K. PETR'S FORMULA OF DOUBLE INTEGRAL AND ESTIMATES OF ITS REMAINDER

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ABSTRACT. In the article, K. Petr's formula of single integral is generalized to that of double integral, some important special cases and estimates of its remainder are established.

1. INTRODUCTION

In [14, p. 218], K. Petr's formula for single integral is given, which can be modified slightly as follows.

Theorem A (K. Petr's formula of single integral)**.** *Let $f(x)$ be a function defined on $[a,b]\subset\mathbb{R}$ such that $f^{(n-1)}(x)$ is absolutely continuous, $P_n(t)$ a polynomial of degree n with coefficient a_n of the term t^n. Then*

$$\int_a^b f(x)\,\mathrm{d}x=\sum_{k=1}^n\frac{(-1)^{k+1}}{n!a_n}\Big[P_n^{(n-k)}(b)f^{(k-1)}(b)-P_n^{(n-k)}(a)f^{(k-1)}(a)\Big]+\frac{(-1)^n}{n!a_n}\int_a^b P_n(x)f^{(n)}(x)\,\mathrm{d}x. \tag{1.1}$$

Remark 1. K. Petr's formula stated in [14, p.218] is a special case of (1.1) by letting $a_n=1$.

Remark 2. If taking $P_n(t)=(t-a)^n$ in (1.1), then we have

$$\int_a^b f(x)\,\mathrm{d}x=\sum_{k=1}^n\frac{(-1)^{k+1}}{k!}(b-a)^kf^{(k-1)}(b)+\frac{(-1)^n}{n!}\int_a^b(x-a)^nf^{(n)}(x)\,\mathrm{d}x. \tag{1.2}$$

In this paper, we will generalize K. Petr's formula (1.1) to the following

Theorem 1 (K. Petr's formula of double integral)**.** *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Let $P_n(t)$ be a polynomial of degree n with coefficient a_n of the term t^n and $Q_m(s)$ a polynomial of degree m with coefficient b_m of the term t^m. Then*

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y=A(f,P_n,Q_m)+B(f,P_n,Q_m)+R(f,P_n,Q_m), \tag{1.3}$$

where

$$\begin{aligned}A(f,P_n,Q_m)&=\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(a)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a,c)\Big]\\&-\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(b)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b,c)\Big],\end{aligned} \tag{1.4}$$

$$\begin{aligned}B(f,P_n,Q_m)&=\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y,\end{aligned} \tag{1.5}$$

$$R(f,P_n,Q_m)=\frac{(-1)^{m+n}}{m!n!a_nb_m}\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{1.6}$$

Further, some important special cases of K. Petr's formula (1.3) are given, some estimates of their remainders are established, and an application is discussed.

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Key words and phrases. K. Petr's formula, Appell polynomial, harmonic polynomial, single integral, double integral, estimate of remainder.

