SOME LANDAU TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATES ARE HÖLDER CONTINUOUS

S.S. DRAGOMIR AND C.I. PREDA

Abstract. Some inequalities of Landau type for functions whose derivates satisfy Hölder's condition are pointed out.

1. Introduction

Let $I = \mathbb{R}_+$ or $I = \mathbb{R}$. If $f : I \to \mathbb{R}$ is twice differentiable and $f, f^{''} \in L_p(I)$, $p \in [1, \infty]$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of the function f, so that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \cdot \|f^{''}\|_{p,I}^{\frac{1}{2}}, \tag{1.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, i.e, we recall

$$\|h\|_{\infty,I} := ess \sup_{t \in I} |h(t)|$$

and

$$\|h\|_{p,I} := \left(\int_I |h(t)|^p dt \right)^{\frac{1}{p}},$$

if $p \in [1, \infty)$.

The investigation of such inequalities was initiated by E. Landau [1] in 1913. He considered the case $p = \infty$ and showed that

$$C_\infty(\mathbb{R}_+) = 2 \qquad \text{and} \qquad C_\infty(\mathbb{R}) = \sqrt{2}, \tag{1.2}$$

are the best constants for which (1.1) holds.

In 1932, G.H. Hardy and J.E. Littlewood [2] proved (1.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \qquad \text{and} \qquad C_2(\mathbb{R}) = 1.$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [3] showed that the best constant $C_p(\mathbb{R}_+)$ in (1.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2 \quad \text{for} \quad p \in [1, \infty), \tag{1.3}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Actually $C_p(\mathbb{R}) \leq \sqrt{2}$ (see [4] by R.R. Kallman and G.-C. Rota and [5] by Z. Ditzian).

For other results concerning this problem, see Chapter I of [7].

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.

Key words and phrases. Landau Inequality, Hardy-Landau-Littlewood inequality, Hölder continuity, Lipschitz-continuity.

2. Some Results for f Bounded and f' Hölder Continuous

The following lemma is useful in what follows.

Lemma 1. *Let $C, D > 0$ and $r, u \in (0,1]$. Consider the function $g_{r,u} : (0,\infty) \to \mathbb{R}$ given by*

$$g_{r,u}(\lambda) = \frac{C}{\lambda^u} + D\lambda^r. \tag{2.1}$$

Define $\lambda_0 := \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0,\infty)$. Then, for $\lambda_1 \in (0,\infty)$ we have the bound

$$\inf_{\lambda\in(0,\lambda_1]} g_{r,u}(\lambda) = \begin{cases} \frac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}}} \cdot C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}} & \text{if } \lambda_1 \geq \lambda_0, \\ \\ \frac{C}{\lambda_1^u} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \lambda_0. \end{cases} \tag{2.2}$$

Proof. We observe that

$$g'_{r,u}(\lambda) = \frac{rD\lambda^{r+u} - Cu}{\lambda^{u+1}}, \quad \lambda \in (0,\infty).$$

The unique solution of the equation $g'_{r,u}(\lambda) = 0$, $\lambda \in (0,\infty)$ is $\lambda_0 = \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0,\infty)$. The function $g_{r,u}$ is decreasing on $(0,\lambda_0)$ and increasing on (λ_0,∞). The global minimum for $g_{r,u}$ on $(0,\infty)$ is

$$\begin{aligned} g_{r,u}(\lambda_0) &= \frac{C}{\left(\frac{uC}{rD}\right)^{\frac{u}{r+u}}} + D\left(\frac{uC}{rD}\right)^{\frac{r}{r+u}} = \frac{C(rD)^{\frac{u}{r+u}}}{(uC)^{\frac{u}{r+u}}} + \frac{D(uC)^{\frac{r}{r+u}}}{(rD)^{\frac{r}{r+u}}} \\ &= \frac{CrD + DuC}{(uC)^{\frac{u}{r+u}}(rD)^{\frac{r}{r+u}}} = \frac{CD(r+u)}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}} \cdot C^{\frac{u}{r+u}} \cdot D^{\frac{r}{r+u}}} \\ &= \frac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}}} C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}}, \end{aligned}$$

which proves that equality (2.2) ∎

The following particular cases are useful:

