IT'S JUST ANOTHER PI

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Abstract. In this paper we consider a particular integral from which we may develop identities for Pi and other numerical constants.

1. Introduction

The ratio of the circumference to the diameter of a circle produces, arguably the most common (famous) mathematical constant known to the human race, Pi, (π).

It appears that Pi was known to the Babylonians circa 2000BC and had a value of about $3\frac{1}{8}$. Throughout the ages Pi has been represented by various formulas and the following are listed for interest.

Vieta (˜1579)

$$\frac{1}{\pi}=\frac{1}{2}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

J. Wallis (˜1650)

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}.$$

Leibnitz (˜1670)

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots.$$

Newton (˜1666)

$$\pi=\frac{3\sqrt{3}}{4}+24\left(\frac{2}{3\cdot 2^3}-\frac{1}{5\cdot 2^5}-\frac{1}{28\cdot 2^7}-\frac{1}{72\cdot 2^9}-\cdots\right).$$

Machin Type Formulae (1706 – 1776)

$$\frac{\pi}{4}=4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right),$$
$$\frac{\pi}{4}=5\operatorname{arccot}(5)-3\operatorname{arccot}(18)-2\operatorname{arccot}(57),$$
$$\frac{\pi}{4}=17\operatorname{arccot}(22)+3\operatorname{arccot}(172)-2\operatorname{arccot}(682)-7\operatorname{arccot}(5357).$$

Euler (˜ 1748)

$$\pi^2=18\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.$$

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Ramanujan (1914)

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\binom{2k}{k}^{3}\frac{4^{2k+5}}{2^{12k+4}}.$$

Comtet (1974)

$$\pi^{4}=\frac{3240}{17}\sum_{k=1}^{\infty}\frac{1}{k^{4}\binom{2k}{k}}.$$

D. and G. Chudnovsky (1989)

$$\frac{1}{\pi}=12\sum_{k=0}^{\infty}(-1)^{k}\frac{(6n)!}{(n!)^{3}(3n)!}\cdot\frac{13591409+545140134k}{(640320^{3})^{k+\frac{1}{2}}}.$$

Bailey, Borwein and Plouffe (1996)

$$\pi=\sum_{k=0}^{\infty}\frac{1}{16^{k}}\left[\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right]. \tag{1.1}$$

Fibonacci type

$$\frac{\pi}{2}=\sum_{k=0}^{\infty}\arctan\left(\frac{1}{F_{2k+1}}\right),$$

where $F_{k+2}=F_{k+1}+F_{k},\ F_{0}=F_{1}=1.$

Bellard (1997)

$$\pi=\frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2^{10k}}\left[\frac{1}{10k+9}-\frac{4}{10k+7}-\frac{4}{10k+5}\right.$$
$$\left.-\frac{64}{10k+3}+\frac{256}{10k+1}-\frac{1}{4k+3}-\frac{32}{4k+1}\right].$$

Lupas (2000)

$$\pi=4+\sum_{k=1}^{\infty}(-1)^{k}\frac{\binom{2k}{k}40k^{2}+16k+1}{\binom{4k}{k}^{2}2k(4k+1)^{2}}.$$

I suspect that the Lupas formula contains an error, although I have not yet been able to find it.

Krattenthaler and Peterson (2000)

$$\pi=\frac{1}{9\cdot 25\cdot 49}\sum_{k=0}^{\infty}\frac{-89286+3875948k-34970134k^{2}+110202472k^{3}-115193600k^{4}}{\binom{8k}{4k}(-4)^{k}}.$$

Borwein and Girgensohn (2003)

$$\pi=\ln 4+10\sum_{k=1}^{\infty}\frac{1}{2^{k}k\binom{3k}{k}}.$$

Many other results of this type exist and recently Chudnovsky and Chudnovsky [4] obtained a master theorem from which they calculate

$$\frac{\pi}{2}=-1+\sum_{r=1}^{\infty}\frac{2^{r}}{\binom{2r}{r}}$$

and using the Taylor series expansion of the $\arcsin x$ function, we can obtain other similar formulae, such as

$$\pi = -3\sqrt{3} + \frac{9\sqrt{3}}{2}\sum_{r=1}^{\infty} \frac{r}{\binom{2r}{r}}.$$

In this paper we consider a general definite integral from which we can develop various other formulae for the representation of Pi and other constants.