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2. Proof of Theorem 1

Integrating by part and using Theorem A yields

$$
\begin{aligned}
\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b P_n(x)\biggl[\int_c^d Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}y\biggr]\mathrm{d}x\\
&= (-1)^m\int_a^b P_n(x)\biggl\{m!b_m\int_c^d f^{(n,0)}(x,y)\,\mathrm{d}y\\
&\quad+\sum_{i=1}^m(-1)^i\Bigl[Q_m^{(m-i)}(d)f^{(n,i-1)}(x,d)-Q_m^{(m-i)}(c)f^{(n,i-1)}(x,c)\Bigr]\biggr\}\mathrm{d}x\\
&= (-1)^m m!b_m\int_a^b\int_c^d P_n(x)f^{(n,0)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(d)\int_a^b P_n(x)f^{(n,i-1)}(x,d)\,\mathrm{d}x\\
&\quad-\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(c)\int_a^b P_n(x)f^{(n,i-1)}(x,c)\,\mathrm{d}x\\
&= (-1)^m m!b_m\int_c^d(-1)^n\biggl\{n!a_n\int_a^b f(x,y)\,\mathrm{d}x\\
&\quad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,0)}(b,y)-P_n^{(n-j)}(a)f^{(j-1,0)}(a,y)\Bigr]\biggr\}\mathrm{d}y\\
&\quad+\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(d)\biggl\{(-1)^n\biggl[n!a_n\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\
&\quad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,i-1)}(b,d)-P_n^{(n-j)}(a)f^{(j-1,i-1)}(a,d)\Bigr]\biggr]\biggr\}\\
&\quad-\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(c)\biggl\{(-1)^n\biggl[n!a_n\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x\\
&\quad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,i-1)}(b,c)-P_n^{(n-j)}(a)f^{(j-1,i-1)}(a,c)\Bigr]\biggr]\biggr\}\\
&= (-1)^{m+n}\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_n^{(n-j)}(a)\Bigl[Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a,c)-Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a,d)\Bigr]\\
&\quad+(-1)^{m+n}\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_n^{(n-j)}(b)\Bigl[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b,c)\Bigr]\\
&\quad-(-1)^{m+n}n!a_n\sum_{i=1}^m(-1)^iQ_m^{(m-i)}(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x\\
&\quad+(-1)^{m+n}n!a_n\sum_{i=1}^m(-1)^iQ_m^{(m-i)}(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\
&\quad+(-1)^{m+n}m!b_m\sum_{j=1}^n(-1)^jP_n^{(n-j)}(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y\\
&\quad-(-1)^{m+n}m!b_m\sum_{j=1}^n(-1)^jP_n^{(n-j)}(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y\\
&\quad+(-1)^{m+n}m!n!a_nb_m\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
\tag{2.1}
$$

Rearranging (2.1) leads to

$$
\begin{aligned}
\int_a^b \int_c^d & f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{(-1)^{i+j}}{m!n!a_nb_m} P_n^{(n-j)}(a)\Big[Q_m^{(m-i)}(d) f^{(j-1,i-1)}(a,d) - Q_m^{(m-i)}(c) f^{(j-1,i-1)}(a,c)\Big] \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{(-1)^{i+j}}{m!n!a_nb_m} P_n^{(n-j)}(b)\Big[Q_m^{(m-i)}(d) f^{(j-1,i-1)}(b,d) - Q_m^{(m-i)}(c) f^{(j-1,i-1)}(b,c)\Big] \\
&\quad + \sum_{i=1}^{m} \frac{(-1)^{i}}{m!b_m} Q_m^{(m-i)}(c) \int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x - \sum_{i=1}^{m} \frac{(-1)^{i}}{m!b_m} Q_m^{(m-i)}(d) \int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x \\
&\quad + \sum_{j=1}^{n} \frac{(-1)^{j}}{n!a_n} P_n^{(n-j)}(a) \int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y - \sum_{j=1}^{n} \frac{(-1)^{j}}{n!a_n} P_n^{(n-j)}(b) \int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y \\
&\quad + \frac{(-1)^{m+n}}{m!n!a_nb_m} \int_a^b \int_c^d P_n(x)Q_m(y) f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{2.2}
$$

The proof of Theorem 1 is complete.

3. Some important special cases of K. Petr's formula for double integral

Definition 1 ([2])**.** Let $P_k(t)$ be a polynomial satisfying

$$P_k'(t) = P_{k-1}(t), \quad P_0(t) = 1, \quad k = 1, 2, \dots, \tag{3.1}$$

then we call $P_k(t)$ an Appell polynomial or a harmonic polynomial.