Corollary 1. *Let $C, D > 0$ and $r \in (0,1]$. Consider the function $g_r : (0,\infty) \to \mathbb{R}$ given by*

$$g_r(\lambda) = \frac{C}{\lambda} + D\lambda^r.$$

Define $\overline{\lambda_0} = \left(\frac{C}{rD}\right)^{\frac{1}{r+1}} \in (0,\infty)$. Then for $\lambda_1 \in (0,\infty)$ one has

$$\inf_{\lambda\in(0,\lambda_1]} g_r(\lambda) = \begin{cases} \frac{r+1}{r^{\frac{r}{r+1}}} \cdot C^{\frac{r}{r+1}} \cdot D^{\frac{1}{r+1}} & \text{if } \lambda_1 \geq \overline{\lambda_0}, \\ \\ \frac{C}{\lambda_1} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \overline{\lambda_0}. \end{cases} \tag{2.3}$$

Corollary 2. *Let $C, D > 0$ and $u \in (0,1]$. Consider the function $g_u : (0,\infty) \to \mathbb{R}$ given by*

$$g_u(\lambda) = \frac{C}{\lambda^u} + D\lambda.$$

Define $\widetilde{\lambda_0} = \left(\frac{uC}{D}\right)^{\frac{1}{1+u}} \in (0,\infty)$. *Then for* $\lambda_1 \in (0,\infty)$ *one has*

$$\inf_{\lambda\in(0,\lambda_1]} g_u(\lambda) = \begin{cases} \frac{1+u}{u^{\frac{u}{1+u}}} \cdot C^{\frac{1}{1+u}} \cdot D^{\frac{u}{1+u}} & \text{if } \lambda_1 \geq \widetilde{\lambda_0}, \\ \\ \frac{C}{\lambda_1^u} + D\lambda_1 & \text{if } 0 < \lambda_1 < \widetilde{\lambda_0}. \end{cases} \tag{2.4}$$

Remark 1. *If* $r = u = 1$ *then the following bound holds*

$$\inf_{\lambda\in(0,\lambda_1]} \left(\frac{C}{\lambda} + D\lambda\right) = \begin{cases} 2\sqrt{CD} & \text{if } \lambda_1 \geq \sqrt{\frac{C}{D}}, \\ \\ \frac{C}{\lambda_1} + D\lambda_1 & \text{if } 0 < \lambda_1 < \sqrt{\frac{C}{D}}. \end{cases} \tag{2.5}$$

The following theorem holds:

Theorem 1. *Let* I *be an interval in* $\mathbb{R}$ *and* $f : I \to \mathbb{R}$ *a locally absolutely continuous function on* I. *If* $f \in L_\infty(I)$ *and the derivative* $f' : I \to \mathbb{R}$ *satisfies the Hölder condition:*

$$|f'(t) - f'(s)| \leq H|t-s|^r \text{ for any } t, s \in I, \tag{2.6}$$

where $H > 0$ *and* $r \in (0,1]$ *are given, then* $f' \in L_\infty(I)$ *and one has the inequalities*

$$\|f'\|_{I,\infty} \leq \begin{cases} 2^{\frac{r}{r+1}} \left(1+\frac{1}{r}\right)^{\frac{r}{r+1}} \|f\|_{I,\infty}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \qquad \text{if} \quad m(I) \geq 2^{\frac{r+2}{r+1}} \left(\frac{\|f\|_{I,\infty}}{H}\right)^{\frac{1}{r+1}} \left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}, \\ \\ \frac{4\|f\|_{I,\infty}}{m(I)} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \text{if} \quad 0 \leq m(I) \leq 2^{\frac{r+2}{r+1}} \left(\frac{\|f\|_{I,\infty}}{H}\right)^{\frac{1}{r+1}} \left(1+\frac{1}{r}\right). \end{cases} \tag{2.7}$$

Proof. We start with the following identity

$$f(t) = f(a) + (t-a)f'(a) + \int_a^t [f'(s) - f'(a)]ds \tag{2.8}$$

to get

$$|f'(a)| \leq \left|\frac{f(t)-f(a)}{t-a}\right| + \frac{1}{|t-a|}\left|\int_a^t |f'(s) - f'(a)|ds\right|, \tag{2.9}$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Since f' is of $r - H$- Hölder type, then

$$\left|\int_a^t |f'(s) - f'(a)ds\right| \leq H\left|\int_a^t |s-a|^r ds\right| = \frac{H}{r+1}|t-a|^{r+1}. \tag{2.10}$$

So then by (2.9) and (2.10) we deduce

$$|f'(a)| \leq \frac{|f(t)-f(a)|}{|t-a|} + \frac{H}{r+1}|t-a|^r, \tag{2.11}$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Since $f \in L_\infty(I)$, then by (2.11) we obviously get that

$$|f'(a)| \leq \frac{2\|f\|_{I,\infty}}{|t-a|} + \frac{H}{r+1}|t-a|^r \tag{2.12}$$

for any $t \in I$ and a.e. $a \in I$, $t \neq a$.