The following integral will be needed for the formulation of Pi.

2. The Integral

Consider the integral

$$I_\infty = \int_0^{\frac{1}{a}} \frac{x^m}{\left(1-x^k\right)^\alpha} dx \tag{2.1}$$
$$= \int_0^{\frac{1}{a}} \sum_{r=0}^{\infty} (-1)^r \binom{-\alpha}{r} x^{kr+m},$$

where we have utilised

$$\frac{1}{(1+z)^\beta} = \sum_{r=0}^{\infty} \binom{-\beta}{r} z^r.$$

Now, from

$$\binom{-\beta}{r} = (-1)^r \binom{\beta+r-1}{r},$$

we have

$$I_\infty = \int_0^{\frac{1}{a}} \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} x^{kr+m}$$

and reversing the order of integration and summation, we obtain

$$I_\infty = \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)\, a^{rk+m+1}} \tag{2.2}$$
$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!\,(rk+m+1)\, a^{rk+m+1}},$$

where $(b)_s$ is Pochhammer's symbol defined by

$$\begin{cases} (b)_0 = 1 \\ (b)_s = b\,(b+1)\cdots(b+s-1) = \frac{\Gamma(b+s)}{\Gamma(b)}. \end{cases} \tag{2.3}$$

Binomial sums are intrinsically associated with generalised hypergeometric functions and if from (2.2) we let

$$T_r = \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)\, a^{rk+m+1}}, \tag{2.4}$$

then the ratio

$$\frac{T_{r+1}}{T_r} = \frac{(\alpha+r)\left(r+\frac{m+1}{k}\right)}{a^k\,(r+1)\left(r+\frac{m+1+k}{k}\right)} \tag{2.5}$$

and

$$T_0 = \frac{1}{(m+1)\, a^{m+1}}. \tag{2.6}$$

From (2.5) and (2.6) we can write

$$I_\infty = T_0\ {}_2F_1\left[\begin{matrix} \frac{m+1}{k}, & \alpha \\ \frac{m+1+k}{k} & \end{matrix} \middle| \frac{1}{a^k}\right], \tag{2.7}$$

where ${}_2F_1[\cdot\,\cdot]$ is the Gauss Hypergeometric function.

We can now match (2.2) and (2.7) so that

$$\sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)\, a^{rk+m+1}} = T_0\ {}_2F_1\left[\begin{matrix} \frac{m+1}{k}, & \alpha \\ \frac{m+1+k}{k} & \end{matrix} \middle| \frac{1}{a^k}\right]. \tag{2.8}$$

It is of interest to note that Bailey, Borwein, Borwein and Plouffe [1] utilised (2.1) for $a = \sqrt{2}$, $\alpha = 1$, $k = 8$ and $m = \beta - 1$, $\beta < 8$; that is

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^{\beta-1}}{1-x^8} dx = \frac{1}{2^{\frac{\beta}{2}}} \sum_{r=0}^{\infty} \frac{1}{16^r\,(8r+\beta)}$$

to prove the new formula (1.1).