Proposition 1. *Let $P_n(t)$ be an Appell polynomial of degree n and the coefficient of the term t^n equal a_n. Then $a_n = \frac{1}{n!}$.*

Theorem 2 (Harmonic K. Petr's formula of double integral)**.** *Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a function such that $f^{(j,i)}(x,y)$ is continuous on D for $0 \le j \le n$ and $0 \le i \le m$. If $P_n(t)$ and $Q_m(s)$ are two harmonic polynomials, then*

$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y = A(f,P_n,Q_m) + B(f,P_n,Q_m) + R(f,P_n,Q_m), \tag{3.2}$$

where

$$
\begin{aligned}
A(f,P_n,Q_m) &= \sum_{i=1}^{m}\sum_{j=1}^{n} (-1)^{i+j} P_j(a)\Big[Q_i(d) f^{(j-1,i-1)}(a,d) - Q_i(c) f^{(j-1,i-1)}(a,c)\Big] \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} (-1)^{i+j} P_j(b)\Big[Q_i(d) f^{(j-1,i-1)}(b,d) - Q_i(c) f^{(j-1,i-1)}(b,c)\Big],
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
B(f,P_n,Q_m) &= \sum_{i=1}^{m} (-1)^{i} Q_i(c) \int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x - \sum_{i=1}^{m} (-1)^{i} Q_i(d) \int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x \\
&\quad + \sum_{j=1}^{n} (-1)^{j} P_j(a) \int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y - \sum_{j=1}^{n} (-1)^{j} P_j(b) \int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y,
\end{aligned} \tag{3.4}
$$

$$R(f,P_n,Q_m) = (-1)^{m+n} \int_a^b \int_c^d P_n(x)Q_m(y) f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.5}$$

Proof. Since $P_n(t)$ and $Q_m(s)$ are harmonic polynomials, then

$$a_n = \frac{1}{n!}, \quad b_m = \frac{1}{m!}, \quad P_n^{(n-j)}(t) = P_j(t), \quad Q_m^{(m-i)}(s) = Q_i(s). \tag{3.6}$$

Substituting (3.6) into Theorem 1 yields Theorem 2. □

Theorem 3. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then for $0\le\lambda\le1$ and $0\le\mu\le1$, we have*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y&=\sum_{i=1}^m\sum_{j=1}^n\frac{(1-\lambda)^j(c-d)^i(b-a)^j}{i!j!}[\mu^if^{(j-1,i-1)}(a,d)-(\mu-1)^if^{(j-1,i-1)}(a,c)]\\&-\sum_{i=1}^m\sum_{j=1}^n\frac{\lambda^j(c-d)^i(a-b)^j}{i!j!}[\mu^if^{(j-1,i-1)}(b,d)-(\mu-1)^if^{(j-1,i-1)}(b,c)]\\&+\sum_{i=1}^m\frac{(1-\mu)^i(d-c)^i}{i!}\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m\frac{\mu^i(c-d)^i}{i!}\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{(1-\lambda)^j(b-a)^j}{j!}\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n\frac{\lambda^j(a-b)^j}{j!}\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y\\&+R(f,a,b,c,d),\end{aligned}\tag{3.7}$$

where

$$R(f,a,b,c,d)=\frac{(-1)^{m+n}}{m!n!}\int_a^b\int_c^d[x-(\lambda a+(1-\lambda)b)]^n[y-(\mu c+(1-\mu)d)]^mf^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.8}$$

Proof. Letting

$$P_n(x)=[x-(\lambda a+(1-\lambda)b)]^n,\quad Q_m(y)=[y-(\mu c+(1-\mu)d)]^m\tag{3.9}$$

in Theorem 1, then $a_n=1$ and $b_m=1$. Further, by direct computation, we have

$$P_n^{(n-j)}(x)=\frac{n!}{j!}[x-(\lambda a+(1-\lambda)b)]^j,\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}[y-(\mu c+(1-\mu)d)]^i.\tag{3.10}$$

Therefore, Theorem 3 follows easily. □

Definition 2 ([1, 23.1.1])**.** Bernoulli's polynomials $B_k(x)$ for k being nonnegative integers are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^\infty\frac{t^k}{k!}B_k(x),\quad |x|<2\pi,\quad t\in\mathbb{R},\tag{3.11}$$

where $B_k(0)=B_k$ is called Bernoulli's numbers.