Now observe that for any $a \in I$ and any $s \in \left(0, \frac{m(I)}{2}\right)$ there exists a $t \in I$ so that $s = |t-a|$ and then, by (2.12), we deduce

$$|f'(a)| \le \frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1}s^r \tag{2.13}$$

for a.e. $a \in I$ and every $s \in \left(0, \frac{m(I)}{2}\right)$. By taking the inequality (2.13) to the infimum over s on $\left(0, \frac{m(I)}{2}\right)$, we get that

$$|f'(a)| \le \inf_{s \in (0, \frac{m(I)}{2})} \left[\frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1}s^r\right] = K \tag{2.14}$$

for a.e. $a \in I$.

If we take the essential supremum over $a \in I$ in (2.14), we conclude that

$$\|f'\|_{I,\infty} \le K. \tag{2.15}$$

Making use of Corollary 1, we get

$$K = \begin{cases} \frac{r+1}{r^{\frac{r}{r+1}}}(2\|f\|_{I,\infty})^{\frac{r}{r+1}}\left(\frac{H}{r+1}\right)^{\frac{1}{r+1}} & \text{if } \frac{m(I)}{2} \ge \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}, \\ \frac{2\|f\|_{I,\infty}}{\frac{m(I)}{2}} + \frac{H}{r+1}\cdot\left(\frac{m(I)}{2}\right)^r & \text{if } 0 < \frac{m(I)}{2} < \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}. \end{cases}$$

giving the desired result (2.7). ∎

The following result also holds

Corollary 3. *With the assumption in Theorem 1 and if f' is L-Lipschitz then*

$$\|f'\|_{I,\infty} \le \begin{cases} 2\sqrt{\|f\|_{I,\infty}\cdot L} & \textit{if } m(I) \ge \sqrt{\frac{\|f\|_{I,\infty}}{L}}; \\ \frac{4\|f\|_{I,\infty}}{m(I)} + \frac{H}{4}m(I) & \textit{if } 0 < m(I) \le \sqrt{\frac{\|f\|_{I,\infty}}{L}}. \end{cases} \tag{2.16}$$

Remark 2. *This result was obtained by Niculescu and Buşe in* [6]*, see Theorem 3.*

3. Some Bounds for f and f' Hölder Continuous

The following result also holds:

Theorem 2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If f is $l-K$-Hölder type, i.e. it satisfies the condition*

$$|f(t) - f(s)| \le K|t-s|^l \quad \textit{for any } t, s, \in \mathring{I}, \tag{3.1}$$

where $K > 0$ and $l \in (0,1)$ are given, and the derivative $f' : I \to \mathbb{R}$ satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality

$$\|f'\|_{I,\infty} \le \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l}{1-l+r}}\cdot r^{\frac{r}{1-l+r}}\cdot(r+1)^{\frac{1-l}{r+1-l}}} K^{\frac{r}{r+1-l}}\cdot H^{\frac{1-l}{r+1-l}} \\ \qquad \textit{if } m(I) \ge 2\left[\frac{(1-l)K}{H}\right]^{\frac{1}{1-l+r}}\left(1+\frac{1}{r}\right)^{\frac{1}{1-l+r}}; \\ \frac{2(1-l)K}{[m(I)]^{1-l}} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \textit{if } 0 < m(I) < 2\left[\frac{(1-l)K}{H}\right]^{\frac{1}{1-l+r}}\left(1+\frac{1}{r}\right)^{\frac{1}{1-l+r}}. \end{cases} \tag{3.2}$$

Proof. We know (see the proof of Theorem 1) that

$$|f'(a)| \le \frac{|f(t)-f(a)|}{|t-a|} + \frac{H}{r+1}|t-a|^r \tag{3.3}$$

for any $t \in I$ and a.e. $a \in I$ with $a \ne t$.

Using the assumption that (3.1) holds, then, by (3.3) we may write that

$$|f'(a)| \le \frac{K}{|t-a|^{1-l}} + \frac{H}{r+1}|t-a|^r \tag{3.4}$$

for any $t \in I$ and a.e. $a \in I$ with $t \ne a$.