Hirschhorn [5] has given a slightly different proof of (1.1) than that given by Bailey, Borwein, Borwein and Plouffe, but it must be mentioned that (1.1) was initially discovered empirically as was the formula

$$\pi^2 = \sum_{r=0}^{\infty} \frac{1}{16^k} \left[\frac{16}{(8k+1)^2} - \frac{16}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{16}{(8k+4)^2} - \frac{4}{(8k+5)^2} - \frac{4}{(8k+6)^2} + \frac{2}{(8k+7)^2} \right].$$

For the case $a = 1$, we notice that from (2.1)

$$I_\infty(1) = \int_0^1 \frac{x^m}{(1-x^k)^\alpha} dx = \frac{1}{k} B\left(1-\alpha, \frac{1+m}{k}\right) \tag{2.9}$$

for $k > 0$, $m > -1$ and $\alpha < 1$, where $B(\cdot,\cdot)$ is the classical Beta function.

Now,

$$B\left(1-\alpha, \frac{1+m}{k}\right) = k \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)},$$

$$\frac{\Gamma(1-\alpha)\,\Gamma\left(\frac{1+m}{k}\right)}{\Gamma\left(1-\alpha+\frac{1+m}{k}\right)} = k \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)},$$

where $\Gamma(\cdot)$ is the classical Gamma function.

From

$$\Gamma(1-\alpha) = \frac{\pi \operatorname{cosec}(\alpha\pi)}{\Gamma(\alpha)} \tag{2.10}$$

for $0 < \alpha < 1$, we have

$$\frac{\pi \operatorname{cosec}(\alpha\pi)\,\Gamma\left(\frac{1+m}{k}\right)}{\Gamma(\alpha)\,\Gamma\left(1-\alpha+\frac{1+m}{k}\right)} = k \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(rk+m+1)}$$

so that

$$\pi=\frac{k\Gamma\left(\alpha\right)\Gamma\left(1-\alpha+\frac{1+m}{k}\right)\sin\left(\alpha\pi\right)}{\Gamma\left(\frac{1+m}{k}\right)}\sum_{r=0}^{\infty}\binom{\alpha+r-1}{r}\frac{1}{(rk+m+1)}. \tag{2.11}$$

Let $m+1=\frac{3}{2}k$, then

$$\pi=\frac{\Gamma\left(\alpha\right)\Gamma\left(\frac{5}{2}-\alpha\right)\sin\left(\alpha\pi\right)}{\Gamma\left(\frac{3}{2}\right)}\sum_{r=0}^{\infty}\binom{\alpha+r-1}{r}\frac{1}{\left(r+\frac{3}{2}\right)}.$$

For $\alpha=\frac{1}{4}$, we have

$$\pi^{\frac{3}{2}}=\frac{5\sqrt{2}}{8}\left(\Gamma\left(\frac{1}{4}\right)\right)^{2}\sum_{r=0}^{\infty}\binom{r-\frac{3}{4}}{r}\frac{1}{(2r+3)}.$$

For $\alpha=\frac{1}{2}$, and using

$$\binom{r-\frac{1}{2}}{r}2^{2r}=\binom{2r}{r}$$

we have

$$\frac{\pi}{4}=\sum_{r=0}^{\infty}\binom{2r}{r}\frac{1}{4^{r}\left(2r+3\right)}.$$

For $\alpha=\frac{2}{3}$ and using the triplication formula

$$\Gamma\left(3z\right)=\frac{3^{3z-\frac{1}{2}}}{2\pi}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$$

we obtain

$$\sqrt{\pi}=\frac{4\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\sum_{r=0}^{\infty}\binom{r-\frac{1}{3}}{r}\frac{1}{(2r+3)}.$$

Other relationships for Pi may be obtained from (2.11), for example for $\alpha=\frac{1}{2}$ and $m+1=\frac{5}{2}k$, then we have

$$\pi=\frac{16}{3}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{1}{4^{r}\left(2r+5\right)}.$$

In general, from (2.11), for $\alpha=\frac{1}{2}$, we can deduce, after some basic algebra, that

$$\pi=\frac{2p!}{\left(\frac{1}{2}\right)_{p}}\sum_{r=0}^{\infty}\binom{2r}{r}\frac{1}{4^{r}\left(2r+2p+1\right)},\quad p=0,1,2,\ldots$$

and the rational number

$$\frac{(p-1)!}{\left(\frac{1}{2}\right)_{p}}=\sum_{r=0}^{\infty}\binom{2r}{r}\frac{1}{4^{r}\left(r+p\right)},\quad p=1,2,3,\ldots.$$