Definition 3 ([1, 23.1.1])**.** Euler's polynomials $E_k(x)$ for k being nonnegative integers are defined by

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^\infty\frac{t^k}{k!}E_k(x),\quad |x|<\pi,\quad t\in\mathbb{R},\tag{3.12}$$

where $2^kE_k(\frac12)=E_k$ is called Euler's numbers.

Remark 3. Notice that Bernoulli's numbers and polynomials and Euler's numbers and polynomials have been generalized by the authors in [4, 5, 6, 7, 9] recently.

Lemma 1 ([1, 23.1.5] and [8])**.** *The following identities hold*

$$B_k'(x)=kB_{k-1}(x),\quad E_k'(x)=kE_{k-1}(x),\quad k=1,2,\dots.\tag{3.13}$$

Lemma 2 ([1, 23.1.6] and [8])**.** *The following identities hold*

$$B_i(t+1)-B_i(t)=it^{i-1},\quad E_i(t+1)+E_i(t)=2t^i,\quad i=0,1,\dots.\tag{3.14}$$

Lemma 3 ([1, 23.1.20] and [8])**.** *The following identities hold*

$$B_k(0)=(-1)^kB_k(1)=B_k,\quad k=0,1,2,\dots,\tag{3.15}$$

$$E_i(0)=-E_i(1)=-\frac{2}{i+1}(2^{i+1}-1)B_{i+1},\quad i=1,2,\dots.\tag{3.16}$$

Theorem 4. *Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0 \le j \le n$ and $0 \le i \le m$. Then*

$$\begin{aligned}\int_a^b \int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{2(2^{i+1}-1)(a-b)^j(c-d)^i}{(i+1)!j!} \\ &\times B_{i+1}B_j\Big[f^{(j-1,i-1)}(a,d) + f^{(j-1,i-1)}(a,c) - (-1)^j\big[f^{(j-1,i-1)}(b,d) + f^{(j-1,i-1)}(b,c)\big]\Big] \\ &+ \sum_{i=1}^{m} \frac{2(1-2^{i+1})(c-d)^i}{(i+1)!} B_{i+1} \int_a^b \Big[f^{(0,i-1)}(x,c) + f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x \\ &+ \sum_{j=1}^{n} \frac{(a-b)^j}{j!} B_j \int_c^d \Big[f^{(j-1,0)}(a,y) + (-1)^{j+1} f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y \\ &+ R(f, B_n, E_m),\end{aligned} \tag{3.17}$$

where

$$R(f,B_n,E_m) = \frac{(a-b)^n(c-d)^m}{m!n!} \int_a^b \int_c^d B_n\Big(\frac{x-a}{b-a}\Big) E_m\Big(\frac{y-c}{d-c}\Big) f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.18}$$

Proof. Taking

$$P_n(x) = B_n\Big(\frac{x-a}{b-a}\Big), \quad Q_m(y) = E_m\Big(\frac{y-c}{d-c}\Big) \tag{3.19}$$

in Theorem 1, then $a_n = \frac{1}{(b-a)^n}$ and $b_m = \frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x) = \frac{n!}{j!}(b-a)^{j-n} B_j\Big(\frac{x-a}{b-a}\Big), \quad Q_m^{(m-i)}(y) = \frac{m!}{i!}(d-c)^{i-m} E_i\Big(\frac{y-c}{d-c}\Big), \tag{3.20}$$

