \big)^{p-1}$ $(k \geq k_0)$, and use (3.1) to obtain

$$0 < \sum_{n=1}^{k} n^{q-1-\lambda} b_n^q(k) = \sum_{n=1}^{k} n^{\lambda(p-1)-1} \big(\sum_{m=1}^{k} \frac{a_m}{Am^\lambda + Bn^\lambda} \big)^p$$

$$= \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{a_m b_n(k)}{Am^\lambda + Bn^\lambda} < \frac{\pi}{A^{1/p} B^{1/q} \lambda sin(\pi/p)}$$

$$\times \big\{ \sum_{n=1}^{k} n^{p-1-\lambda} a_n^p \big\}^{1/p} \big\{ \sum_{n=1}^{k} n^{q-1-\lambda} b_n^q(k) \big\}^{1/q}. \tag{3.7}$$

Thus we find

$$\big\{ \sum_{n=1}^{k} n^{q-1-\lambda} b_n^q(k) \big\}^{1/p} < \frac{\pi}{A^{1/p} B^{1/q} \lambda sin(\pi/p)} \big\{ \sum_{n=1}^{k} n^{p-1-\lambda} a_n^p \big\}^{1/p}. \tag{3.8}$$

It follows that $0 < \sum_{n=1}^{\infty} n^{q-1-\lambda} b_n^q(\infty) < \infty$. Hence (3.7) is valid as $k \to \infty$ by (3.1). So is (3.8). Thus, inequality (3.5) holds.

For the equivalence, we need show that (3.5) implies (3.1). By Holder's inequality, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{Am^\lambda + Bn^\lambda} = \sum_{n=1}^{\infty} \big[n^{(\lambda+1-q)/q} \sum_{m=1}^{\infty} \frac{a_m}{Am^\lambda + Bn^\lambda} \big] \big[n^{(q-1-\lambda)/q} b_n \big]$$

$$\leq \big\{ \sum_{n=1}^{\infty} n^{\lambda(p-1)-1} \big(\sum_{m=1}^{\infty} \frac{a_m}{Am^\lambda + Bn^\lambda} \big)^p \big\}^{1/p} \big\{ \sum_{n=1}^{\infty} n^{q-1-\lambda} b_n^q \big\}^{1/q}. \tag{3.9}$$

Hence by (3.5), we have (3.1). It follows that inequality (3.5) is equivalent to (3.1) .

If the constant factor in (3.5) is not the best possible, we may get a contradiction that the constant factor in (3.1) is not the best possible by using (3.9). The theorem is proved.

For $\lambda = 1$, reducing (3.2) and (3.6), we have

Corollary 3.3. If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, such that $0 < \sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-2} b_n^q < \infty$,then we have the following two equivalent inequalities:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{sin(\pi/p)} \big\{ \sum_{n=1}^{\infty} n^{p-2} a_n^p \big\}^{1/p} \big\{ \sum_{n=1}^{\infty} n^{q-2} b_n^q \big\}^{1/q}; \tag{3.10}$$

$$\sum_{n=1}^{\infty} n^{p-2} \big(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \big)^p < \big[\frac{\pi}{sin(\pi/p)} \big]^p \sum_{n=1}^{\infty} n^{p-2} a_n^p, \tag{3.11}$$

where both the constant factors in (3.10) and (3.11) are the best possible.

Since for $A = B = \lambda = 1, \omega_1(1,1,r,n) = \varpi_1(r,n)$, by (3.3) and (1.4), we have

Corollary 3.4. If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, such that $0 < \sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-2} b_n^q < \infty$,then we have a strengthened version of (3.10) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\pi/p)} - \frac{1-\gamma}{n^{1/q}}]n^{p-2}a_n^p\}^{1/p}$$
$$\times\{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\pi/p)} - \frac{1-\gamma}{n^{1/p}}]n^{q-2}b_n^q\}^{1/q}, \quad (3.12)$$

where $1-\gamma = 0.42278433^+$(γ is Euler constant).

Remark 3.5. (a) For p=q=2, both (3.2) and (1.6) reduce to the same inequality as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} < \frac{\pi}{\lambda}\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\}^{1/2}\,(0<\lambda\leq 2), \quad (3.13)$$

and inequality (3.10) reduces to (1.3). It follows that (3.2) and (1.6) are different generalizations of (3.13) and (1.3), and (3.10) is a new generalization of (1.3) with (p, q)-parameters form, but other than (1.1).

(b) Inequality (3.1) is also a generalization of (1.3), (3.10) and (3.2), but not (1.1)

(c) Since all the given inequalities and equivalent form are with best constant factors, we obtain some new results.

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