Some other results are:

- For $m=5$, $k=24$, $\alpha=\frac{7}{8}$

$$\frac{1}{\sqrt{\pi}}=\frac{4\left(\sqrt{2}-1\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{4}\right)^{2}}\sum_{r=0}^{\infty}\binom{r-\frac{1}{8}}{r}\frac{1}{(4r+1)}$$

and using the duplication formula for $\Gamma\left(\frac{1}{4}\right)$, we have

$$\pi^{\frac{3}{2}} = 2\left(\sqrt{2}-1\right)^{\frac{1}{2}} \Gamma\left(\frac{3}{4}\right)^{2} \sum_{r=0}^{\infty} \binom{r-\frac{1}{8}}{r} \frac{1}{(4r+1)}.$$

- For $m = \frac{23}{7}$, $k = 5$, $\alpha = \frac{6}{7}$

$$\pi = 7\sin\left(\frac{6\pi}{7}\right) \sum_{r=0}^{\infty} \frac{\left(\frac{6}{7}\right)_r}{r!\,(7r+6)}.$$

- For $m = 18$, $k = 19$, $\alpha = \frac{8}{9}$

$$\pi = \frac{1}{9} \sum_{r=0}^{\infty} \binom{r-\frac{1}{9}}{r} \frac{1}{(r+1)}.$$

In the case when $\frac{m+1}{k}$ =integer= s, say then from (2.11),

$$\pi = \frac{\Gamma(\alpha)\,\Gamma(1+s-\alpha)\sin(\alpha\pi)}{\Gamma(s)} \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(r+s)}$$

and using (2.10), then we obtain the numerical constant

$$B(s, 1-\alpha) = \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} \frac{1}{(r+s)}.$$

For $\alpha = \frac{1}{2}$ and $s = 6$ then

$$\frac{512}{693} = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{1}{4^r\,(r+6)}.$$

For other cases of the value of a in the integral (2.1) we may also obtain identities for π. In these cases the integral is a little more difficult to handle and these results will be reported in another forum. We will show that we can obtain remarkable identities such as

$$\pi = \frac{243}{3153920\sqrt{3}} \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+1}{r+1} \times \frac{(2r+3)(2r+5)(2r+7)(2r+9)(2r+11)}{(2r+13)} \left(\frac{3}{16}\right)^{r} - \frac{52488}{385}$$

and

$$\pi = \frac{1076778408885389 \times 34359738368}{242992069738496\sqrt{3} \times 27981667175} - \frac{34359738368}{27981667175 \cdot 2^{39}} \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+1}{r+1} \frac{1}{(2r+39)\,(16)^{r}}. \tag{2.12}$$

The first term of the right hand side of (2.12) estimates π to 12 significant digits. We will also obtain a formula for other constants like

$$\sqrt{11} = \frac{10673289}{50000000} \sum_{r=0}^{\infty} \binom{r+1}{r} \binom{2r+1}{r+1} \frac{(2r+3)(2r+5)}{(20)^{2r}}$$

and

$$\sqrt{14} = \frac{7}{2}\sum_{r=0}^{\infty}\binom{r+1}{r}\binom{2r+1}{r+1}\frac{1}{(2r+1)\,2^{5r}}.$$

For the sake of completeness, we now consider the 'finite' case of the integral (2.1) and obtain some nice closed form identities of sums.