Theorem 4 follows. □

Theorem 5. *Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0 \le j \le n$ and $0 \le i \le m$. Then*

$$\begin{aligned}\int_a^b \int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{(a-b)^j(c-d)^i}{i!j!} B_iB_j \\ &\times \Big[(-1)^i f^{(j-1,i-1)}(a,d) - f^{(j-1,i-1)}(a,c) - (-1)^j[(-1)^i f^{(j-1,i-1)}(b,d) - f^{(j-1,i-1)}(b,c)]\Big] \\ &+ \sum_{i=1}^{m} \frac{(c-d)^i}{i!} B_i \int_a^b \Big[f^{(0,i-1)}(x,c) + (-1)^{i+1} f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x \\ &+ \sum_{j=1}^{n} \frac{(a-b)^j}{j!} B_j \int_c^d \Big[f^{(j-1,0)}(a,y) + (-1)^{j+1} f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y \\ &+ R(f, B_n, B_m),\end{aligned} \tag{3.21}$$

where

$$R(f,B_n,B_m) = \frac{(a-b)^n(c-d)^m}{m!n!} \int_a^b \int_c^d B_n\Big(\frac{x-a}{b-a}\Big) B_m\Big(\frac{y-c}{d-c}\Big) f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.22}$$

Proof. Setting

$$P_n(x) = B_n\Big(\frac{x-a}{b-a}\Big), \quad Q_m(y) = B_m\Big(\frac{y-c}{d-c}\Big) \tag{3.23}$$

in Theorem 1, then $a_n = \frac{1}{(b-a)^n}$ and $b_m = \frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x) = \frac{n!}{j!}(b-a)^{j-n} B_j\Big(\frac{x-a}{b-a}\Big), \quad Q_m^{(m-i)}(y) = \frac{m!}{i!}(d-c)^{i-m} B_i\Big(\frac{y-c}{d-c}\Big), \tag{3.24}$$

Theorem 5 follows. □

Theorem 6. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y=&-\sum_{i=1}^m\sum_{j=1}^n\frac{4(2^{i+1}-1)(2^{j+1}-1)(a-b)^j(c-d)^i}{(i+1)!(j+1)!}B_{i+1}B_{j+1}\\&\times\Big[f^{(j-1,i-1)}(b,d)+f^{(j-1,i-1)}(b,c)+f^{(j-1,i-1)}(a,d)+f^{(j-1,i-1)}(a,c)\Big]\\&+\sum_{i=1}^m\frac{2(c-d)^i(1-2^{i+1})}{(i+1)!}B_{i+1}\int_a^b\Big[f^{(0,i-1)}(x,c)+f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{2(a-b)^j(1-2^{j+1})}{(j+1)!}B_{j+1}\int_c^d\Big[f^{(j-1,0)}(a,y)+f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y\\&+R(f,E_n,E_m),\end{aligned}\tag{3.25}$$

where

$$R(f,E_n,E_m)=\frac{(a-b)^n(c-d)^m}{m!n!}\int_a^b\int_c^d E_n\Big(\frac{x-a}{b-a}\Big)E_m\Big(\frac{y-c}{d-c}\Big)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.26}$$

Proof. Letting

$$P_n(x)=E_n\Big(\frac{x-a}{b-a}\Big),\quad Q_m(y)=E_m\Big(\frac{y-c}{d-c}\Big)\tag{3.27}$$

in Theorem 1, then $a_n=\frac{1}{(b-a)^n}$ and $b_m=\frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x)=\frac{n!}{j!}(b-a)^{j-n}E_j\Big(\frac{x-a}{b-a}\Big),\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}(d-c)^{i-m}E_i\Big(\frac{y-c}{d-c}\Big),\tag{3.28}$$

Theorem 6 follows. □

Remark 4. If taking the following harmonic polynomials

$$P_n(x)=\frac{(x-b)^n}{n!},\qquad Q_m(y)=\frac{(y-d)^m}{m!},\tag{3.29}$$

$$P_n(x)=\frac{(x-a)^n}{n!}B_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}E_m\Big(\frac{y-c}{d-c}\Big),\tag{3.30}$$

$$P_n(x)=\frac{(x-a)^n}{n!}B_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}B_m\Big(\frac{y-c}{d-c}\Big),\tag{3.31}$$

$$P_n(x)=\frac{(x-a)^n}{n!}E_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}E_m\Big(\frac{y-c}{d-c}\Big)\tag{3.32}$$

in Theorem 2, then the theorems in this section can be obtained again.