Using a similar argument to the one in Theorem 1, we may conclude that $\|f'\|_{I,\infty} \le S$, where

$$\begin{aligned} S &= \inf_{\lambda \in (0, \frac{m(I)}{2})} \left[\frac{K}{\lambda^{1-l}} + \frac{H}{r+1}\lambda^r \right] \\ &= \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l}{1-l+r}} \cdot r^{\frac{r}{1-l+r}}} K^{\frac{r}{r+1-l}} \cdot \left(\frac{H}{r+1}\right)^{\frac{1-l}{r+1-l}} & \text{if } \frac{m(I)}{2} \ge \left[\frac{(1-l)K}{r\frac{H}{r+1}}\right]^{\frac{1}{1-l+r}} \\ \frac{K}{(\frac{m(I)}{2})^{1-l}} + \frac{H}{r+1}\left(\frac{m(I)}{2}\right)^r & \text{if } 0 < \frac{m(I)}{2} \le \left[\frac{(1-l)K}{r \cdot \frac{H}{r+1}}\right]^{\frac{1}{1-l+r}} \end{cases} \end{aligned}$$

from where we deduce the desired inequality (3.2). ∎

The following corollary is useful.

Corollary 4. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If $f' \in L_p(I)$, $p > 1$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality:*

$$\|f'\|_{I,\infty} \le \begin{cases} \frac{pr+1}{p^{\frac{pr}{pr+1}}} \cdot \frac{1}{r^{\frac{pr}{pr+1}} \cdot (r+1)^{\frac{1}{pr+1}}} \|f'\|_{I,p}^{\frac{pr}{pr+1}} H^{\frac{1}{pr+1}} \\ \quad \text{if } m(I) \ge 2\left[\frac{\|f'\|_{I,p}}{pH}\right]^{\frac{p}{pr+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{p}{pr+1}}; \\ \frac{\|f'\|_{I,p} \cdot 2^{\frac{1}{p}}}{[m(I)]^{\frac{1}{p}}} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \quad \text{if } 0 < m(I) < 2\left[\frac{\|f'\|_{I,p}}{pH}\right]^{\frac{p}{pr+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{p}{pr+1}}. \end{cases} \tag{3.5}$$

Proof. If $f' \in L_p(I)$, then we have

$$\begin{aligned} |f(b)-f(a)| &= \left|\int_a^b f'(s)ds\right| \le \left|\int_a^b |f'(s)|ds\right| \\ &\le |b-a|^{\frac{1}{q}} \left|\int_a^b |f'(s)|^p ds\right|^{\frac{1}{p}} \\ &\le |b-a|^{1-\frac{1}{p}} \cdot \|f'\|_{I,p}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, for a.e. $a, b \in I$.

Using Theorem 2 for $l = 1 - \frac{1}{p}$ and $K = \|f'\|_{I,p}$ we deduce the desired result (3.5). ∎

Finally we may state the following corollary as well.

Corollary 5. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If $f' \in L_1(I)$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality*

$$\|f'\|_{I,\infty} \leq \begin{cases} \left(1+\frac{1}{r}\right)^{\frac{r}{r+1}} \cdot \|f'\|_{I,1}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \qquad \text{if } m(I) \geq 2\left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}; \\ \\ \frac{2\|f'\|_{I,1}}{m(I)} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \text{if } 0 < m(I) < 2\left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}. \end{cases} \tag{3.6}$$

References

[1] E. LANDAU, Einige Ungleichungen für zweimal differentzierban funktionen, *Proc. London Math. Soc.*, **13** (1913), 43-49.

[2] G.H. HARDY and J.E. LITTLEWOOD, Some integral inequalities connected with the calculus of variations, *Quart. J. Math. Oxford Ser.,* **3** (1932), 241-252.

[3] G.H. HARDY, E. LANDAU and J.E. LITTLEWOOD, Some inequalities satisfied by the integrals or derivatives of real or analytic functions, *Math. Z.*, **39** (1935), 677-695.

[4] R.R. KALLMAN and G.-C. ROTA, On the inequality $\|f'\|^2 \leq 4\|f\| \cdot \|f''\|$, in "*Inequalities*", vol. II (O. Shisha, Ed) pp. 187-192. Academic Press, New York, 1970.

[5] Z. DITZIAN, Remarks, questions and conjections on Landau-Kolmogorov-type inequalities, *Math. Ineq. Appl.*, **3** (2000), 15-24.

[6] C.P. NICULESCU and C. BUŞE, The Hardy-Landau-Littlewood inequalities with less smoothness, *J. Inequal. in Pure and Appl. Math.*, to appear.

[7] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

School of Computer Science and Mathematics, Victoria University of Technology, Melbourne City MC, Victoria 8001, Australia.

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`

Department of Mathematics, University of the West, B-dul V. Parvan 4, Timisoara, Ro-1900, Romania