3. The Finite Case

Consider

$$I_n = \int_0^{\frac{1}{a}} x^m\left(1-x^k\right)^n dx \tag{3.1}$$

and from calculations as in the previous section, we have

$$I_n = \sum_{r=0}^{n}\frac{(-1)^r\binom{n}{r}}{(rk+m+1)\,a^{rk+m+1}} \tag{3.2}$$

and

$$I_n = T_0\;{}_2F_1\left[\begin{matrix}\frac{m+1}{k},\quad -n\\ \frac{m+1+k}{k}\end{matrix}\;\middle|\;\frac{1}{a^k}\right], \tag{3.3}$$

where T_0 is given by (2.6), hence

$$\sum_{r=0}^{n}\frac{(-1)^r\binom{n}{r}}{(rk+m+1)\,a^{rk+m+1}} = T_0\;{}_2F_1\left[\begin{matrix}\frac{m+1}{k},\quad -n\\ \frac{m+1+k}{k}\end{matrix}\;\middle|\;\frac{1}{a^k}\right]. \tag{3.4}$$

We can also integrate (3.1) by parts and after laborious but straightforward algebra we obtain

$$I_n = \sum_{r=0}^{n}\frac{r!k^r\binom{n}{r}a^{-(rk+m+1)}\left(1-a^{-k}\right)^{n-r}}{\prod_{j=0}^{r}(jk+m+1)}. \tag{3.5}$$

Now,

$$\prod_{j=0}^{r}(jk+m+1) = k^{r+1}\left(\frac{m+1}{k}\right)_{r+1},$$

where $(b)_s$ is Pochhammer's symbol defined previously. From (3.5)

$$\begin{aligned} I_n &= \sum_{r=0}^{n}\frac{r!\binom{n}{r}a^{-(rk+m+1)}\left(1-a^{-k}\right)^{n-r}}{k\left(\frac{m+1}{k}\right)_{r+1}} \\ &= \frac{1}{k}\sum_{r=0}^{n}\binom{n}{r}a^{-(rk+m+1)}\left(1-a^{-k}\right)^{n-r}B\left(\frac{m+1}{k},r+1\right), \end{aligned} \tag{3.6}$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the classical Beta function.

From (3.4) and (3.6)

$$\begin{aligned} \sum_{r=0}^{n}\frac{(-1)^r\binom{n}{r}}{(rk+m+1)\,a^{rk+m+1}} &= \frac{1}{k}\sum_{r=0}^{n}\frac{\binom{n}{r}\left(1-a^{-k}\right)^{n-r}}{a^{rk+m+1}}B\left(\frac{m+1}{k},r+1\right) \\ &= \frac{\left(1-a^{-k}\right)^{n-r}}{(m+1)\,a^{m+1}}\;{}_2F_1\left[\begin{matrix}1,\quad -n\\ \frac{m+1+k}{k}\end{matrix}\;\middle|\;\frac{1}{1-a^k}\right]. \end{aligned} \tag{3.7}$$

When $a = 1$, then from (3.2)

$$I_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{rk + m + 1}. \tag{3.8}$$

From (3.5) the only contribution is the $r = n$ term, so that

$$I_n = \frac{n!k^n}{\prod_{j=0}^{n} (jk + m + 1)} = \frac{n!}{k \left(\frac{m+1}{k}\right)_{n+1}}. \tag{3.9}$$

From (3.8) and (3.9)

$$\begin{aligned}\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{rk + m + 1} &= \frac{n!k^n}{\prod_{j=0}^{n} (jk + m + 1)} \\ &= \frac{1}{k} B\left(n+1, \frac{m+1}{k}\right) \\ &= \frac{1}{(m+1)\binom{n+\frac{m+1}{k}}{n}}.\end{aligned}$$

An interesting case is when $m = np$, hence

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{rk + np + 1} = \frac{1}{(np+1)\binom{\frac{1+n(k+p)}{k}}{n}}$$

and for $k = 1$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{r + np + 1} = \frac{1}{(np+1)\binom{np+n+1}{n}} = \frac{1}{(pn+n+1)\binom{pn+n}{n}}.$$

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