4. Estimates of remainders

In this section, we will give some estimates of the remainders mentioned above.

Lemma 4 ([1, 23.1.12])**.** *We have the following*

$$\int_0^1 B_n(x)B_m(x)\,\mathrm{d}x=(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n},\quad m,n=1,2,\dots,\tag{4.1}$$

$$\int_0^1 E_n(x)E_m(x)\,\mathrm{d}x=4(-1)^n(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2},\quad m,n=0,1,\dots.\tag{4.2}$$

Theorem 7. *Under conditions of Theorem 1, the remainder (1.6) can be estimated as follows*

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\max_{x\in[a,b]}\{|P_n(x)|\}\max_{y\in[c,d]}\{|Q_m(y)|\}\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\tag{4.3}$$

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}\left|\int_a^b P_n(x)\,\mathrm{d}x\right|\left|\int_c^d Q_m(y)\,\mathrm{d}y\right|,\tag{4.4}$$

$$|R(f,P_n,Q_m)|\le\frac{(b-a)(d-c)}{m!n!\,|a_nb_m|}\max_{x\in[a,b]}\{|P_n(x)|\}\max_{y\in[c,d]}\{|Q_m(y)|\}\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\},\tag{4.5}$$

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\left[\int_a^b|P_n(x)|^q\,\mathrm{d}x\int_c^d|Q_m(y)|^q\,\mathrm{d}y\right]^{\frac1q}\left[\int_a^b\int_c^d\left|f^{(n,m)}(x,y)\right|^p\mathrm{d}x\,\mathrm{d}y\right]^{\frac1p},\tag{4.6}$$

where $p>1$ and $\frac1p+\frac1q=1$.

Proof. The estimates (4.3), (4.4) and (4.5) follows from standard arguments. The estimate (4.6) follows from Hölder's inequality of double integral. □

Theorem 8. *Under conditions of Theorem 3, we have the following estimates for remainder* (3.8)

$$|R(f,a,b,c)| \le \frac{(b-a)^n(d-c)^m}{m!n!}\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|, \tag{4.7}$$

$$\begin{aligned} |R(f,a,b,c)| &\le \frac{(b-a)^{n+1}(d-c)^{m+1}\left|\lambda^{n+1}-(1-\lambda)^{n+1}\right|\left|\mu^{m+1}-(1-\mu)^{m+1}\right|}{(m+1)!(n+1)!} \\ &\quad \times \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} |R(f,P_n,Q_m)| &\le \frac{(b-a)^{n+\frac1q}(d-c)^{m+\frac1q}[\lambda^{nq+1}+(1-\lambda)^{nq+1}]^{\frac1q}[\mu^{mq+1}+(1-\mu)^{mq+1}]^{\frac1q}}{m!n![(nq+1)(mq+1)]^{\frac1q}} \\ &\quad \times\left(\int_a^b\int_c^d\left|f^{(n,m)}(x,y)\right|^p\mathrm{d}x\,\mathrm{d}y\right)^{\frac1p}, \end{aligned} \tag{4.9}$$

where $p>1$ *and* $\frac1p+\frac1q=1$.

Proof. Estimates (4.7) and (4.8) follows from taking $P_n(x)=[x-(\lambda a+(1-\lambda)b)]^n$ and $Q_m(y)=[y-(\mu c+(1-\mu)d)]^m$ in (4.3) and (4.4) and using $a_n=1$ and $b_m=1$. Estimate (4.9) follows from Hölder's inequality of double integral. □

Theorem 9. *Under conditions of Theorem 4, we can estimate* (3.18) *as*

$$\begin{aligned} |R(f,B_n,E_m)| &\le \frac{(b-a)^n(d-c)^m}{m!n!}\max_{x\in[a,b]}\left\{\left|B_n\left(\frac{x-a}{b-a}\right)\right|\right\}\max_{y\in[c,d]}\left\{\left|E_m\left(\frac{y-c}{d-c}\right)\right|\right\} \\ &\quad\times\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|, \end{aligned} \tag{4.10}$$

$$\begin{aligned} |R(f,B_n,E_m)| &\le \frac{n!(b-a)^{n+1}(d-c)^{m+1}\left[(2n^2+3n+1)\left|B_{2n}\right|+(2^{2n+3}-2)\left|B_{2n+2}\right|\right]}{m!(2n+2)!} \\ &\quad\times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}. \end{aligned} \tag{4.11}$$

Proof. The estimate (4.10) is straightforward.

Taking $m=n$ in (4.1) and (4.2) of Lemma 4 yields

$$\begin{aligned} \int_0^1 B_n^2(x)\,\mathrm{d}x &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}B_{2n} = \frac{(n!)^2}{(2n)!}\left|B_{2n}\right|, \\ \int_0^1 E_m^2(x)\,\mathrm{d}x &= \frac{4(-1)^n(4^{n+1}-1)(n!)^2}{(2n+2)!}B_{2n+2} = \frac{4(4^{n+1}-1)(n!)^2}{(2n+2)!}\left|B_{2n+2}\right|. \end{aligned} \tag{4.12}$$

From (4.12), we have

$$\begin{aligned} |R(f,B_n,E_m)| &\le \frac{(b-a)^n(d-c)^m}{m!n!}\int_a^b\int_c^d\left|B_n\left(\frac{x-a}{b-a}\right)\right|\left|E_m\left(\frac{y-c}{d-c}\right)\right|\left|f^{(n,m)}(x,y)\right|\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{(b-a)^n(d-c)^m}{2m!n!}\int_a^b\int_c^d\left[B_n^2\left(\frac{x-a}{b-a}\right)+E_m^2\left(\frac{y-c}{d-c}\right)\right]\left|f^{(n,m)}(x,y)\right|\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{(b-a)^{n+1}(d-c)^{m+1}}{2m!n!}\left[\int_0^1 B_n^2(x)\,\mathrm{d}x+\int_0^1 E_m^2(y)\,\mathrm{d}y\right]\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\} \\ &\le \frac{n!(b-a)^{n+1}(d-c)^{m+1}\left[(2n^2+3n+1)\left|B_{2n}\right|+(2^{2n+3}-2)\left|B_{2n+2}\right|\right]}{m!(2n+2)!} \\ &\quad\times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}. \end{aligned} \tag{4.13}$$

The proof is complete. □

Theorem 10. *Under conditiions of Theorem 5, the remainder (3.22) can be estimated as*

$$\begin{aligned}|R(f,B_n,B_m)| \le \frac{(b-a)^n(d-c)^m}{m!n!} \max_{x\in[a,b]}\left\{\left|B_n\left(\frac{x-a}{b-a}\right)\right|\right\} \max_{y\in[c,d]}\left\{\left|B_m\left(\frac{y-c}{d-c}\right)\right|\right\} \\ \times \left|\int_a^b \int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\end{aligned} \tag{4.14}$$

$$\begin{aligned}|R(f,B_n,B_m)| \le \frac{(b-a)^{n+1}(d-c)^{m+1}}{2}\left[\frac{n!}{m!(2n)!}|B_{2n}| + \frac{m!}{n!(2m)!}|B_{2m}|\right] \\ \times \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.15}$$

Proof. The proof of (4.14) is easy. The proof of (4.15) is similar to that of (4.11). □

Theorem 11. *Under conditions of Theorem 6, the remainder (3.26) can be estimated as*

$$\begin{aligned}|R(f,E_n,E_m)| \le \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!} \max_{x\in[a,b]}\left\{\left|E_n\left(\frac{x-a}{b-a}\right)\right|\right\} \max_{y\in[c,d]}\left\{\left|E_m\left(\frac{y-c}{d-c}\right)\right|\right\} \\ \times \left|\int_a^b \int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\end{aligned} \tag{4.16}$$

$$\begin{aligned}|R(f,E_n,E_m)| \le 4(b-a)^{n+1}(d-c)^{m+1}\left[\frac{n!(4^{n+1}-1)}{m!(2n+2)!}|B_{2n+2}| + \frac{m!(4^{m+1}-1)}{n!(2m+2)!}|B_{2m+2}|\right] \\ \times \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\},\end{aligned} \tag{4.17}$$

$$\begin{aligned}|R(f,E_n,E_m)| \le \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1}(d-c)^{m+1}}{(n+2)!(m+2)!}|B_{n+2}||B_{m+2}| \\ \times \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.18}$$

Proof. The estimate of (4.16) is evident. The proof of (4.17) is similar to (4.11).

Using $\int_0^1 E_n(x)\,\mathrm{d}x = \frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}$ in [1, 23.1.11] and (3.16) in Lemma 3, we have

$$\begin{aligned}&|R(f,E_n,E_m)| \\ &\le \frac{(b-a)^n(d-c)^m}{m!n!}\left|\int_a^b E_n\left(\frac{y-c}{d-c}\right)\mathrm{d}x\right|\left|\int_c^d E_m\left(\frac{y-c}{d-c}\right)\mathrm{d}y\right| \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\} \\ &= \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!}\int_0^1 E_n(x)\,\mathrm{d}x\int_0^1 E_m(y)\,\mathrm{d}y \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\} \\ &= \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!}\left|\frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}\right|\left|\frac{E_{m+1}(1)-E_{m+1}(0)}{m+1}\right| \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\} \\ &= \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1}(d-c)^{m+1}}{(n+2)!(m+2)!}|B_{n+2}||B_{m+2}|\Big] \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.19}$$

The proof is complete. □

5. Examples

Now we apply K. Petr's formula to the function e^{x+y}.

Example 1. If taking $[a,b]\times[c,d]=[0,1]\times[0,1]$ and $f(x,y)=e^{x+y}$ in Theorem 3, then we have

$$\begin{aligned}\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = -\sum_{i=1}^m\sum_{j=1}^n \frac{1}{i!j!} + (e-1)\left[\sum_{i=1}^m \frac{1}{i!} + \sum_{j=1}^n \frac{1}{j!}\right] \\ + \frac{1}{m!n!}\int_0^1 (1-x)^n e^x\,\mathrm{d}x \int_0^1 (1-y)^m e^y\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{5.1}$$

Example 2. If taking $[a,b]\times[c,d]=[0,1]\times[0,1]$ and $f(x,y)=e^{x+y}$ in Theorem 4, then we have

$$
\begin{aligned}
\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = (e+1)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{2(-1)^{i+j}(2^{i+1}-1)B_{i+1}B_j}{(i+1)!j!}\left[1+(-1)^{j+1}e\right]\\
+(e^2-1)\sum_{i=1}^{m}\frac{2(-1)^{i}(1-2^{i+1})B_{i+1}}{(i+1)!}\\
+(e-1)\sum_{j=1}^{n}\frac{(-1)^{j}B_j}{j!}\Big[1+(-1)^{j+1}e\Big]\,\mathrm{d}y\\
+\frac{(-1)^{m+n}}{m!n!}\int_0^1 B_n(x)e^x\,\mathrm{d}x\int_0^1 E_m(y)e^y\,\mathrm{d}y.
\end{aligned}
\tag{5.2}
$$

By emploiting Theorem 5 and Theorem 6, we can obtain more other expansions of the double integral $\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y$.

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